Relaxation limits for compressible one-velocity Baer-Nunziato multi-fluid systems: the overdamping case

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Abstract

In this paper we investigate two types of relaxation processes in the context of one-velocity multi-fluid flows with two different pressure laws. First, we justify the pressure-relaxation limit from a one-velocity Baer-Nunziato system to a Kapila model, in a uniform manner with respect to the time-relaxation parameter associated to the friction forces modeled in the equation of the velocity. This uniformity allows us to further address the friction-relaxation limit associated to the well-known *overdamping* phenomena. More precisely, we show that the diffusely time-rescaled solution of the Kapila system converges strongly to the solution of a porous media type system, as the relaxation time tends to zero, at an explicit rate of convergence. Our analysis is done in the framework of critical homogeneous Besov spaces and the differentiation of the low and high-frequency behaviors of the solution is crucial to capture the subtle dissipative properties of the system. To that matter, as in the context of partially dissipative hyperbolic system [15], we set the threshold between the two frequency regimes depending on the time-relaxation parameter so as to overcome the overdamping phenomena in this multiphasic flow context.

Keywords— Multi-fluid system, relaxation limit, overdamping, critical regularity, porous media system, Kapila system, Baer-Nunziato System

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1 Introduction

1.1 Models

Multi-fluid systems have been used to simulate a wide range of physical mixing phenomena, from engineering to biological systems (cf. [1, 8, 28, 38] and the references therein). In the present paper we investigate a one-velocity Baer-Nunziato system, that can be found in the mathematical work [7] of Bresch and Hillairet, in the presence of drag forces and in its following inviscid formulation:

$$\begin{cases} \alpha_+^{\varepsilon,\tau} + \alpha_-^{\varepsilon,\tau} = 1, \\ \partial_t \alpha_+^{\varepsilon,\tau} + u^{\varepsilon,\tau} \cdot \nabla \alpha_+^{\varepsilon,\tau} = \dfrac{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}}{\varepsilon}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big), \\ \partial_t(\alpha_+^{\varepsilon,\tau}\rho_+^{\varepsilon,\tau}) + \operatorname{div}(\alpha_+^{\varepsilon,\tau}\rho_+^{\varepsilon,\tau}u^{\varepsilon,\tau}) = 0, \\ \partial_t \rho^{\varepsilon,\tau} + \operatorname{div}(\rho^{\varepsilon,\tau}u^{\varepsilon,\tau}) = 0, \\ \partial_t(\rho^{\varepsilon,\tau}u^{\varepsilon,\tau}) + \operatorname{div}(\rho^{\varepsilon,\tau}u^{\varepsilon,\tau}\otimes u^{\varepsilon,\tau}) + \nabla P^{\varepsilon,\tau} + \dfrac{\rho^{\varepsilon,\tau}u^{\varepsilon,\tau}}{\tau} = 0, \\ \rho^{\varepsilon,\tau} = \alpha_+^{\varepsilon,\tau}\rho_+^{\varepsilon,\tau} + \alpha_-^{\varepsilon,\tau}\rho_-^{\varepsilon,\tau}, \\ P^{\varepsilon,\tau} = \alpha_+^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \alpha_-^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau}), \qquad x \in \mathbb{R}^d, \quad t > 0, \end{cases} \tag{BN}$$

where the unknowns $\alpha_\pm^{\varepsilon,\tau} = \alpha_\pm^{\varepsilon,\tau}(t,x) \in [0,1]$, $\rho_\pm^{\varepsilon,\tau} = \rho_\pm^{\varepsilon,\tau}(t,x) \geq 0$ and $u^{\varepsilon,\tau} = u^{\varepsilon,\tau}(t,x) \in \mathbb{R}^d$ stand for the volume fractions, the densities and the common velocity of two fluids (denoted by $+$ and $-$), respectively. The two positive constants ε and τ are (small) relaxation parameters, associated respectively to the pressure-relaxation and large friction limits. Finally, the two pressures P_+, P_- take the gamma-law forms

$$P_\pm(s) = A_\pm s^{\gamma_\pm} \quad \text{with constants} \quad A_\pm > 0, \qquad 1 \leq \gamma_- < \gamma_+. \tag{1.1}$$

The *Baer-Nunziato* terminology refers to the pressure-relaxation mechanism in the equations of the volume fractions. Numerically, such relaxation procedure can simplify its resolution as it reduces the number of constraints by introducing new unknowns: *two pressures instead of one*. The readers can see [7] and references therein for more discussions on this pressure-relaxation process. Very recently, the one-dimensional version of System (BN) was rigorously derived by Bresch, Burtea and Lagoutiére in [5].

At the formal level, if the solution $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ of System (BN) tends to some limit $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ as $\varepsilon \to 0$, then $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ satisfies the following so-called one-velocity Kapila system (cf. [26]):

$$\begin{cases} \alpha_+^\tau + \alpha_-^\tau = 1, \\ \partial_t(\alpha_+^\tau\rho_+^\tau) + \operatorname{div}(\alpha_+^\tau\rho_+^\tau u^\tau) = 0, \\ \partial_t\rho^\tau + \operatorname{div}(\rho^\tau u^\tau) = 0, \\ \partial_t(\rho^\tau u^\tau) + \operatorname{div}(\rho^\tau u^\tau \otimes u^\tau) + \nabla P^\tau + \dfrac{\rho^\tau u^\tau}{\tau} = 0, \\ \rho^\tau = \alpha_+^\tau\rho_+^\tau + \alpha_-^\tau\rho_-^\tau, \\ P^\tau = P_+(\rho_+^\tau) = P_-(\rho_-^\tau). \end{cases} \tag{K}$$

In terms of the new variables $(\alpha_+^\tau\rho_+^\tau, \alpha_-^\tau\rho_-^\tau, u^\tau)$, System (K) can be written as a mono-velocity multi-fluid model with an algebraic closure law (common pressure depending only on $\alpha_+^\tau\rho_+^\tau$ and $\alpha_-^\tau\rho_-^\tau$) (cf. [10,33]).

Then, we further investigate the friction-relaxation limit of System (K) as $\tau \to 0$. Inspired by the works [13,25,41] concerning the relaxation problems for the compressible Euler system with damping, we introduce a large time-scale $O(\frac{1}{\tau})$ and define the following scaling

$$(\beta_\pm^\tau, \varrho_\pm^\tau, v^\tau)(s,x) := \left(\alpha_\pm^\tau, \rho_\pm^\tau, \frac{u^\tau}{\tau}\right)\left(\frac{s}{\tau}, x\right). \tag{1.2}$$

Under the diffusive scaling (1.2), System (K) reads

$$\begin{cases}\beta_+^\tau + \beta_-^\tau = 1,\\ \partial_s(\beta_+^\tau \varrho_+^\tau) + \mathrm{div}\,(\beta_\pm^\tau \varrho_+^\tau v^\tau) = 0,\\ \partial_s \varrho^\tau + \mathrm{div}\,(\varrho^\tau v^\tau) = 0,\\ \tau^2 \partial_s(\varrho^\tau v^\tau) + \tau^2 \mathrm{div}\,(\varrho^\tau v^\tau \otimes v^\tau) + \nabla \Pi^\tau + \varrho^\tau v^\tau = 0,\\ \varrho^\tau = \beta_+^\tau \rho_+^\tau + \beta_-^\tau \varrho_-^\tau,\\ \Pi^\tau = P_+(\varrho_+^\tau) = P_-(\varrho_-^\tau).\end{cases} \tag{K_τ}$$

As $\tau \to 0$, one then expects that $(\beta_\pm^\tau, \varrho_\pm^\tau, v^\tau)$ converges to some limit $(\beta_\pm, \varrho_\pm, v)$ which is the solution of the following two-phase system:

$$\begin{cases}\beta_+ + \beta_- = 1,\\ \partial_s(\beta_+ \varrho_+) + \mathrm{div}\,(\beta_+ \varrho_+ v) = 0,\\ \partial_s \varrho + \mathrm{div}\,(\varrho v) = 0,\\ \nabla \Pi + \varrho v = 0,\\ \varrho = \beta_+ \varrho_+ + \beta_- \varrho_-,\\ \Pi = P_+(\varrho_+) = P_-(\varrho_-).\end{cases} \tag{1.3}$$

System (1.3) can be viewed as the porous media type equations

$$\begin{cases}\beta_+ + \beta_- = 1,\\ \partial_s \beta_+ + v \cdot \nabla \beta_+ = -\dfrac{(\gamma_+ - \gamma_-)\beta_+\beta_-}{\gamma_+\beta_- + \gamma_-\beta_+}\,\mathrm{div}\, v,\\ \partial_s \Pi + v \cdot \nabla \Pi = \dfrac{\gamma_+\gamma_-\Pi}{\gamma_+\beta_- + \gamma_-\beta_+}\mathrm{div}\,(\dfrac{\nabla \Pi}{\beta_+\varrho_+ + \beta_-\varrho_-}),\\ \Pi = P_+(\varrho_+) = P_-(\varrho_-),\end{cases} \tag{PM}$$

coupled with the Darcy's law $(1.3)_4$.

The main goals of this article is to justify rigorously the above pressure-relaxation limit as $\varepsilon \to 0$ from System (BN) to System (K) and the friction-relaxation limit as $\tau \to 0$ from System (K_τ) to System (1.3). To achieve such results, we will:

- Extend the analysis perform in [11] that does not provide the uniform-in-τ a-priori estimates needed to pass to the limit as $\tau \to 0$.
- Generalize the result and the techniques developed in [15] that can not be applied directly to System (BN) due to the complex forms of the pressure $P^{\varepsilon,\tau}$ and the lack of symmetry.

For both relaxation limits, our proofs rely on the construction of global-in-time strong small solutions that are close to some constant equilibrium state. In other words, we consider solutions $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ to System (BN) (resp. $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ to System (K)) with positive densities and volume fractions which, as $|x| \to \infty$, tend to some thermodynamically stable equilibrium state $(\bar\alpha_\pm, \bar\rho_\pm, 0)$ of system (K) fulfilling

$$0 < \bar\alpha_\pm < 1, \qquad \bar\alpha_+ + \bar\alpha_- = 1, \qquad \bar\rho_\pm > 0, \qquad P_+(\bar\rho_+) = P_-(\bar\rho_-). \tag{1.4}$$

We define the corresponding equilibrium state for the total density and the total pressure as

$$\bar{\rho} := \bar{\alpha}_+\bar{\rho}_+ + \bar{\alpha}_-\bar{\rho}_-, \qquad \bar{P} := P_+(\bar{\rho}_+) = P_-(\bar{\rho}_-). \tag{1.5}$$

We will prove the uniform in ε and τ (such that $\varepsilon \leq \tau$) existence of global-in-time strong solutions for System (BN) subject to the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ close to the equilibrium state satisfying (1.4). And as a consequence of these uniform estimates we justify the strong convergence of the global solutions to System (BN) when the relaxation parameters ε and τ go to zero continuously, in this specific order.

On the other hand, it is natural to ask what happens if one reversed the order of the limits. To investigate this process, we introduce a diffusive scaling similar to (1.2) as follows

$$(\beta_\pm^{\varepsilon,\tau}, \varrho_\pm^{\varepsilon,\tau}, v^{\varepsilon,\tau})(s,x) := \left(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, \frac{u^{\varepsilon,\tau}}{\tau}\right)\left(\frac{s}{\tau}, x\right). \tag{1.6}$$

Under such scaling, System (BN) reads:

$$\begin{cases} \partial_s \beta_\pm^{\varepsilon,\tau} + v^{\varepsilon,\tau}\cdot\nabla\beta_\pm^{\varepsilon,\tau} = \pm\dfrac{\beta_+^{\varepsilon,\tau}\beta_-^{\varepsilon,\tau}}{\varepsilon\tau}\big(P_+(\varrho_+^{\varepsilon,\tau}) - P_-(\varrho_-^{\varepsilon,\tau})\big), \\ \partial_s(\beta_\pm^{\varepsilon,\tau}\varrho_\pm^{\varepsilon,\tau}) + \mathrm{div}\,(\beta_\pm^{\varepsilon,\tau}\varrho_\pm^{\varepsilon,\tau}v^{\varepsilon,\tau}) = 0, \\ \tau^2\partial_s(\varrho^{\varepsilon,\tau}v^{\varepsilon,\tau}) + \tau^2\mathrm{div}\,(\varrho^{\varepsilon,\tau}v^{\varepsilon,\tau}\otimes v^{\varepsilon,\tau}) + \nabla\Pi^{\varepsilon,\tau} + \varrho^{\varepsilon,\tau}v^{\varepsilon,\tau} = 0, \\ \varrho^{\varepsilon,\tau} = \beta_+^{\varepsilon,\tau}\varrho_+^{\varepsilon,\tau} + \beta_-^{\varepsilon,\tau}\varrho_-^{\varepsilon,\tau}, \\ \Pi^{\varepsilon,\tau} = \beta_+^{\varepsilon,\tau}P_+(\varrho_+^{\varepsilon,\tau}) + \beta_-^{\varepsilon,\tau}P_-(\varrho_-^{\varepsilon,\tau}). \end{cases} \tag{BN$_\tau$}$$

The crucial observation is that the parameter τ now also appears under the pressure-relaxation term in the equation of the volume fractions. This reveals that, as $\tau \to 0$, the two pressures in System (BN$_\tau$) converge to a common pressure, and the Kapila system (K) is therefore not involved in the limit processes, in contrast to the reverse limit sequence. This is in accordance with our results as we are only obtain to justify the uniform well-posedness for $\varepsilon \leq \tau$, which implies that $\varepsilon \to 0$ when $\tau \to 0$.

Such a condition, $\varepsilon \leq \tau$, appears in the spectral analysis of the system, see Section 1.4, so as to the ensure the *natural* dissipative behavior of the system. Let us now give another explanation for this condition on the parameters. First of all, ε and τ are both time-relaxation parameters and the diffusive scaling associated to the infinite damping limit focuses on the large time-scale $\mathcal{O}(\frac{t}{\tau})$, hence on the long-time behavior of the solution when $\tau \ll 1$. Moreover, such scaling is adapted to study solutions which converges to constant steady states (i.e. natural mechanic equilibrium) at infinity. And since such steady states have to satisfy System (BN), to avoid initial time-layer, they must verify $\bar{P}_+ = \bar{P}_-$ to satisfy the equations of the volume fractions. Now, as we know that our solution is close to the steady state at infinity, it is clear that the pressures should agree at infinity. It might seem that τ takes the role of the parameter ε but actually it is just that with such diffusive scaling we are looking at a time-scale in which the pressure need to agree due to the specific but mandatory choice of steady states.

Figure 1 summarizes the limit processes that we tackle in this article.

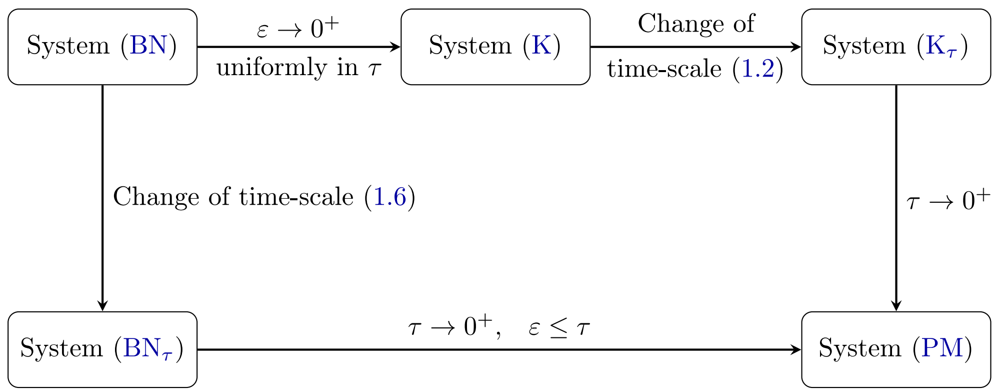

Figure 1: Relaxation limits diagram.

1.2 State of the art and key points of our analysis

There is an extensive literature on the mathematical analysis of multi-fluid systems. For example, in the one-velocity case, the global existence of weak solutions has been studied in [10, 29, 33, 37], and the global well-posedness and optimal time-decay rates of strong solutions has been established in the framewok of Sobolev spaces [23, 43, 44] and critical Besov spaces [11, 24]. We also refer to [6, 9, 20, 27] on the study of multi-fluid systems in the two-velocity case. Complete reviews on multi-fluid systems are presented in [4, 40].

The study of relaxation problems associated to systems of conservation laws can be tracked back to the work [12] by Chen, Levermore and Liu. Recently, Giovangigli and Yong in [21, 22] studied a relaxation problem arising in the dynamics of perfect gases out of thermodynamical-equilibrium.

The present paper is a follow-up to the paper [11] by Burtea, Crin-Barat and Tan where the authors justified the pressure-relaxation limit ($\varepsilon \to 0$) for System (BN) to System (K) in the context of vanishing viscosity. As a by-product, it was shown that the diffusion effect in the velocity equation $(\mathrm{BN})_5$ is not necessary to ensure the global well-posedness of the Cauchy problem for System (BN). And for this reason, we omit the beneficial diffusion effect here, even though the original derivation of (BN) in [7] relies on the viscosity term. Moreover, in [11], the smallness condition on initial data for the global well-posedness result depends on $\min\{\tau, \frac{1}{\tau}\}$ and therefore does not allow to further investigate the limit when $\tau \to 0$. This is due to the overdamping phenomena: *as the friction coefficient* $\frac{1}{\tau}$ *gets larger after a threshold, the decay rates of hyperbolic relaxation systems do not necessarily increase and on the contrary follow* $\min\{\tau, \frac{1}{\tau}\}$ (cf. Remark 3.1 in [11] and [45] for more details).

The main novelty of the present paper is to solve the issue related to the overdamping phenomenon in the context of multi-fluid flows and establish global a-priori estimates which are uniform with respect to both the relaxation parameters ε, τ for $\varepsilon \leq \tau \leq 1$. This improves significantly the global well-posedness result for System (BN) obtained in [11] and enables us to consider the relaxation limit as $\tau \to 0$.

Recently, in [14, 15], the issue concerning the overdamping phenomena has been well solved in the context of the compressible Euler system with damping in the critical space $\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}$, where the relaxation limit toward the porous media equation has been justified rigorously and an explicit convergence rate of the process was exhibited. The readers also can refer to [16] for the application of a hyperbolic-parabolic chemotaxis system. As explained in the notes [19], the regularity index $\frac{d}{2}+1$ is called critical for initial data of general hyperbolic systems since $\dot{B}^{\frac{d}{2}+1}_{2,1}$ is embedded in the set of globally Lipschitz functions.

Let us recall briefly the main ideas from [14, 15]. By a careful analysis of the spectral matrix of the compressible Euler system with damping, one sees that the solutions have very different behaviors in the low and high frequencies. Indeed, we expect that the low-frequency part of the density behaves as the heat equation while the other part is purely damped, and the high frequencies part of the solution decays exponentially. Therefore, it is natural to consider different regularity settings for the low and high frequencies of the solutions and determining a suitable threshold between both regime turns out to be crucial.

The spectral analysis for the linearized compressible Euler system with damping (cf. [36, 39]) suggests that the frequency space shall be separated into the low-frequency region $|\xi| \lesssim \frac{1}{\tau}$ and the high-frequency region $|\xi| \gtrsim \frac{1}{\tau}$ (with the suitable threshold $-[\log_2 \tau]$ between both frequency-regimes in the Besov dyadic formulation) to recover uniform estimates and the optimal regularity of the solution. Formally, this implies that as $\tau \to 0$, the low-frequency region cover the whole frequency space and will therefore be dominant at the limit. Such considerations allows to comprehend totally the overdamping phenomena and can be justified rigorously thanks to:

- The functional framework of critical Besov spaces allows us to reproduce perfectly the behaviors observed in the spectral analysis.
- A purely damped mode corresponding to Darcy's law shall be introduced in the low-frequency setting to partially diagonalize System (BN) and extract $\mathcal{O}(\varepsilon)$ bounds on crucial quantities.
- The construction of a Lyapunov functional in the spirit of Beauchard and Zuazua in [3] allows us to study the partially dissipative structures in high frequencies.

Nevertheless, the above method developed in [14, 15] based on the functional space $\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}$ is not suitable to derive qualitative regularity properties which are uniform with respect to the relaxation parameter τ. This is mainly due to the *complex form of the total pressure* $P^{\varepsilon,\tau}$ in the velocity equation $(\mathrm{BN})_5$, which leads to a lack of time integrability on $\mathbb{R}_+$ for $(\alpha_\pm^{\varepsilon,\tau} - \bar{\alpha}_\pm, \rho_\pm^{\varepsilon,\tau} - \bar{\rho}_\pm)$ (see Remarks 3.2-3.3). To overcome this difficulty, in the current paper we assume that the initial data are uniformly bounded in $\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$ and establish additional uniform estimates for the solutions compared with the works [14, 15]. Another difficulty is that owing to the parameter ε in $(\mathrm{BN})_1$, one can not perform a rescaling as in [14, 15] to reduce the proof to the case $\tau = 1$ and then recover the corresponding uniform estimates with respect to τ thanks to the homogeneity of Besov norms. To handle it, mixed L^2 and L^1-time estimates, with elaborate dependencies in ε and τ, at several regularity levels will be performed. Moreover, we will introduce a nonlinear damped mode involving the pressure term for low frequencies and adapt the Lyapunov functional techniques developed in [3] for all frequencies.

The rest of the paper is organized as follows. In Subsection 1.3, we state our main results. In

Subsection 1.4, we recall the reformulation of System (BN) from [11] and do a spectral analysis of the linearized equations. Section 2 is devoted to some notations of Besov spaces and Littlewood-Paley decomposition. In Section 3, we establish the uniform a-priori estimates of the linear problem. Then in Section 4, we prove the global existence of solutions for Systems (BN), (K) and (PM), respectively. In Section 5, we justify the strong relaxation limits from both and System (BN) to System (K) as $\varepsilon \to 0$ and System (K_τ) to System (PM) as $\tau \to 0$ with explicit convergence rates. A few technical results are recalled in Appendix.

1.3 Main results

Our first theorem concerns the uniform-in-ε-τ global well-posedness of System (BN) in the critical regularity framework.

Theorem 1.1. *Let $d \geq 2$ and $0 < \varepsilon \leq \tau \leq 1$. Given the constants $\bar{\alpha}_\pm, \bar{\rho}_\pm$ verifying (1.4)-(1.5), assume that the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ satisfies $(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$. There exists a positive constant c_0 independent of τ and ε such that if*

$$\|(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq c_0, \tag{1.7}$$

then the Cauchy problem of System (BN) with the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ has a unique global solution $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ satisfying

$$\begin{cases} (\alpha_\pm^{\varepsilon,\tau} - \bar{\alpha}_\pm, \rho_\pm^{\varepsilon,\tau} - \bar{\rho}_\pm, u^{\varepsilon,\tau}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\ P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}) \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\ P^{\varepsilon,\tau} - \bar{P} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}), \\ u^{\varepsilon,\tau} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}). \end{cases} \tag{1.8}$$

Moreover, the following uniform estimates holds:

$$\begin{aligned} &\|(\alpha_\pm^{\varepsilon,\tau} - \bar{\alpha}_\pm, \rho_\pm^{\varepsilon,\tau} - \bar{\rho}_\pm, u^{\varepsilon,\tau})\|_{L^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \alpha_\pm^{\varepsilon,\tau}, \partial_t \rho_\pm^{\varepsilon,\tau}, \partial_t u^{\varepsilon,\tau})\|_{L^1(\dot{B}^{\frac{d}{2}})} \\ &\quad + \frac{1}{\varepsilon}\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{L^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\ &\quad + \tau\|P^{\varepsilon,\tau} - \bar{P}\|_{L^1(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau}\|P^{\varepsilon,\tau} - \bar{P}\|_{L^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\ &\quad + \|u^{\varepsilon,\tau}\|_{L^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u^{\varepsilon,\tau}\|_{L^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|\frac{\rho^{\varepsilon,\tau} u^{\varepsilon,\tau}}{\tau} + \nabla P^{\varepsilon,\tau}\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \\ &\leq C\|(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}, \end{aligned} \tag{1.9}$$

where $C > 0$ is a uniform constant independent of τ, ε and time.

Remark 1.1. *It should be emphasized that the regularity and decay-in-τ properties of the effective unknown $\frac{\rho^{\varepsilon,\tau} u^{\varepsilon,\tau}}{\tau} + \nabla P^{\varepsilon,\tau}$ is better than the one satisfied by the whole solution $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$. This is consistent with Darcy's law and plays key role in the justification of the relaxation limits.*

By compactness arguments, the uniform estimates (1.9) imply the relaxation processes mentioned before. We obtain the following global well-posedness theorems for Systems (K) and (PM) in the critical regularity framework.

Theorem 1.2. *Let $d \geq 2$ and $0 < \tau \leq 1$. Given the constants $\bar{\alpha}_\pm, \bar{\rho}_\pm$ verifying* (1.4)-(1.5)*, assume that the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ satisfies $(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$. There exists a positive constant c_1 independent of τ such that if*

$$\|(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq c_1, \tag{1.10}$$

then the Cauchy problem of System (K) *with the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ admits a unique global solution $(\alpha^\tau_\pm, \rho^\tau_\pm, u^\tau)$ satisfying*

$$\begin{cases} (\alpha^\tau_\pm - \bar{\alpha}_\pm, \rho^\tau_\pm - \bar{\rho}_\pm, u^\tau) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\ P^\tau - \bar{P} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}), \\ u^\tau \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}). \end{cases} \tag{1.11}$$

Moreover, the following uniform estimates holds:

$$\begin{aligned} &\|(\alpha^\tau_\pm - \bar{\alpha}_\pm, \rho^\tau_\pm - \bar{\rho}_\pm, u^\tau)\|_{L^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \alpha^\tau_\pm, \partial_t \rho^\tau_\pm, \partial_t u^\tau)\|_{L^1(\dot{B}^{\frac{d}{2}})} \\ &\quad + \tau \|P^\tau - \bar{P}\|_{L^1(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau}\|P^\tau - \bar{P}\|_{L^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\ &\quad + \|u^\tau\|_{L^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u^\tau\|_{L^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|\frac{\rho^{\varepsilon,\tau} u^\tau}{\tau} + \nabla P^\tau\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \\ &\leq C\|(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}, \end{aligned} \tag{1.12}$$

where $C > 0$ is a uniform constant independent of τ and time.

Theorem 1.3. *Let $d \geq 2$. Given the constants $\bar{\alpha}_\pm, \bar{\rho}_\pm$ verifying* (1.4)-(1.5)*, assume that the initial data $(\beta_{\pm,0}, \varrho_{\pm,0})$ satisfies $(\beta_{\pm,0} - \bar{\alpha}_\pm, \varrho_{\pm,0} - \bar{\rho}_\pm) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$ and*

$$\|(\beta_{\pm,0} - \bar{\alpha}_\pm, \varrho_{\pm,0} - \bar{\rho}_\pm)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq c_2, \tag{1.13}$$

for a positive constant c_2 independent of τ, then the Cauchy problem of System (PM) *with the initial data $(\beta_{\pm,0}, \varrho_{\pm,0})$ admits a unique global solution (β, ϱ), which satisfies*

$$\begin{cases} \beta_\pm - \bar{\alpha}_\pm \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\ \varrho_\pm - \bar{\rho}_\pm \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}). \end{cases} \tag{1.14}$$

Moreover, the following uniform estimates holds:

$$\begin{aligned} &\|(\beta_\pm - \bar{\alpha}_\pm, \varrho_\pm - \bar{\rho}_\pm)\|_{L^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \beta_\pm, \partial_t \varrho_\pm)\|_{L^1(\dot{B}^{\frac{d}{2}})} + \|\varrho_\pm - \bar{\rho}_\pm\|_{L^1(\dot{B}^{\frac{d}{2}+1} \cap \dot{B}^{\frac{d}{2}+3})} \\ &\leq C\|(\beta_{\pm,0} - \bar{\alpha}_\pm, \varrho_{\pm,0} - \bar{\rho}_\pm)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}, \end{aligned} \tag{1.15}$$

where $C > 0$ is a uniform constant independent of time.

Remark 1.2. *Theorems 1.2 and 1.3 are direct consequences of the uniform estimates* (1.9) *obtained in Theorem 1.1 via weak compactness arguments and Fatou's property* (*see Section 4.2 and 4.3, respectively*)*. We mention that these theorems can also be proved independently of System* (BN)*, and the a-priori estimates for Systems* (K) *and* (PM) *follows the computations carried out in Section 3.*

Finally, we justify rigorously the relaxation limit from System (BN) to System (K) as $\varepsilon \to 0$, and further the relaxation limit from System (K) to System (1.3) as $\tau \to 0$, with explicit convergence rates.

Theorem 1.4. *Let $d \geq 2$ and $0 < \varepsilon \leq \tau \leq 1$. Given the constants $\bar{\alpha}_{\pm}, \bar{\rho}_{\pm}$ verifying (1.4)-(1.5), let $(\alpha_{\pm}^{\varepsilon,\tau}, \rho_{\pm}^{\varepsilon,\tau}, u^{\varepsilon,\tau})$, $(\alpha_{\pm}^{\tau}, \rho_{\pm}^{\tau}, u^{\tau})$ and $(\beta_{\pm}, \varrho_{\pm})$ be the global solution to the Cauchy problems of Systems (BN), (K) and (PM) obtained from Theorems 1.1-1.3 associated to their corresponding initial data $(\alpha_{\pm,0}^{\varepsilon,\tau}, \rho_{\pm,0}^{\varepsilon,\tau}, u_0^{\varepsilon,\tau})$, $(\alpha_{\pm,0}^{\tau}, \rho_{\pm,0}^{\tau}, u_0^{\tau})$ and $(\beta_{\pm,0}, \varrho_{\pm,0})$, respectively.*

- *Let the initial quantities $P_0^{\varepsilon,\tau} - P_0^{\tau}$ and $Y_0^{\varepsilon,\tau} - Y_0^{\tau}$ be denoted by (5.1) and (5.3), respectively. If $d \geq 3$ and*

$$\|\big(P_+(\rho_{+,0}^{\varepsilon,\tau}) - P_-(\rho_{-,0}^{\varepsilon,\tau}), Y_0^{\varepsilon,\tau} - Y_0^{\tau}, P_0^{\varepsilon,\tau} - P_0^{\tau}, u_0^{\varepsilon,\tau} - u_0^{\tau}\big)\|_{\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1}} \leq \sqrt{\varepsilon\tau}, \tag{1.16}$$

  then there exists a positive constant C_1 independent of ε, τ and time such that the following estimates hold:

$$\begin{aligned}\|(\alpha_{\pm}^{\varepsilon,\tau} - \alpha_{\pm}^{\tau}, \rho_{\pm}^{\varepsilon,\tau} - \rho_{\pm}^{\tau}, u^{\varepsilon,\tau} - u^{\tau})\|_{L^\infty(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})} + \sqrt{\tau}\|\rho_{\pm}^{\varepsilon,\tau} - \rho_{\pm}^{\tau}\|_{L^2(\dot{B}^{\frac{d}{2}-1})} \\ + \frac{1}{\sqrt{\tau}}\|u^{\varepsilon,\tau} - u^{\tau}\|_{L^2(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})} + \|u^{\varepsilon,\tau} - u^{\tau}\|_{L^1(\dot{B}^{\frac{d}{2}-1})} \leq C_1\sqrt{\varepsilon\tau}.\end{aligned} \tag{1.17}$$

  Therefore, as $\varepsilon \to 0$, $(\alpha_{\pm}^{\varepsilon,\tau}, \rho_{\pm}^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ converges strongly to $(\alpha_{\pm}^{\tau}, \rho_{\pm}^{\tau}, u^{\tau})$ in $L^\infty(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})$.

- *Furthermore, define $(\beta_{\pm}^{\tau}, \varrho_{\pm}^{\tau}, v^{\tau})$ by the diffusive scaling (1.2) and v by Darcy's law $(1.3)_4$. Let the initial quantity $Z_0^{\tau} - Z_0$ be denoted by (5.29). If*

$$\|Z_0^{\tau} - Z_0\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}}} + \|\varrho_{\pm,0}^{\tau} - \varrho_{\pm,0}\|_{\dot{B}^{\frac{d}{2}-1}} \leq \tau, \tag{1.18}$$

  then there exists a positive constant C_2 independent of τ and time such that the following estimates hold:

$$\|(\beta_{\pm}^{\tau} - \beta_{\pm}, \varrho_{\pm}^{\tau} - \varrho_{\pm})\|_{L^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\varrho_{\pm}^{\tau} - \varrho_{\pm}\|_{L^1(\dot{B}^{\frac{d}{2}+1})} + \|v^{\tau} - v\|_{L^1(\dot{B}^{\frac{d}{2}})} \leq C_2\tau. \tag{1.19}$$

  Therefore, as $\tau \to 0$, $(\beta_{\pm}^{\tau}, \varrho_{\pm}^{\tau})$ converges strongly to $(\beta_{\pm}, \varrho_{\pm})$ in $L^\infty(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1})$, and v^{τ} converges strongly to v in $L^1(\mathbb{R}; \dot{B}^{\frac{d}{2}})$.

Remark 1.3 (Precision on the initial data I)**.** *The choice of the initial data in Theorem 1.4 has to be clarified since we show the global existence of solutions in Theorems 1.2 and 1.3 by passing to the limit on the parameters. In Theorem 1.4, once the solutions for Systems (K) and (PM) are constructed, we can choose a solution for System (BN) associated to an initial data which is not necessarily the same as the ones used to prove the existence of (K) and (PM).*

Remark 1.4 (Precision on the initial data II)**.** *To prove (1.17), we impose the stability condition on $Y_0^{\varepsilon,\tau} - Y_0^{\tau}$ in (1.16) instead of the original data $\alpha_{\pm,0}^{\varepsilon,\tau} - \alpha_{\pm,0}$. Indeed, one may not estimate $\alpha_{\pm}^{\varepsilon,\tau} - \alpha_{\pm}^{\tau}$ directly due to some nonlinear terms in the equations of $\alpha_{\pm}^{\varepsilon,\tau}, \alpha_{\pm}^{\tau}$. To overcome it, we make use of the purely transported variables subject to the initial quantities $Y_0^{\varepsilon,\tau}$ and Y_0^{τ} to reformulate the system and thus isolate this transported unknown from the other variables. In addition, since the dissipation structures of $u^{\varepsilon,\tau} - u^{\tau}$ require us to consider the equation of $P^{\varepsilon,\tau} - P^{\tau}$, we need the stability condition of $P_0^{\varepsilon,\tau} - P_0^{\tau}$ instead of $\rho_{0\pm}^{\varepsilon,\tau} - \rho_{0\pm}^{\tau}$.*

Remark 1.5. *Compared with the result in* [11]*, we establish the convergence estimates* (1.17) *uniformly with respect to* τ.

Remark 1.6. *We are able to derive the higher-order convergence estimates in Theorem 1.4. For example, by* (1.19) *and interpolation between* $\dot{B}^{\frac{d}{2}-1}$ *and* $\dot{B}^{\frac{d}{2}+1}$*, we have*

$$\|(\beta_{\pm}^{\tau}-\beta_{\pm},\varrho_{\pm}^{\tau}-\varrho_{\pm})\|_{L^{\infty}(\dot{B}^{\sigma})}\lesssim\varepsilon^{\frac{1}{2}(\frac{d}{2}+1-s)},\qquad \sigma\in(\frac{d}{2}-1,\frac{d}{2}+1).$$

By the convergence estimates (1.17) and (1.19), we can justify directly the relaxation limit from System (BN$_\tau$) to System (PM) as both $\varepsilon,\tau\to 0$.

Corollary 1.1. *Let* $d\geq 3$ *and* $0<\varepsilon\leq\tau\leq 1$*. Given the constants* $\bar{\alpha}_{\pm},\bar{\rho}_{\pm}$ *verifying* (1.4)-(1.5)*, let* $(\beta_{\pm}^{\varepsilon,\tau},\varrho_{\pm}^{\varepsilon,\tau},v^{\varepsilon,\tau})$ *be the global solution to the Cauchy problem of System* (BN) *with the initial data* $(\alpha_{\pm,0}^{\varepsilon,\tau},\rho_{\pm,0}^{\varepsilon,\tau},u_0^{\varepsilon,\tau})$ *obtained from Theorem 1.1 and the diffusive scaling* (1.6)*, and* $(\beta_{\pm},\varrho_{\pm})$ *be the global solution to the Cauchy problem of System* (PM) *with the initial data* $(\beta_{\pm,0},\varrho_{\pm,0})$ *derived from Theorem 1.3. Then under the assumptions* (1.16) *and* (1.18)*, there is a positive constant* C_3 *independent of* ε,τ *and time such that*

$$\|(\beta_{\pm}^{\varepsilon,\tau}-\beta_{\pm},\varrho_{\pm}^{\varepsilon,\tau}-\varrho_{\pm})\|_{L^{\infty}(\dot{B}^{\frac{d}{2}-1})}\leq C_3(\sqrt{\varepsilon\tau}+\tau).$$

1.4 Reformulation and spectral analysis

The first difficulty in the study of System (BN) is its lack of direct dissipativity and symmetrizability. Indeed, the linearization of (BN) does not satisfy the well-known "Shizuta-Kawashima" stability condition for partially dissipative hyperbolic systems (cf. [35]) due to the fact that the total pressure $P^{\varepsilon,\tau}$ depends on the multiple variables $\varrho_{\pm}$, $\alpha_{\pm}$. The first crucial part in the analysis is to partially symmetrize System (BN). To this matter, as explained in [11], we define the new unknowns

$$\begin{cases} y^{\varepsilon,\tau}:=\dfrac{\alpha_{+}^{\varepsilon,\tau}\rho_{+}^{\varepsilon,\tau}}{\alpha_{+}^{\varepsilon,\tau}\rho_{+}^{\varepsilon,\tau}+\alpha_{-}^{\varepsilon,\tau}\rho_{-}^{\varepsilon,\tau}}-\dfrac{\bar{\alpha}_{+}\bar{\rho}_{+}}{\bar{\alpha}_{+}\bar{\rho}_{+}+\bar{\alpha}_{-}\bar{\rho}_{-}},\\ w^{\varepsilon,\tau}:=\dfrac{\alpha_{+}^{\varepsilon,\tau}\alpha_{-}^{\varepsilon,\tau}}{\gamma_{+}\alpha_{-}^{\varepsilon,\tau}+\gamma_{-}\alpha_{+}^{\varepsilon,\tau}}\big(P_{+}(\rho_{+}^{\varepsilon,\tau})-P_{-}(\rho_{-}^{\varepsilon,\tau})\big),\\ r^{\varepsilon,\tau}:=P^{\varepsilon,\tau}-\bar{P}-(\gamma_{+}-\gamma_{-})w^{\varepsilon,\tau}, \end{cases}\tag{1.20}$$

and the corresponding initial data

$$\begin{cases} y_0:=\dfrac{\alpha_{+,0}\rho_{+,0}}{\alpha_{+,0}\rho_{+,0}+\alpha_{-,0}\rho_{-,0}}-\dfrac{\bar{\alpha}_{+}\bar{\rho}_{+}}{\bar{\alpha}_{+}\bar{\rho}_{+}+\bar{\alpha}_{-}\bar{\rho}_{-}},\\ w_0:=\dfrac{\alpha_{+,0}\alpha_{-,0}}{\gamma_{+}\alpha_{-,0}+\gamma_{-}\alpha_{+,0}}\big(P_{+}(\rho_{+,0})-P_{-}(\rho_{-,0})\big),\\ r_0:=\alpha_{+,0}P_{+}(\rho_{+,0})+\alpha_{-,0}P_{-}(\rho_{-,0})-\bar{P}-(\gamma_{+}-\gamma_{-})w_0, \end{cases}$$

so that the Cauchy problem of System (BN) subject to the initial data $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$ is reformulated as

$$
\begin{cases}
\partial_t y^{\varepsilon,\tau}+u^{\varepsilon,\tau}\cdot\nabla y^{\varepsilon,\tau}=0,\\
\partial_t w^{\varepsilon,\tau}+u^{\varepsilon,\tau}\cdot\nabla w^{\varepsilon,\tau}+(\bar F_1+G_1^{\varepsilon,\tau})\mathrm{div}\, u^{\varepsilon,\tau}+(\bar F_2+G_2^{\varepsilon,\tau})\dfrac{w^{\varepsilon,\tau}}{\varepsilon}=0,\\
\partial_t r^{\varepsilon,\tau}+u^{\varepsilon,\tau}\cdot\nabla r^{\varepsilon,\tau}+(\bar F_3+G_3^{\varepsilon,\tau})\mathrm{div}\, u^{\varepsilon,\tau}=F_4^{\varepsilon,\tau}\dfrac{(w^{\varepsilon,\tau})^2}{\varepsilon},\\
\partial_t u^{\varepsilon,\tau}+u^{\varepsilon,\tau}\cdot\nabla u^{\varepsilon,\tau}+\dfrac{u^{\varepsilon,\tau}}{\tau}+(\bar F_0+G_0^{\varepsilon,\tau})\nabla r^{\varepsilon,\tau}+(\gamma_+-\gamma_-)(\bar F_0+G_0^{\varepsilon,\tau})\nabla w^{\varepsilon,\tau}=0,\\
(y^{\varepsilon,\tau},w^{\varepsilon,\tau},r^{\varepsilon,\tau},u^{\varepsilon,\tau})(0,x)=(y_0,w_0,r_0,u_0)(x),
\end{cases}
\tag{1.21}
$$

where $F_i^{\varepsilon,\tau}$ $(i=0,1,2,3,4)$ are the nonlinear terms

$$
\begin{cases}
F_0^{\varepsilon,\tau}:=\dfrac{1}{\alpha_+^{\varepsilon,\tau}\rho_+^{\varepsilon,\tau}+\alpha_-^{\varepsilon,\tau}\rho_-^{\varepsilon,\tau}},\\
F_1^{\varepsilon,\tau}:=\dfrac{(\gamma_+-\gamma_-)\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}}{\gamma_+\alpha_-^{\varepsilon,\tau}+\gamma_-\alpha_+^{\varepsilon,\tau}}(\bar P+r^{\varepsilon,\tau})+\dfrac{\gamma_+^2\alpha_-^{\varepsilon,\tau}+\gamma_-^2\alpha_+^{\varepsilon,\tau}}{\gamma_+\alpha_-^{\varepsilon,\tau}+\gamma_-\alpha_+^{\varepsilon,\tau}}w^{\varepsilon,\tau},\\
F_2^{\varepsilon,\tau}:=(\gamma_+\alpha_-^{\varepsilon,\tau}+\gamma_-\alpha_+^{\varepsilon,\tau})(\bar P+r^{\varepsilon,\tau})-\dfrac{(\gamma_+-\gamma_+^2)(\alpha_-^{\varepsilon,\tau})^2-(\gamma_--\gamma_-^2)(\alpha_+^{\varepsilon,\tau})^2}{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}}w^{\varepsilon,\tau},\\
F_3^{\varepsilon,\tau}:=\dfrac{\gamma_+\gamma_-}{\gamma_+\alpha_-^{\varepsilon,\tau}+\gamma_-\alpha_+^{\varepsilon,\tau}}P^{\varepsilon,\tau},\\
F_4^{\varepsilon,\tau}:=\dfrac{\gamma_+\gamma_-}{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}}(1-\gamma_+\alpha_-^{\varepsilon,\tau}-\gamma_-\alpha_+^{\varepsilon,\tau}),
\end{cases}
\tag{1.22}
$$

$\bar F_i$ $(i=0,1,2,3)$ are the constants

$$
\begin{cases}
\bar F_0:=\dfrac{1}{\bar\alpha_+\bar\rho_++\bar\alpha_-\bar\rho_-}>0,\\
\bar F_1:=\dfrac{(\gamma_+-\gamma_-)\bar\alpha_+\bar\alpha_-}{\gamma_+\bar\alpha_-+\gamma_-\bar\alpha_+}\bar P>0,\\
\bar F_2:=(\gamma_+\bar\alpha_-+\gamma_-\bar\alpha_+)\bar P>0,\\
\bar F_3:=\dfrac{\gamma_+\gamma_-}{\gamma_+\bar\alpha_-+\gamma_-\bar\alpha_+}\bar P>0,
\end{cases}
\tag{1.23}
$$

and $G_i^{\varepsilon,\tau}$ $(i=0,1,2,3)$ are defined by

$$
G_i^{\varepsilon,\tau}:=F_i^{\varepsilon,\tau}-\bar F_i. \tag{1.24}
$$

Since the equation $(1.21)_1$ is purely transport and the partially dissipative subsystem $(1.21)_2$-$(1.21)_4$ satisfies the "Shizuta-Kawashima" stability condition, we will estimate the undamped unknown $y^{\varepsilon,\tau}$ and the dissipative components $(w^{\varepsilon,\tau},r^{\varepsilon,\tau},u^{\varepsilon,\tau})$ separately. However, it is not obvious how to apply the classical theorems from [11, 14, 15, 35, 41, 42] to analyze the above dissipative structures with explicit dependence of the two relaxation parameters ε,τ.

In order to understand the behaviors of the solution to (1.21) with respect to ε,τ, we perform a spectral analysis of the linearized system for (1.21). For simplicity we set $\bar F_i=1$ $(i=0,1,2,3)$. In terms of Hodge decomposition, we denote the "compressible" part $m=\Lambda^{-1}\mathrm{div}\, u$ and the "incompressible" part $\omega=\Lambda^{-1}\nabla\times u$ with $\Lambda^\sigma:=\mathcal{F}^{-1}(|\xi|^\sigma\mathcal{F}(\cdot))$ and rewrite the linearized system of (1.21) as

$$
\partial_t\begin{pmatrix}w\\ r\\ m\end{pmatrix}=\mathbb{A}\begin{pmatrix}w\\ r\\ m\end{pmatrix},\quad \mathbb{A}:=\begin{pmatrix}-\frac{1}{\varepsilon}&0&-\Lambda\\ 0&0&-\Lambda\\ (\gamma_+-\gamma_-)\Lambda&\Lambda&-\frac{1}{\tau}\end{pmatrix},\qquad \partial_t\omega+\frac{1}{\tau}\omega=0.
$$

The eigenvalues of the matrix $\widehat{\mathbb{A}}(\xi)$ satisfy

$$|\widehat{\mathbb{A}}(\xi)-\lambda\mathbb{I}_{3\times 3}|=\lambda^3+(\frac{1}{\tau}+\frac{1}{\varepsilon})\lambda^2+\big[\frac{1}{\varepsilon\tau}+\big(\gamma_+-\gamma_-+1\big)|\xi|^2\big]\lambda+\frac{1}{\varepsilon}|\xi|^2=0.$$

Under the condition $0<\varepsilon<<\tau$, the behaviors of λ_i $(i=1,2,3)$ can be analyzed as follows:

- In the low-frequency region $|\xi|<<\frac{1}{\tau}$, by Taylor's expansion near $|\tau\xi|<<1$ as in [32], all the eigenvalues are real, and we have $\lambda_1=-\frac{1}{\varepsilon}+\frac{1}{\tau}O(|\tau\xi|^2)$, $\lambda_2=-\tau|\xi|^2+\frac{1}{\tau}O(|\tau\xi|^3)$ and $\lambda_3=-\frac{1}{\tau}+\frac{1}{\tau}O(|\tau\xi|^2)$.

- In the medium-frequency region $\frac{1}{\tau}<<|\xi|<<\frac{1}{\varepsilon}$, according to Cardano's formula, λ_1 is real and λ_i $(i=2,3)$ are conjugated complex, and Re $\lambda_i\lesssim-\frac{1}{\tau}$ holds for all $i=1,2,3$.

- In the high-frequency region $|\xi|>>\frac{1}{\varepsilon}$, by Taylor's expansion near $|\varepsilon\xi|^{-1}<<1$, the real eigenvalue λ_1 and the conjugated complex eigenvalues λ_i $(i=2,3)$ satisfy $\lambda_1=-\frac{1}{(\gamma_+-\gamma_-+1)\varepsilon}+\frac{1}{\varepsilon}O(\frac{1}{|\varepsilon\xi|^2})$ and $\lambda_{2,3}=-\frac{1}{2\tau}-\frac{\gamma_+-\gamma_-}{2(\gamma_+-\gamma_-+1)\varepsilon}\pm\sqrt{\gamma_+-\gamma_-+1}|\xi|i+(\frac{1}{\tau}+\frac{\gamma_+-\gamma_-}{\varepsilon})O(\frac{1}{|\varepsilon\xi|})$.

It should be noted that compared with the eigenvalues of the compressible Euler equations with damping (cf. [36,39]), λ_2 and λ_3 have the same behaviors in low frequencies and satisfy a stronger damping effect in high frequencies.

The above spectral analysis suggests us to separate the whole frequencies into two parts $|\xi|\lesssim\frac{1}{\tau}$ and $|\xi|\gtrsim\frac{1}{\tau}$ so as to capture the qualitative properties of solutions for System (1.21). Indeed, the time-decay rates (determined by λ_2) achieve the fastest rate in the low-frequency region $|\xi|\lesssim\frac{1}{\tau}$. Moreover this region recover the whole frequency-space when $\tau\to 0$, as expected from the well-known overdamping phenomenon mentioned in the introduction. To this end, the threshold J_τ between these two regions is chosen to be (2.1) in next section.

2 Besov spaces and Littlewood-Play decomposition

In this section, we recall the notations of the Littlewood-Paley decomposition and Besov spaces. The reader can refer to Chapter 2 in [2] for a complete overview. Choose a smooth radial non-increasing function $\chi(\xi)$ with compact supported in $B(0,\dfrac{4}{3})$ and $\chi(\xi)=1$ in $B(0,\dfrac{3}{4})$ such that

$$\varphi(\xi):=\chi(\frac{\xi}{2})-\chi(\xi),\quad\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\cdot)=1,\quad\text{Supp }\varphi\subset\{\xi\in\mathbb{R}^d\mid\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}.$$

For any $j\in\mathbb{Z}$, the homogeneous dyadic blocks $\dot{\Delta}_j$ and the low-frequency cut-off operator $\dot{S}_j$ are defined by

$$\dot{\Delta}_j u:=\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u),\qquad\dot{S}_j u:=\mathcal{F}^{-1}(\chi(2^{-j}\cdot)\mathcal{F}u),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse. From now on, we use the shorthand notation

$$\dot{\Delta}_j u=u_j.$$

Let $\mathcal{S}'_h$ be the set of tempered distributions on $\mathbb{R}^d$ such that every $u \in \mathcal{S}'_h$ satisfies $u \in \mathcal{S}'$ and $\lim_{j\to-\infty} \|\dot{S}_j u\|_{L^\infty} = 0$. Then it follows that

$$u = \sum_{j\in\mathbb{Z}} u_j \quad \text{in } \mathcal{S}', \qquad \dot{S}_j u = \sum_{j'\leq j-1} u_{j'}, \quad \forall u \in \mathcal{S}'_h,$$

With the help of these dyadic blocks, the homogeneous Besov space $\dot{B}^s$ for $s \in \mathbb{R}$ is defined by

$$\dot{B}^s := \{u \in \mathcal{S}'_h \mid \|u\|_{\dot{B}^s} := \sum_{j\in\mathbb{Z}} 2^{js}\|u_j\|_{L^2} < \infty\}.$$

We denote the Chemin-Lerner type space $\widetilde{L}^\varrho(0,T;\dot{B}^s)$ $s \in \mathbb{R}$ and $T > 0$:

$$\widetilde{L}^\varrho(0,T;\dot{B}^s) := \{u \in L^\varrho(0,T;\mathcal{S}'_h) \mid \|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s)} := \sum_{j\in\mathbb{Z}} 2^{js}\|u_j\|_{L^\varrho_T(L^2)} < \infty\}.$$

By the Minkowski inequality, it holds that

$$\|u\|_{L^\varrho_T(\dot{B}^s)} \leq \|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s)},$$

where $\|\cdot\|_{L^\varrho_T(\dot{B}^s_{p,r})}$ is the usual Lebesgue-Besov norm. Moreover, we write

$$\mathcal{C}_b(\mathbb{R}_+;\dot{B}^s) := \{u \in \mathcal{C}(\mathbb{R}_+;\dot{B}^s) \mid \|f\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^s)} < \infty\}.$$

In order to restrict our Besov norms to the low and high frequencies regions, we set the threshold

$$J_\tau := -[\log_2 \tau] + k, \tag{2.1}$$

for suitably small integer k to be determined by the estimates (3.16)-(3.17) below. Denote the following notations for $p \in [1,\infty]$ and $s \in \mathbb{R}$:

$$\|u\|^\ell_{\dot{B}^s} := \sum_{j\leq J_\tau} 2^{js}\|u_j\|_{L^2}, \qquad \|u\|^h_{\dot{B}^s} := \sum_{j\geq J_\tau-1} 2^{js}\|u_j\|_{L^\varrho_T(L^2)},$$
$$\|u\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^s)} := \sum_{j\leq J_\tau} 2^{js}\|u_j\|_{L^2}, \qquad \|u\|^h_{\widetilde{L}^\varrho_T(\dot{B}^s)} := \sum_{j\geq J_\tau-1} 2^{js}\|u_j\|_{L^\varrho_T(L^2)}.$$

For any $u \in \mathcal{S}'_h$, we also define the low-frequency part u^ℓ and the high-frequency part u^h by

$$u^\ell := \sum_{j\leq J_\tau-1} u_j, \qquad u^h := u - u^\ell = \sum_{j\geq J_\tau} u_j.$$

It is easy to check for any $s' > 0$ that

$$\begin{cases} \|u^\ell\|_{\dot{B}^s} \leq \|u\|^\ell_{\dot{B}^s} \leq 2^{J_\tau s'}\|u\|^\ell_{\dot{B}^{s-s'}} \leq \dfrac{2^{s'}2^{ks'}}{\tau^{s'}}\|u\|^\ell_{\dot{B}^{s-s'}}, \\ \|u^h\|_{\dot{B}^s} \leq \|u\|^h_{\dot{B}^s} \leq 2^{-(J_\tau-1)s'}\|u\|^h_{\dot{B}^{s+s'}} \leq 2^{s'}2^{-ks'}\tau^{s'}\|u\|^h_{\dot{B}^{s+s'}}. \end{cases} \tag{2.2}$$

3 Analysis of the linear system

We now consider the linear problem associated to (1.21), which reads

$$\begin{cases} \partial_t y + v\cdot\nabla y = 0, \\ \partial_t w + v\cdot\nabla w + (h_1+H_1)\mathrm{div}\, u + (h_2+H_2)\dfrac{w}{\varepsilon} = S_1, \\ \partial_t r + v\cdot\nabla r + (h_3+H_3)\mathrm{div}\, u = S_2, \\ \partial_t u + v\cdot\nabla u + \dfrac{u}{\tau} + (h_4+H_4)\nabla r + (h_5+H_5)\nabla w = S_3, \\ (y,w,r,u)(0,x) = (y_0,w_0,r_0,u_0)(x), \end{cases} \tag{3.1}$$

where h_i $(i=1,...,5)$ are given positive constants and $H_i = H_i(t,x)$ $(1,...,5)$, $S_i = S_i(t,x)$ $(i=1,2,3)$ are given smooth functions.

We first establish the following a-priori estimates for solutions of the linear problem (3.1) uniformly with respect to the parameters ε,τ. This improves the result in [11] without the uniformity with respect to τ. As explained before, the threshold J_τ between low and high frequencies given by (2.1) is the key to our analysis.

Proposition 3.1. *Let $d\geq 2$, $0<\varepsilon\leq\tau<1$, $T>0$, and the threshold J_ε be given by (2.1). Assume that $(w_0,r_0,u_0)\in \dot B^{\frac d2-1}\cap \dot B^{\frac d2+1}$, $S_1,S_2,S_3\in L^1(0,T;\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})$, $H_i\in C([0,T];\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})$ and $\partial_t H_i\in L^1(0,T;\dot B^{\frac d2})$, $i=1,2,...5$. There exists a constant $c>0$ independent of T, ε and τ such that if*

$$\mathcal{Z}(t) := \sum_{i=1}^{5} \|H_i\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})} \leq c, \quad t\in(0,T), \tag{3.2}$$

then for $t\in(0,T)$, the solution (y,w,r,u) of the Cauchy problem (3.1) satisfies

$$\begin{aligned} \mathcal{X}(t) :=& \|(y,w,r,u)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})} + \|(\partial_t y,\partial_t w,\partial_t r,\partial_t u)\|_{L^1_t(\dot B^{\frac d2})} \\ &+\frac1\varepsilon\|w\|_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})} + \frac{1}{\sqrt\varepsilon}\|w\|_{\widetilde L^2_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})} \\ &+\tau\|r\|^\ell_{L^1_t(\dot B^{\frac d2+1}\cap\dot B^{\frac d2+2})} + \|r\|^h_{L^1_t(\dot B^{\frac d2+1})} + \tau\|r\|_{L^1_t(\dot B^{\frac d2+1})} + \sqrt\tau\|r\|_{\widetilde L^2_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})} \\ &+\|u\|_{L^1_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})} + \frac{1}{\sqrt\tau}\|u\|_{\widetilde L^2_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})} + \|\frac u\tau + (h_4+H_4)\nabla r\|_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})} \\ \leq& C_1 e^{C_1\int_0^t\mathcal V(\tau)d\tau}\Big(\|(y_0,w_0,r_0,u_0)\|_{\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1}} + \|(S_1,S_2,S_3)\|_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})} + \int_0^t\mathcal V(\tau)d\tau\Big), \end{aligned} \tag{3.3}$$

where $C_1>1$ is a constant independent of T, ε and τ, and $\mathcal V(t)$ is denoted by

$$\mathcal V(t) := \|v(t)\|_{\dot B^{\frac d2}\cap\dot B^{\frac d2+1}} + \sum_{i=1}^5\|\partial_t H_i(t)\|_{\dot B^{\frac d2}}. \tag{3.4}$$

Proof. First, we deal with the "no dissipation" unknown y. By direct computations for the transport equation $(3.1)_1$ (cf. [2][Theorem 3.14]), one has

$$\|y\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1})} \lesssim e^{\int_0^t\mathcal V(\tau)d\tau}\|y_0\|_{\dot B^{\frac d2-1}\cap\dot B^{\frac d2+1}}. \tag{3.5}$$

And direct produce law (6.2) for $(3.1)_1$ gives that

$$\|\partial_t y\|_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \int_0^t \|v(\tau)\|_{\dot{B}^{\frac{d}{2}}} \|y(\tau)\|_{\dot{B}^{\frac{d}{2}+1}} d\tau. \tag{3.6}$$

Thanks to the product law (6.2) for $(3.1)_3$, we also get

$$\|\partial_t r\|_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \int_0^t \|v(\tau)\|_{\dot{B}^{\frac{d}{2}}} \|r(\tau)\|_{\dot{B}^{\frac{d}{2}+1}} d\tau + (1+\|H_3\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})})\|u\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|S_2\|_{L^1_t(\dot{B}^{\frac{d}{2}})}, \tag{3.7}$$

and similarly,

$$\begin{aligned} \|(\partial_t w, \partial_t u)\|_{L^1_t(\dot{B}^{\frac{d}{2}})} &\lesssim \int_0^t \|v(\tau)\|_{\dot{B}^{\frac{d}{2}}} \|(w,u)(\tau)\|_{\dot{B}^{\frac{d}{2}+1}} d\tau + \frac{1}{\tau}\|u+\tau(h_5+H_5)\nabla r\|_{L^1_t(\dot{B}^{\frac{d}{2}})} \\ &\quad + (1+\sum_{i=1}^5 \|H_i\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})})(\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot{B}^{\frac{d}{2}})} + \|u\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}) + \|(S_1,S_3)\|_{L^1_t(\dot{B}^{\frac{d}{2}})}. \end{aligned} \tag{3.8}$$

The conclusion of the proof will follow from Lemmas 3.1-3.3 given and proven in the next three subsections. Indeed, combining (3.5)-(3.8) and the uniform estimates of (w,r,u) from Lemmas 3.1-3.3 together, we obtain

$$\begin{aligned} \mathcal{X}(t) &\lesssim \|(y_0,w_0,r_0,u_0)\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1}} + \|(S_1,S_2,S_3)\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1})} \\ &\quad + (\eta+\sqrt{\mathcal{Z}(t)})\mathcal{X}(t) + \frac{1}{\eta}\int_0^t \mathcal{V}(\tau)\mathcal{X}(\tau)d\tau. \end{aligned}$$

Then, choosing a suitable small constant $\eta\in(0,1)$ and making use of the Grönwall inequality and the smallness assumption (3.2) of $\mathcal{Z}(t)$ and η, we derive the uniform a-priori estimates(3.3). $\square$

3.1 $\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}$-estimates in low frequencies

Motivated by Darcy's law $(1.3)_4$, we introduce the following effective velocity

$$z := u+\tau(h_4+H_4)\nabla r, \tag{3.9}$$

which undergoes a purely damped effect in the low-frequency region $|\xi|\leq\frac{2^k}{\tau}$ and allows us to diagonalize the subsystem $(3.1)_2$-$(3.1)_4$ up to some higher linear terms that can be absorbed in terms of a suitable small constant k. Indeed, substituting (3.9) into (3.1), we obtain

$$\begin{cases} \partial_t w + \dfrac{h_2}{\varepsilon}w = L_1+R_1+S_1, \\ \partial_t r - h_3h_4\tau\Delta r = L_2+R_2+S_2, \\ \partial_t z + \dfrac{z}{\tau} = L_3+R_3+S_3, \\ (w,r,z)(x,0) = (w_0,r_0,z_0)(x), \quad z_0 := u_0+\tau(h_4+H_4(x,0))\nabla r_0, \end{cases} \tag{3.10}$$

where the linear terms L_i $(i=1,2,3)$ are denoted as

$$\begin{cases} L_1 := h_1\big(h_4\tau\Delta r - \operatorname{div} z\big), \\ L_2 := -h_3\operatorname{div} z, \\ L_3 := h_3h_4\tau\nabla\big(h_4\tau\Delta r - \operatorname{div} z\big) - h_5\nabla w, \end{cases} \tag{3.11}$$

and the nonlinear terms R_i $(i=1,2,3)$ are defined by

$$\begin{cases} R_1 := -v\cdot\nabla w - H_1\mathrm{div}\, u + h_1\tau\mathrm{div}\,(H_4\nabla r) - \dfrac{1}{\varepsilon}H_2 w,\\ R_2 := -v\cdot\nabla r - H_3\mathrm{div}\, u + h_3\tau\mathrm{div}\,(H_4\nabla r),\\ R_3 := -v\cdot\nabla u - H_5\nabla w - \tau\partial_t(H_4\nabla r) + h_4\tau\nabla R_2. \end{cases} \tag{3.12}$$

Since the equations in (3.10) are decoupled, we can thus establish the low-frequency a-priori estimates, which is more precise than previous results [11] and plays a key role in our proofs.

Remark 3.1. *An alternative way to obtain the low-frequency estimates could be to use the Lyapunov functional techniques as in* [11]*. However, such method does not lead to the desired estimates uniform with respect to* τ*. Moreover, it should be noted that the effective unknown* z *given by* (3.9) *enables us to capture the heat-like behavior of the unknown* r *in low frequencies directly, which is consistent with the parabolicity of the limiting porous media equations.*

To establish the $\dot B^{\frac d2-1}\cap\dot B^{\frac d2}$-estimates to the solution of System (3.1) uniformly in both ε and τ, we treat the equations of w and z as damped equations and r as a heat equation.

Lemma 3.1. *Let* $T>0$*, and the threshold* J_ε *be given by* (2.1)*. Then for* $t\in(0,T)$*, the solution* (y,w,r,u) *to the linear problem* (3.1) *satisfies*

$$\begin{aligned} &\|(w,r,u)\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})} + \frac1\varepsilon\|w\|^{\ell}_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})} + \frac{1}{\sqrt\varepsilon}\|w\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}\\ &\quad+\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1}\cap\dot B^{\frac d2+2})} + \sqrt\tau\|r\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})}\\ &\quad+\|u\|^{\ell}_{L^1_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})} + \frac{1}{\sqrt\tau}\|u\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})} + \frac1\tau\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}\\ &\lesssim \|(w_0,r_0,u_0)\|_{\dot B^{\frac d2-1}\cap\dot B^{\frac d2}} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})} + \mathcal Z(t)\mathcal X(t) + \int_0^t \mathcal V(\tau)\mathcal X(\tau)d\tau, \end{aligned} \tag{3.13}$$

where $\mathcal Z(t)$*,* $\mathcal X(t)$*,* $\mathcal V(t)$ *and* z *are defined by* (3.2)*,* (3.3)*,* (3.4) *and* (3.9)*, respectively.*

3.1.1 $\dot B^{\frac d2}$-estimates

Following the works [15, 16], we first perform $\dot B^{\frac d2}$-estimates for low frequencies. It follows from the regularity estimates for the heat equation $(3.10)_2$ in Lemma 6.5 that

$$\begin{aligned} &\|r\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2})} + \tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+2})} + \sqrt\tau\|r\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2+1})}\\ &\quad\lesssim \|r_0\|^{\ell}_{\dot B^{\frac d2}} + \|L_2\|^{\ell}_{L^1_t(\dot B^{\frac d2})} + \|R_2\|^{\ell}_{L^1_t(\dot B^{\frac d2})} + \|S_2\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\\ &\quad\lesssim \|r_0\|^{\ell}_{\dot B^{\frac d2}} + 2^{J_\tau}\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2})} + \|R_2\|^{\ell}_{L^1_t(\dot B^{\frac d2})} + \|S_2\|^{\ell}_{L^1_t(\dot B^{\frac d2})}. \end{aligned} \tag{3.14}$$

Applying Lemma 6.6 to the damped equation $(3.10)_2$, we get

$$\begin{aligned}
&\|w\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}+\frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\frac{1}{\sqrt{\varepsilon}}\|w\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|w_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+\|L_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|R_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|S_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|w_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})}+2^{J_\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|R_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|S_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|(w_0,r_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+2^{J_\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|(R_1,R_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|(S_1,S_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}
\end{aligned}\tag{3.15}$$

where we have used (3.14) to bound the linear term involving r.

Similarly, by virtue of (3.14) and Lemmas 6.5-6.6 on the equation $(3.10)_3$, there holds

$$\begin{aligned}
&\|z\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}+\frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|z_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+\|L_3\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|(R_3,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|z_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+2^{J_\tau}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\tau^2 2^{J_\tau}\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})}+\tau 2^{2J_\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})}+\|(R_3,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}.
\end{aligned}\tag{3.16}$$

Since the threshold J_τ satisfies $\tau 2^{J_\tau}\sim 2^k << 1$ for a suitable small positive constant k, and $2^{J_\tau}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\leq \frac{2^{k+1}}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}$ due to $\varepsilon\leq\tau$, we have by (3.14)-(3.16) that

$$\begin{aligned}
&\|(w,r,z)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}+\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})}+\sqrt{\tau}\|r\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\\
&\quad+\frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\frac{1}{\sqrt{\varepsilon}}\|w\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|(w_0,r_0,z_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+\|(R_1,R_2,R_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}.
\end{aligned}\tag{3.17}$$

The terms on the right-hand side of (3.17) can be estimated as follows. First, one derives from (2.2), the product law (6.2) and the composition estimate (6.4) that

$$\|z_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}}\lesssim \|(r_0,u_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}}+\|H_4(0)\|_{\dot{B}^{\frac{d}{2}}}\|r_0\|_{\dot{B}^{\frac{d}{2}}}\lesssim \|(r_0,u_0)\|_{\dot{B}^{\frac{d}{2}}}\tag{3.18}$$

By the product law (6.2), we also get

$$\begin{cases}
\|v\cdot\nabla w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\lesssim \displaystyle\int_0^t \|v(\tau)\|_{\dot{B}^{\frac{d}{2}}}\|w(\tau)\|_{\dot{B}^{\frac{d}{2}+1}}d\tau,\\
\|H_1\operatorname{div} u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\lesssim \|H_1\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})},\\
\dfrac{1}{\varepsilon}\|H_2 w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\lesssim \|H_2\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\dfrac{1}{\varepsilon}\|w\|_{L^1_t(\dot{B}^{\frac{d}{2}})}.
\end{cases}\tag{3.19}$$

According to (2.2) and (6.1)-(6.2), the tricky nonlinear term $H_4\nabla r$ in (3.12) can be controlled by

$$\begin{aligned}
&\tau\|\operatorname{div}(H_4\nabla r)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \tau\|H_4\nabla r^{\ell}\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}+\tau\|H_4\nabla r^{h}\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\\
&\lesssim \tau\|H_4\nabla r^{\ell}\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}+\|H_4\nabla r^{h}\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})}+\|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}+1})}\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}+\|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|r\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}.
\end{aligned}\tag{3.20}$$

Remark 3.2. *The above estimate* (3.20) *for $H_4\nabla r$ arising from two pressures implies that one needs to perform the uniform $\dot B^{\frac d2-1}$-estimates for low frequencies. Indeed, as H_4 does not have the either L^1-in-time or L^2-in-time integrability property, the product law* (6.1) *in $\dot B^{\frac d2+1}$ in forces us to control $\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}$ which can not be obtained from the $\dot B^{\frac d2}$-estimates in this subsection.*

Remark 3.3. *It is also one of the reasons why we need to perform the $\dot B^{\frac d2+1}$-estimates in the both low and high frequencies* (*see Subsection 3.3*)*. Indeed, in the low-frequency setting, the uniform $\widetilde L^\infty_t(\dot B^{\frac d2})$-norm is not enough to infer the uniform $\widetilde L^\infty_t(\dot B^{\frac d2+1})$-norm required in* (3.20) *due to the inclusion* (2.2).

Therefore, one derives from (3.18)-(3.20) that

$$
\begin{aligned}
\|R_1\|^{\ell}_{L^1_t(\dot B^{\frac d2})}&\lesssim \|v\cdot\nabla w\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\|H_1\mathrm{div}\,u\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\tau\|H_4\nabla r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\frac1\varepsilon\|H_2w\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\\
&\lesssim \mathcal Z(t)\mathcal X(t)+\int_0^t\mathcal V(\tau)\mathcal X(\tau)d\tau.
\end{aligned}\tag{3.21}
$$

Similarly, we have

$$
\begin{aligned}
\|R_2\|^{\ell}_{L^1_t(\dot B^{\frac d2})}&\lesssim \|v\cdot\nabla r\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\|H_3\mathrm{div}\,u\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\tau\|H_4\nabla r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}\\
&\lesssim \mathcal Z(t)\mathcal X(t)+\int_0^t\mathcal V(\tau)\mathcal X(\tau)d\tau.
\end{aligned}\tag{3.22}
$$

To estimate R_3, we notice that (2.2) together with (6.2) implies

$$
\begin{aligned}
\tau\|\partial_t(H_4\nabla r)\|^{\ell}_{L^1_t(\dot B^{\frac d2})}&\lesssim \|\partial_t(H_4\nabla r)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim \int_0^t\|\partial_tH_4(\tau)\|_{\dot B^{\frac d2}}\|r(\tau)\|_{\dot B^{\frac d2}}d\tau+\|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|\partial_t r\|_{L^1_t(\dot B^{\frac d2})}.
\end{aligned}\tag{3.23}
$$

and

$$
\|H_5\nabla w\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\lesssim\frac1\tau\|H_5\nabla w\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\lesssim\|H_5\|_{\widetilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\frac1\varepsilon\|w\|_{L^1_t(\dot B^{\frac d2})},
$$

where we have used the fact $\varepsilon\le\tau$. Thus, it holds that

$$
\begin{aligned}
\|R_3\|^{\ell}_{L^1_t(\dot B^{\frac d2})}&\lesssim\int_0^t\|v(\tau)\|_{\dot B^{\frac d2}}\|u(\tau)\|_{\dot B^{\frac d2+1}}d\tau+\tau\|H_4\nabla r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\|H_5\nabla w\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\\
&\quad+\tau\|\partial_t(H_4\nabla r)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}+\|R_2\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim \mathcal Z(t)\mathcal X(t)+\int_0^t\mathcal V(\tau)\mathcal X(\tau)d\tau.
\end{aligned}\tag{3.24}
$$

We substitute (3.18), (3.21)-(3.22) and (3.24) into (3.17) to derive

$$
\begin{aligned}
&\|(w,r,z)\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2})}+\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+2})}+\sqrt\tau\|r\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2+1})}\\
&\quad+\frac1\varepsilon\|w\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\frac1{\sqrt\varepsilon}\|w\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2})}+\frac1\tau\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\\
&\lesssim\|(w_0,r_0,u_0)\|_{\dot B^{\frac d2}}+\|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\mathcal Z(t)\mathcal X(t)+\int_0^t\mathcal V(\tau)\mathcal X(\tau)d\tau.
\end{aligned}\tag{3.25}
$$

Thence, we rewrite the form (3.9) and use (2.2) and (3.20) to obtain the $L^1_t(\dot B^{\frac d2})$-estimate of u as follows:

$$
\begin{aligned}
\|u\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})} &\lesssim \|z\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})} + \tau\|\nabla r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})} + \tau\|H_4\nabla r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}\\
&\lesssim \frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2})} + \tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+2})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+2})}\\
&\quad + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2+1})}\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|r\|^{h}_{L^1_t(\dot B^{\frac d2+1})}.
\end{aligned}
$$

Similarly, we have

$$
\|u\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2})} \lesssim \|(z,r)\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|r\|_{\widetilde L^\infty_t(\dot B^{\frac d2})},
$$

and

$$
\frac{1}{\sqrt{\tau}}\|u\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2})} \lesssim \frac{1}{\sqrt{\tau}}\|z\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2})} + \sqrt{\tau}\|r\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2+1})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\sqrt{\tau}\|r\|_{\widetilde L^2_t(\dot B^{\frac d2+1})}.
$$

We thus obtain from (3.25) that

$$
\begin{aligned}
&\|u\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2})} + \frac{1}{\sqrt{\tau}}\|u\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2})} + \|u\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}\\
&\quad\lesssim \|(w_0,r_0,u_0)\|_{\dot B^{\frac d2}} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2})} + \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.26}
$$

3.1.2 $\dot B^{\frac d2-1}$-estimates

We perform the $\dot B^{\frac d2-1}$-estimates so as to control $\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}$, as explained in Remark 3.2. Arguing similarly as in (3.14)-(3.17), we have

$$
\begin{aligned}
&\|(w,r,z)\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac d2-1})} + \tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})} + \sqrt{\tau}\|r\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2})}\\
&\quad + \frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})} + \frac{1}{\sqrt{\varepsilon}}\|w\|^{\ell}_{\widetilde L^2_t(\dot B^{\frac d2-1})} + \frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim \|(w_0,r_0,u_0)\|_{\dot B^{\frac d2-1}} + \|(R_1,R_2,R_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}.
\end{aligned}
\tag{3.27}
$$

Direct calculations give

$$
\begin{aligned}
\|(R_1,R_2)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})} &\lesssim \int_0^t \|v(\tau)\|_{\dot B^{\frac d2}}\|(w,r)(\tau)\|_{\dot B^{\frac d2}}d\tau + \|(H_1,H_3)\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|u\|_{L^1_t(\dot B^{\frac d2})}\\
&\quad + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\tau\|r\|_{L^1_t(\dot B^{\frac d2+1})} + \|H_2\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.28}
$$

By (3.23), (3.28) and (6.2), the term R_3 can be bounded by

$$
\begin{aligned}
\|R_3\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})} &\lesssim \int_0^t \|v(\tau)\|_{\dot B^{\frac d2}}\|u(\tau)\|_{\dot B^{\frac d2}}d\tau + \|H_5\|_{\widetilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot B^{\frac d2})}\\
&\quad + \tau\|\partial_t(H_4\nabla r)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})} + \|R_2\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.29}
$$

Inserting (3.28) and (3.29) into (3.27), we obtain

$$
\begin{aligned}
&\|(w,r,z)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1})}+\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\sqrt{\tau}\|r\|^{\ell}_{\widetilde{L}^2_t(\dot B^{\frac d2})}\\
&\quad+\frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\varepsilon}}\|w\|^{\ell}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim\|(w_0,r_0,u_0)\|_{\dot B^{\frac d2-1}}+\|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}+\mathcal{Z}(t)\mathcal{X}(t)+\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.30}
$$

This together with (2.2) and the fact $u=z-\tau(h_4+H_4)\nabla r$ leads to

$$
\|u\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1})}\lesssim\|(z,r)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1})}+\|H_4\|_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2})}\|r\|_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1})}.
$$

Similarly, one gets

$$
\frac{1}{\sqrt{\tau}}\|u\|^{\ell}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\lesssim\frac{1}{\sqrt{\tau}}\|z\|^{\ell}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\sqrt{\tau}\|r\|^{\ell}_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\|H_4\|_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2})}\sqrt{\tau}\|r\|_{\widetilde{L}^2_t(\dot B^{\frac d2})},
$$

and

$$
\|u\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\lesssim\frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}+\tau\|r\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\|H_4\|_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2})}\tau\|r\|_{L^1_t(\dot B^{\frac d2+1})}.
$$

Combining the above three estimates, we obtain

$$
\begin{aligned}
&\|u\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|u\|^{\ell}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\|u\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\\
&\lesssim\|(w_0,r_0,u_0)\|_{\dot B^{\frac d2-1}}+\|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}+\mathcal{Z}(t)\mathcal{X}(t)+\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.31}
$$

Putting the estimates (3.13) (3.25)-(3.26) and (3.30)-(3.31) together, we complete the proof of Lemma 3.1.

3.2 $\dot B^{\frac d2-1}\cap\dot B^{\frac d2}$-estimates in high frequencies

In this subsection, we establish some uniform high-frequency estimates of the solution to the linear problem (3.1) in terms of the Lyapunov functional. More precisely, we establish the $\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})$-norm estimates, and furthermore obtain the higher-order $L^1_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})$-norm estimates.

Lemma 3.2. *Let $T>0$, and the threshold J_ε be given by (2.1). Then for any $t\in(0,T)$, the solution (y,w,r,u) to the linear problem (3.1) satisfies*

$$
\begin{aligned}
&\|(w,r,u)\|^{h}_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}+\|(w,r,u)\|^{h}_{L^1_t(\dot B^{\frac d2+1})}\\
&\quad+\frac{1}{\varepsilon}\|w\|^{h}_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}+\frac{1}{\sqrt{\varepsilon}}\|w\|^{h}_{\widetilde{L}^2_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}+\sqrt{\tau}\|r\|^{h}_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}+\frac{1}{\tau}\|z\|^{h}_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}\\
&\lesssim\|(w_0,r_0,u_0)\|^{h}_{\dot B^{\frac d2+1}}+\|(S_1,S_2,S_3)\|^{h}_{L^1_t(\dot B^{\frac d2+1})}+\mathcal{Z}(t)\mathcal{X}(t)+\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau,
\end{aligned}
\tag{3.32}
$$

where $\eta\in(0,1)$ is a constant to be chosen, and $\mathcal{Z}(t)$, $\mathcal{X}(t)$, $\mathcal{V}(t)$ and z are defined by (3.2), (3.3), (3.4) and (3.9), respectively.

Proof. To prove of Lemma 3.2, we localize in frequencies for the equations $(3.1)_2$-$(3.1)_4$ as

$$\begin{cases}\partial_t w_j + v\cdot\nabla w_j + (h_1+H_1)\mathrm{div}\, u_j + (h_2+H_2)\dfrac{w_j}{\varepsilon} = \dot{\Delta}_j S_1 + T_j^1,\\ \partial_t r_j + v\cdot\nabla r_j + (h_3+H_3)\mathrm{div}\, u_j = \dot{\Delta}_j S_2 + T_j^2,\\ \partial_t u_j + v\cdot\nabla u_j + \dfrac{u_j}{\tau} + (h_4+H_4)\nabla r_j + (h_5+H_5)\nabla w_j = \dot{\Delta}_j S_3 + T_j^3,\end{cases} \tag{3.33}$$

with the commutator terms

$$\begin{cases}T_j^1 := [v,\dot{\Delta}_j]\nabla w + [H_1,\dot{\Delta}_j]\mathrm{div}\, u + \dfrac{1}{\varepsilon}[H_2,\dot{\Delta}_j]w,\\ T_j^2 := [v,\dot{\Delta}_j]\nabla r + [H_3,\dot{\Delta}_j]\mathrm{div}\, u,\\ T_j^3 := [v,\dot{\Delta}_j]\nabla u + [H_4,\dot{\Delta}_j]\nabla r + [H_5,\dot{\Delta}_j]\nabla w.\end{cases} \tag{3.34}$$

Multiplying $(3.33)_3$ by u_j and integrating the resulting equation by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}|u_j|^2dx + \int_{\mathbb{R}^d}\frac{1}{\tau}|u_j|^2dx\\ &\quad - \int_{\mathbb{R}^d}\big((h_4+H_4)r_j\mathrm{div}\, u_j + (h_5+H_5)w_j\mathrm{div}\, u_j\big)dx\\ &\lesssim \|\mathrm{div}\, v\|_{L^\infty}\|u_j\|_{L^2}^2 + \big(\|\dot{\Delta}_j S_3\|_{L^2} + \|T_j^3\|_{L^2}\big)\|u_j\|_{L^2} + \|\nabla H_4\|_{L^\infty}\|(w_j,r_j)\|_{L^2}\|u_j\|_{L^2}.\end{aligned} \tag{3.35}$$

Thence, we multiply $(3.33)_1$ by $\frac{h_5+H_5}{h_1+H_1}w_j$ and integrate the resulting equation by parts to show

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\frac{h_5+H_5}{h_1+H_1}|w_j|^2dx + \int_{\mathbb{R}^d}\Big((h_5+H_5)w_j\mathrm{div}\, u_j + \frac{(h_2+H_2)(h_5+H_5)}{\varepsilon(h_1+H_1)}|w_j|^2\Big)dx\\ &\lesssim \Big(\|\partial_t\Big(\frac{h_5+H_5}{h_1+H_1}\Big)\|_{L^\infty} + \|\frac{h_5+H_5}{h_1+H_1}\|_{L^\infty}\|\mathrm{div}\, v\|_{L^\infty} + \|\nabla\Big(\frac{h_5+H_5}{h_1+H_1}\Big)\|_{L^\infty}\|v\|_{L^\infty}\Big)\|w_j\|_{L^2}^2\\ &\quad + \|\frac{h_5+H_5}{h_1+H_1}\|_{L^\infty}\big(\|\dot{\Delta}_j S_1\|_{L^2} + \|T_j^1\|_{L^2}\big)\|w_j\|_{L^2}.\end{aligned} \tag{3.36}$$

Similarly, direct computations on $(3.33)_2$ yield

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\frac{h_4+H_4}{h_3+H_3}|r_j|^2dx + \int_{\mathbb{R}^d}(h_4+H_4)r_j\mathrm{div}\, u_jdx\\ &\le \Big(\|\partial_t\Big(\frac{h_4+H_4}{h_3+H_3}\Big)\|_{L^\infty} + \|\frac{h_4+H_4}{h_3+H_3}\|_{L^\infty}\|\mathrm{div}\, v\|_{L^\infty} + \|\nabla\Big(\frac{h_4+H_4}{h_3+H_3}\Big)\|_{L^\infty}\|v\|_{L^\infty}\Big)\|r_j\|^2\\ &\quad + \|\frac{h_4+H_4}{h_3+H_3}\|_{L^\infty}\big(\|\dot{\Delta}_j S_2\|_{L^2} + \|T_j^2\|_{L^2}\big)\|r_j\|_{L^2}.\end{aligned} \tag{3.37}$$

To derive the cross estimate and capture the dissipative property of r_j, we gain by taking the L^2-inner product of $(3.33)_3$ with ∇r_j that

$$\begin{aligned}&\int_{\mathbb{R}^d}\partial_t u_j\cdot\nabla r_jdx + \int_{\mathbb{R}^d}(h_4+H_4)|\nabla r_j|^2dx + \int_{\mathbb{R}^d}\Big((h_5+H_5)\nabla w_j\cdot\nabla r_j + \frac{1}{\tau}u_j\cdot\nabla r_j\Big)dx\\ &\quad\lesssim \big(\|v\|_{L^\infty}\|\nabla u_j\|_{L^2} + \|\dot{\Delta}_j S_3\|_{L^2} + \|T_j^3\|_{L^2}\big)\|\nabla r_j\|_{L^2},\end{aligned} \tag{3.38}$$

and taking the L^2-inner product of $(3.33)_2$ with $\mathrm{div}\, u_j$ that

$$\begin{aligned}&\int_{\mathbb{R}^d}u_j\cdot\nabla\partial_t r_jdx - \int_{\mathbb{R}^d}(h_3+H_3)|\mathrm{div}\, u_j|^2dx\\ &\quad\lesssim \big(\|v\|_{L^\infty}\|\nabla r_j\|_{L^2} + \|\dot{\Delta}_j S_2\|_{L^2} + \|T_j^2\|_{L^2}\big)\|\mathrm{div}\, u_j\|_{L^2}.\end{aligned} \tag{3.39}$$

In the spirit of the work [3] by Beauchard and Zuazua, we define the Lyapunov functional with some nonlinear weights

$$\mathcal{L}_j(t) := \int_{\mathbb{R}^d} \frac{1}{2}\Big(\frac{h_5+H_5}{h_1+H_1}|w_j|^2 + \frac{h_4+H_4}{h_3+H_3}|r_j|^2 + |u_j|^2\Big)dx + \frac{\eta_*}{\tau}2^{-2j}\int_{\mathbb{R}^d} u_j\cdot\nabla r_j dx,$$

and its dissipation rate

$$\begin{aligned}\mathcal{H}_j(t) :=& \int_{\mathbb{R}^d}\Big(\frac{1}{\tau}|u_j|^2 + \frac{(h_2+H_2)(h_5+H_5)}{\varepsilon(h_1+H_1)}|w_j|^2\Big)dx\\ &+ \frac{\eta_*}{\tau}2^{-2j}\int_{\mathbb{R}^d}\Big((h_4+H_4)|\nabla r_j|^2 + (h_5+H_5)\nabla w_j\cdot\nabla r_j + \frac{1}{\tau}u_j\cdot\nabla r_j\Big)dx,\end{aligned}$$

for a small constant $\eta_*>0$ to be determined. One derives from (3.2) and the embedding $L^\infty \hookrightarrow \dot{B}^{\frac{d}{2}}$ that

$$\|H_i\|_{L^\infty_t(L^\infty)} + \|\nabla H_i\|_{L^\infty_t(L^\infty)} \lesssim \|H_i\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}\cap\dot{B}^{\frac{d}{2}+1})} \lesssim c << 1, \tag{3.40}$$

which together with (3.35)-(3.39) and the fact $2^{-j}\lesssim\tau\leq 1$ yields following Lyapunov inequality:

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_j(t)+\mathcal{H}_j(t) \lesssim& (\|\mathrm{div}\, v\|_{L^\infty} + \|v\|_{L^\infty} + \sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty})\|(r_j,w_j,u_j)\|_{L^2}^2\\ &+\Big(\sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty} + \|\dot{\Delta}_j(S_1,S_2,S_3)\|_{L^2} + \|(T_j^1,T_j^2,T_j^3)\|_{L^2}\Big)\|(r_j,w_j,u_j)\|_{L^2}.\end{aligned} \tag{3.41}$$

For any $j\geq J_\varepsilon-1$, it follows from the smallness condition (3.40), the Bernstein inequality in Lemma 6.1 and the fact $2^{-j}\lesssim\tau$ that

$$(1-\eta_*)\|(w_j,r_j,u_j)\|_{L^2}^2 \lesssim \mathcal{L}_j(t) \lesssim (1+\eta_*)\|(w_j,r_j,u_j)\|_{L^2}^2,$$

and

$$\begin{aligned}\mathcal{H}_j(t) &\gtrsim \frac{1}{\tau}\|u_j\|_{L^2}^2 + \frac{1}{\varepsilon}\|w_j\|_{L^2}^2 + \frac{\eta_*}{\tau}2^{-2j}(\|\nabla r_j\|_{L^2} - \|\nabla w_j\|_{L^2}^2 - \frac{1}{\tau^2}\|u_j\|_{L^2})\\ &\gtrsim \frac{1}{\tau}(1-\eta_*)\|u_j\|_{L^2}^2 + \frac{1}{\varepsilon}(1-\eta_*)\|w_j\|_{L^2}^2 + \frac{\eta_*}{\tau}\|r_j\|_{L^2}.\end{aligned}$$

where one has used the condition $\varepsilon\leq\tau$. Thus, we can choose a sufficiently small constant $\eta_*>0$ independent of ε and τ so that

$$\mathcal{L}_j(t)\sim\|(w_j,r_j,u_j)\|_{L^2}^2, \qquad \mathcal{H}_j(t)\gtrsim\frac{1}{\tau}\|(w_j,r_j,u_j)\|_{L^2}^2 \gtrsim \frac{1}{\tau}\mathcal{L}_j(t). \tag{3.42}$$

Dividing the two sides of (3.41) by $\sqrt{\mathcal{L}_j(t)+\eta}$ for any $\eta>0$, we have

$$\begin{aligned}&\frac{d}{dt}\sqrt{\mathcal{L}_j(t)+\eta} + \frac{1}{\tau}\sqrt{\mathcal{L}_j(t)+\eta}\\ &\lesssim (\|\mathrm{div}\, v\|_{L^\infty} + \|v\|_{L^\infty} + \sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty})\|(r_j,w_j,u_j)\|_{L^2} + \|\dot{\Delta}_j(S_1,S_2,S_3)\|_{L^2} + \|(T_j^1,T_j^2,T_j^3)\|_{L^2},\end{aligned}$$

which together with (3.42) and the embedding $\dot{B}^{\frac{d}{2}}\hookrightarrow L^\infty$ gives rise to

$$\begin{aligned}&\tau\|(w,r,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})} + \|(w,r,u)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\\ &\lesssim \tau\|(w_0,r_0,u_0)\|^h_{\dot{B}^{\frac{d}{2}+1}} + \int_0^t(\|v(\tau)\|_{\dot{B}^{\frac{d}{2}}\cap\dot{B}^{\frac{d}{2}+1}} + \sum_{i=1}^{5}\|\partial_t H_i(\tau)\|_{\dot{B}^{\frac{d}{2}}})\tau\|(w,r,u)(\tau)\|^h_{\dot{B}^{\frac{d}{2}+1}}d\tau\\ &\quad+\tau\sum_{j\geq J_\tau-1}2^{j(\frac{d}{2}+1)}\|(T_j^1,T_j^2,T_j^3)\|_{L^1_t(L^2)} + \tau\|(S_1,S_2,S_3)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})}.\end{aligned} \tag{3.43}$$

According to the commutator estimate (6.3), it follows that

$$
\begin{aligned}
&\tau\sum_{j\geq J_\tau-1}2^{j(\frac{d}{2}+1)}\|(T_j^1,T_j^2,T_j^3)\|_{L^1_t(L^2)}\\
&\quad\lesssim\int_0^t\|v(\tau)\|_{\dot B^{\frac{d}{2}+1}}\|(w,r,u)(\tau)\|_{\dot B^{\frac{d}{2}+1}}d\tau\\
&\quad\quad+\sum_{i=1}^4\|H_i\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}+1})}\big(\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot B^{\frac{d}{2}})}+\tau\|r\|_{L^1_t(\dot B^{\frac{d}{2}+1})}+\|u\|_{L^1_t(\dot B^{\frac{d}{2}+1})}\big)\lesssim\mathcal{Z}(t)\mathcal{X}(t).
\end{aligned}
$$

This as well as (3.43) leads to

$$
\begin{aligned}
&\tau\|(w,r,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}+1})}+\|(w,r,u)\|^h_{L^1_t(\dot B^{\frac{d}{2}+1})}\\
&\quad\lesssim\tau\|(w_0,r_0,u_0)\|^h_{\dot B^{\frac{d}{2}+1}}+\tau\|(S_1,S_2,S_3)\|^h_{L^1_t(\dot B^{\frac{d}{2}+1})}+\mathcal{Z}(t)\mathcal{X}(t)+\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.44}
$$

Furthermore, for any $\eta>0$, we deduce from (3.36) that

$$
\begin{aligned}
&\frac{d}{dt}\sqrt{\|w_j\|_{L^2}^2+\eta}+\frac{1}{\varepsilon}\sqrt{\|w_j\|_{L^2}^2+\eta}\\
&\lesssim 2^j\|u_j\|_{L^2}+\|(\partial_tH_1,\partial_tH_5)\|_{L^\infty}\|w_j\|_{L^2}+\|\mathrm{div}\,v\|_{L^\infty}\|w_j\|_{L^2}+\|v\|_{L^\infty}\|w_j\|_{L^2}+\|\dot\Delta_jS_1\|_{L^2}+\|T_j^1\|_{L^2},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{1}{\varepsilon}\|w\|^h_{L^1_t(\dot B^{\frac{d}{2}})}&\lesssim\|w_0\|^h_{\dot B^{\frac{d}{2}}}+\|u\|^h_{L^1_t(\dot B^{\frac{d}{2}+1})}+\|S_1\|^h_{L^1_t(\dot B^{\frac{d}{2}})}+\tau\sum_{j\geq J_\tau-1}2^{j(\frac{d}{2}+1)}\|T_j^1\|_{L^1_t(L^2)}\\
&\quad+\int_0^t\big(\|(\partial_tH_1,\partial_tH_5)(\tau)\|_{\dot B^{\frac{d}{2}}}+\|v(\tau)\|_{\dot B^{\frac{d}{2}}\cap\dot B^{\frac{d}{2}+1}}\big)\|w(\tau)\|_{\dot B^{\frac{d}{2}}}d\tau\\
&\lesssim\tau\|(w_0,r_0,u_0)\|^h_{\dot B^{\frac{d}{2}+1}}+\tau\|(S_1,S_2,S_3)\|^h_{L^1_t(\dot B^{\frac{d}{2}+1})}+\mathcal{Z}(t)\mathcal{X}(t)+\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
\tag{3.45}
$$

Finally, the other estimates in (3.32) can be achieved by making full use of (2.2) and (3.44), for example, $\|(w,r,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}})}\lesssim\tau\|(w,r,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}+1})}$ and $\|(w,r,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-1})}\lesssim\tau\|(w,r,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}})}\lesssim\tau\|(w,r,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}+1})}$. We omit the details here. The proof of Lemma 3.2 is complete. □

3.3 $\dot B^{\frac{d}{2}+1}$-estimates

According to Remark 3.3, we need to establish the uniform $L^\infty_t(\dot B^{\frac{d}{2}+1})$-estimate of (w,r,u) which also leads to the uniform $\widetilde L^2_t(\dot B^{\frac{d}{2}+1})$-estimates of $(\frac{1}{\sqrt{\varepsilon}}w,\frac{1}{\sqrt{\tau}}u)$ for both low and high frequencies.

Lemma 3.3. *Let $T>0$, and the threshold J_ε be given by (2.1). Then for any $t\in(0,T)$, the solution (y,w,r,u) to the linear problem (3.1) satisfies*

$$
\begin{aligned}
&\|(w,r,u)\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}+1})}+\frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}+1})}+\frac{1}{\sqrt{\tau}}\|u\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}+1})}\\
&\quad\lesssim\|(w_0,r_0,u_0)\|_{\dot B^{\frac{d}{2}+1}}+\|(S_1,S_2,S_3)\|_{L^1_t(\dot B^{\frac{d}{2}+1})}+(\eta+\sqrt{\mathcal{Z}(t)})\mathcal{X}(t)+\frac{1}{\eta}\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau,
\end{aligned}
\tag{3.46}
$$

where $\eta\in(0,1)$ is a constant to be chosen, and $\mathcal{Z}(t)$, $\mathcal{X}(t)$ and $\mathcal{V}(t)$ are defined by (3.2), (3.3) and (3.4), respectively.

Proof. It is difficult to consider the L^1-time type estimates due to the growth-in-τ of the nonlinear term in $[H_4, \dot{\Delta}_j]\nabla r$ for L^1 time-integrability. To overcome this difficulty, we perform the L^2-time type estimates and make use of the better decay-in-τ estimates of u. Indeed, for any $j \in \mathbb{Z}$, combining (3.35)-(3.36) together, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\big(\frac{h_5+H_5}{h_1+H_1}|w_j|^2+\frac{h_4+H_4}{h_3+H_3}|r_j|^2+|u_j|^2\big)dx\\
&\quad+\int_{\mathbb{R}^d}\big(\frac{1}{\tau}|u_j|^2+\frac{(h_2+H_2)(h_5+H_5)}{\varepsilon(h_1+H_1)}|w_j|^2\big)dx\\
&\lesssim \|\mathrm{div}\, v\|_{L^\infty}\|(w_j,r_j,u_j)\|_{L^2}^2+\big(\sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty}+\|\mathrm{div}\, v\|_{L^\infty}+\|v\|_{L^\infty}\big)\|(w_j,r_j)\|_{L^2}^2\\
&\quad+\|\nabla H_4\|_{L^\infty}\|(w_j,r_j)\|_{L^2}\|u_j\|_{L^2}+\|T_j^1\|_{L^2}\|w_j\|_{L^2}+\|T_j^2\|_{L^2}\|r_j\|_{L^2}+\|T_j^3\|_{L^2}\|u_j\|_{L^2}\\
&\quad+\|\dot{\Delta}_j(S_1,S_2,S_3)\|_{L^2}\|(w_j,r_j,u_j)\|_{L^2}.
\end{aligned}
\tag{3.47}
$$

By (3.47), we get

$$
\begin{aligned}
&\|(w,r,u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}+\frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}+\frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\\
&\quad\lesssim \|(w_0,r_0,u_0)\|_{\dot{B}^{\frac{d}{2}+1}}+\|v\|^{\frac12}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\|(w,r,u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\\
&\qquad+\Big(\int_0^t\big(\sum_{i=1}^{5}\|\partial_t H_i(\tau)\|_{\dot{B}^{\frac{d}{2}}}+\|v(\tau)\|_{\dot{B}^{\frac{d}{2}}\cap\dot{B}^{\frac{d}{2}+1}}\big)\|(w,r)(\tau)\|_{\dot{B}^{\frac{d}{2}+1}}d\tau\Big)^{\frac12}\|(w,r)\|^{\frac12}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\\
&\qquad+\Big(\|H_4\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\sqrt{\tau}\|(w,r)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\Big)^{\frac12}\\
&\qquad+\sum_{j\in\mathbb{Z}}2^{j(\frac{d}{2}+1)}\Big(\int_0^t\big(\|T_j^1\|_{L^2}\|w_j\|_{L^2}+\|T_j^2\|_{L^2}\|r_j\|_{L^2}+\|T_j^3\|_{L^2}\|u_j\|_{L^2}\big)d\tau\Big)^{\frac12}\\
&\qquad+\Big(\|(S_1,S_2,S_3)\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\|(w,r,u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\Big)^{\frac12}.
\end{aligned}
\tag{3.48}
$$

The right-hand side of (3.48) can be estimated as follows. By the commutator estimate (6.3), we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}2^{j(\frac{d}{2}+1)}\int_0^t\|T_j^1\|_{L^2}\|w_j\|_{L^2}d\tau\\
&\quad\lesssim \|w\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\int_0^t\|v(\tau)\|_{\dot{B}^{\frac{d}{2}+1}}\|w(\tau)\|_{\dot{B}^{\frac{d}{2}+1}}d\tau+\|H_1\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\|w\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\\
&\qquad+\|H_2\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\frac{1}{\varepsilon}\|w\|^2_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\\
&\quad\lesssim \mathcal{Z}(t)\mathcal{X}^2(t)+\mathcal{X}(t)\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
$$

Similarly, it holds

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}2^{j(\frac{d}{2}+1)}\int_0^t\|T_j^2\|_{L^2}\|r_j\|_{L^2}d\tau\\
&\quad\lesssim \|r\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\int_0^t\|v(\tau)\|_{\dot{B}^{\frac{d}{2}+1}}\|r(\tau)\|_{\dot{B}^{\frac{d}{2}+1}}d\tau+\|H_3\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\|r\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\\
&\quad\lesssim \mathcal{Z}(t)\mathcal{X}^2(t)+\mathcal{X}(t)\int_0^t\mathcal{V}(\tau)\mathcal{X}(\tau)d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} 2^{j(\frac{d}{2}+1)} \int_0^t \|T_j^3\|_{L^2}\|u_j\|_{L^2} d\tau \\
&\quad\lesssim \|u\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}+1})} \int_0^t \|v(\tau)\|_{\dot B^{\frac{d}{2}+1}}\|u(\tau)\|_{\dot B^{\frac{d}{2}+1}} d\tau + \|(H_4,H_5)\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}+1})}\|r\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}+1})}\|u\|_{L^1_t(\dot B^{\frac{d}{2}+1})} \\
&\quad\lesssim \mathcal{Z}(t)\mathcal{X}^2(t) + \mathcal{X}(t)\int_0^t \mathcal{V}(\tau)\mathcal{X}(\tau)d\tau.
\end{aligned}
$$

We conclude from the above estimates that

$$
\begin{aligned}
&\|(w,r,u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}+1})} \\
&\quad\lesssim \|(w_0,r_0,u_0)\|_{\dot B^{\frac{d}{2}+1}} + \|(S_1,S_2,S_3)\|_{L^1_t(\dot B^{\frac{d}{2}+1})} + \sqrt{\mathcal{Z}(t)}\mathcal{X}(t) + \Big(\int_0^t \mathcal{V}(\tau)\mathcal{X}(\tau)d\tau\Big)^{\frac{1}{2}}\sqrt{\mathcal{X}(t)}.
\end{aligned}
$$

This together with Hölder's inequality leads to (3.46). □

4 Proof of global existence

4.1 Global existence of System (BN)

In this section, we prove the global existence of solutions to the Cauchy problem for (BN) subject to the initial data $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$, i.e. Theorem 1.1.

Proof of Theorem 1.1: Let (y_0,w_0,r_0,u_0) satisfy (1.7) and set $(y^0,w^0,r^0,u^0) := (y_0,w_0,r_0,u_0)$. For any $n\geq 0$, we consider the approximate scheme for (1.21) as follows:

$$
\begin{cases}
\partial_t y^{n+1} + u^n\cdot\nabla y^{n+1} = 0,\\
\partial_t w^{n+1} + u^n\cdot\nabla w^{n+1} + (\bar F_1 + G_1^n)\mathrm{div}\, u^{n+1} + (\bar F_2 + G_2^n)\dfrac{w^{n+1}}{\varepsilon} = 0,\\
\partial_t r^{n+1} + u^n\cdot\nabla r^{n+1} + (\bar F_3 + G_3^n)\mathrm{div}\, u^{n+1} = F_4^n\dfrac{(w^n)^2}{\varepsilon},\\
\partial_t u^{n+1} + u^n\cdot\nabla u^{n+1} + \dfrac{u^{n+1}}{\tau} + (\bar F_0 + G_0^n)\nabla r^{n+1} + (\gamma_+ - \gamma_-)(\bar F_0 + G_0^n)\nabla w^{n+1} = 0,\\
(y^{n+1},w^{n+1},r^{n+1},u^{n+1})(0,x) = (\dot S_n y_0, \dot S_n w_0, \dot S_n r_0, \dot S_n u_0)(x),
\end{cases}
\tag{4.1}
$$

with $F_i^n = F_i^{\varepsilon,\tau}(y^n,w^n,r^n)$, $\bar F i$ and $G_i^n = G^i(y^n,w^n,r^n)$ defined in (1.22), (1.23) and (1.24), respectively.

We define $\mathbb{E}_t$ the functional space associated to the following norm:

$$
\begin{aligned}
\|(y,w,r,u)\|_{\mathbb{E}_t} :=\ & \|(y,w,r,u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})} + \|(\partial_t y,\partial_t w,\partial_t r,\partial_t u)\|_{L^1_t(\dot B^{\frac{d}{2}})} \\
&+ \frac{1}{\varepsilon}\|w\|_{L^1_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})} \\
&+ \tau\|r\|^\ell_{L^1_t(\dot B^{\frac{d}{2}+1}\cap\dot B^{\frac{d}{2}+2})} + \sqrt{\tau}\|r\|^\ell_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}}\cap\dot B^{\frac{d}{2}+3})} + \|r\|^h_{L^1_t(\dot B^{\frac{d}{2}+1})} + \|r\|^h_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}+1})} \\
&+ \|u\|_{L^1_t(\dot B^{\frac{d}{2}}\cap\dot B^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})} \\
&+ \frac{1}{\tau}\|u+\tau(\bar F_0+G_0)\nabla r\|_{L^1_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}})} + \frac{1}{\sqrt{\tau}}\|u+\tau(\bar F_0+G_0)\nabla r\|_{\widetilde{L}^2_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})}.
\end{aligned}
$$

For any fixed $n \geq 1$, we assume that (y^n, w^n, r^n, u^n) satisfies

$$\|(y^n, w^n, r^n, u^n)\|_{\mathbb{E}_t} \leq 2C_0 \|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}, \quad t > 0, \tag{4.2}$$

where the constant $C_1 > 1$ is given by (3.3). Then by virtue of classical theorems for linear transport and parabolic equations (cf. [2, 18]), there exists a unique global solution $(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$. Our goal is to show that $(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})$ also satisfies the estimate (4.2) uniformly in $n \geq 0$. To this end, it follows from (1.7), (4.2) and the composition estimates in Lemma 6.4 that

$$\sum_{i=0}^{4} \|G_i^n\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \lesssim \|(y^n, r^n, w^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} << 1, \quad t > 0,$$

and similarly,

$$\sum_{i=0}^{4} \|\partial_t G_i^n\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \lesssim (1 + \|(y^n, r^n, w^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})}) \|(\partial_t y^n, \partial_t w^n, \partial_t r^n)\|_{L_t^1(\dot{B}^{\frac{d}{2}})} << 1, \quad t > 0.$$

Thus, (3.2) follows when $\|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}$ is sufficiently small, and we are able to employ the uniform a-priori estimates established in Proposition 3.1, (4.2) and the product laws (6.1)-(6.2) to obtain

$$\begin{aligned} \|(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})\|_{\mathbb{E}_t} &\leq C_1 e^{C_1 \int_0^t (\|u^n(\tau)\|_{\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}} + \sum_{i=0}^4 \|\partial_t G_i^n(\tau)\|_{\dot{B}^{\frac{d}{2}}}) d\tau} \\ &\quad \times \left(\|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} + \frac{C}{\varepsilon} \|w^n\|^2_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \right) \\ &\leq 2C_1 \|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}, \qquad t > 0, \end{aligned} \tag{4.3}$$

as long as $\|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}$ is suitably small. Thus, the uniform estimates (4.2) hold for any $n \geq 0$.

Then, for any given time $T > 0$ and $\chi \in C_c(\mathbb{R}^d \times (0, T))$, in view of the Aubin-Lions lemma and the cantor diagonal process, there is a limit $(y^{\varepsilon,\tau}, w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ such that as $n \to \infty$, up to a subsequence, (y^n, w^n, r^n, u^n) converges strongly to $(y^{\varepsilon,\tau}, w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ in $C([0, T]; \dot{B}^s)$ $(s < \frac{d}{2} + 1)$. In addition, taking the advantage of Fatou property, we have $\|(y^{\varepsilon,\tau}, w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau})\|_{\mathbb{E}_t} \lesssim \|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}$, and as in [17], one can obtain $(y^{\varepsilon,\tau}, w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$. Therefore, the limit $(y^{\varepsilon,\tau}, w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ is indeed a global classical solution to the Cauchy problem (1.21). The uniqueness can be shown by repeating same arguments as in [11]. Finally, applying the inverse function theorem, we can see that if $\alpha_\pm^{\varepsilon,\tau}$ and $\rho_\pm^{\varepsilon,\tau}$ are determined uniquely by (1.20), then $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$ is the unique global classical solution to the original Cauchy problem of System (BN) with the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ satisfying the properties (1.8)-(1.9) which concludes the proof of Theorem 1.1.

4.2 Global existence of System (K)

We provide a brief explanation of the global existence of System (K). The uniformity of the estimates (1.9) for System (BN) essentially ensures us to construct solutions of System (K) by passing to the limit on the relaxation parameter ε.

Proof of Theorem 1.2: Let the initial data $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$ satisfy the assumption (1.10). For any $\varepsilon\in(0,1)$, we consider the approximate problem, namely, System (BN) and the regularized initial data

$$(\alpha^{\varepsilon}_{\pm,0},\rho^{\varepsilon}_{\pm,0},u^{\varepsilon}_0)(x):=\dot S_{[\frac{1}{\varepsilon}]}(\alpha_{\pm,0},\rho_{\pm,0},u_0)(x). \tag{4.4}$$

It is clear that the sequence $(\alpha^{\varepsilon}_{\pm,0},\rho^{\varepsilon}_{\pm,0},u^{\varepsilon}_0)$ converges to $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$ strongly in $\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1}$ as $\varepsilon\to 0$. By virtue of Theorem 1.1, the approximate problem of System (BN) and the initial data $(\alpha^{\varepsilon}_{\pm,0},\rho^{\varepsilon}_{\pm,0},u^{\varepsilon}_0)$ has a unique global solution $(\alpha^{\varepsilon,\tau}_{\pm},\rho^{\varepsilon,\tau}_{\pm},u^{\varepsilon,\tau})$ satisfying the uniform estimates (1.9). In view of the Aubin-Lions lemma and the cantor diagonal process, there exists a limit $(\alpha^{\tau}_{\pm},\rho^{\tau}_{\pm},u^{\tau})$ such that as $\varepsilon\to 0$, up to a subsequence, $\chi(\alpha^{\varepsilon,\tau}_{\pm},\rho^{\varepsilon,\tau}_{\pm},u^{\varepsilon,\tau})$ converges strongly to $\chi(\alpha^{\tau}_{\pm},\rho^{\tau}_{\pm},u^{\tau})$ in $C([0,T];\dot B^{s})$ $(s<\frac{d}{2}+1)$ for any given time $T>0$ and $\chi\in C^{\infty}_c(\mathbb{R}^d\times[0,T])$. Thus, we can check that $(\alpha^{\tau}_{\pm},\rho^{\tau}_{\pm},u^{\tau})$ satisfies System (K) with the initial data $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$ in the sense of distributions. Taking the advantage of the Fatou property, one can show that $(\alpha^{\tau}_{\pm},\rho^{\tau}_{\pm},u^{\tau})$ satisfies the properties (1.11)-(1.12), and therefore is indeed a classical solution of System (K) associated to the initial data $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$. For brevity and because it follows from the exact same arguments as in [11], we omit here the proof of uniqueness.

4.3 Global existence of System (PM)

The following lemma states the uniform estimates of the solution of (K) diffusely rescaled by (1.2).

Lemma 4.1. *The global solution $(\beta^{\tau}_{\pm},\varrho^{\tau}_{\pm},v^{\tau})$ to the Cauchy problem of System (K_{τ}) subject to the initial data $(\alpha_{\pm,0},\rho_{\pm,0},u_0)$ given by Theorem 1.4 and the diffusive scaling (1.2) satisfies the uniform estimates*

$$\begin{aligned}
&\|(\beta^{\tau}_{\pm}-\bar\alpha_{\pm},\varrho^{\tau}_{\pm}-\bar\rho_{\pm})\|_{\widetilde L^{\infty}_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})}\\
&\quad+\|(\Pi^{\tau}-\bar P,\varrho^{\tau}_{\pm}-\bar\rho_{\pm})\|_{L^1_t(\dot B^{\frac{d}{2}+1})}+\|(\Pi^{\tau}-\bar P,\varrho^{\tau}_{\pm}-\bar\rho)\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}}\cap\dot B^{\frac{d}{2}+1})}\\
&\quad+\|v^{\tau}\|_{L^1_t(\dot B^{\frac{d}{2}}\cap\dot B^{\frac{d}{2}+1})}+\|v^{\tau}\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})}+\frac{1}{\tau}\|z^{\tau}\|_{L^1_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}})}+\|z^{\tau}\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1})}\\
&\leq C\|(\alpha_{\pm,0}-\bar\alpha_{\pm},\rho_{\pm,0}-\bar\rho_{\pm},u_0)\|_{\dot B^{\frac{d}{2}-1}\cap\dot B^{\frac{d}{2}+1}},\qquad t>0,
\end{aligned}\tag{4.5}$$

with $z^{\tau}:=v^{\tau}+\frac{1}{\varrho^{\tau}}\nabla\Pi^{\tau}$, and $C>0$ a constant independent of τ and time.

With the help of the uniform estimates presented in (4.1), we have the global existence of System (PM).

Proof of Theorem 1.3: Assume that the initial data $(\beta_{\pm,0},\varrho_{\pm,0})$ satisfies (1.13). For any $\tau\in(0,1)$, we regularize the initial data as follows:

$$(\alpha^{\varepsilon}_{\pm,0},\rho^{\varepsilon}_{\pm,0})(x):=\dot S_{[\frac{1}{\tau}]}(\beta_{\pm,0},\varrho_{\pm,0})(x).$$

Thence employing Theorem 1.2 and the diffusive scaling (1.2), we can find a sequence $(\beta^{\tau}_{\pm},\varrho^{\tau}_{\pm},v^{\tau})$, which is the global solution to System (K_{τ}) subject to the initial data $(\alpha^{\varepsilon}_{\pm,0},\rho^{\varepsilon}_{\pm,0},\tau)$. In view of the uniform estimates (4.5) established in Lemma 4.1, the Aubin-Lions lemma and the cantor diagonal process, there exists a limit $(\beta_{\pm},\varrho_{\pm})$ such that as $\tau\to 0$, up to a subsequence, $\chi(\beta^{\tau}_{\pm},\varrho^{\tau}_{\pm})$ converges strongly to $\chi(\beta_{\pm},\varrho_{\pm})$ in $C([0,T];\dot B^{s})$ $(s<\frac{d}{2}+1)$ for any given time $T>0$ and $\chi\in C^{\infty}_c(\mathbb{R}^d\times[0,T])$. Thus, we can check that $(\beta_\pm, \varrho_\pm)$ solves System (PM) in the sense of distributions. Furthermore, taking the advantage of the Fatou property and the optimal regularity estimates in Lemma 6.5 for the equation of Π, we can conclude (1.14). Finally, the uniqueness can be obtained in a simple fashion. The details are omitted.

5 Relaxation limits

5.1 Strong convergence of System (BN) to System (K)

In this subsection, we prove Theorem 1.4 related to the convergence rate of the relaxation process between System (BN) and System (K). Let $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ and $(\alpha_\pm^{\tau}, \rho_\pm^{\tau}, u^{\tau})$ be the global solutions to System (BN) with the initial data $(\alpha_{\pm,0}^{\varepsilon,\tau}, \rho_{\pm,0}^{\varepsilon,\tau}, u_0^{\varepsilon,\tau})$ and System (K) with the initial data $(\alpha_{\pm,0}^{\varepsilon,\tau}, \rho_{\pm,0}^{\varepsilon,\tau}, u_0^{\varepsilon,\tau})$ given by Theorems 1.1 and 1.2, respectively. Denote the error variables

$$(\delta\alpha_\pm, \delta\rho_\pm, \delta\rho, \delta P_\pm, \delta P, \delta u) := (\alpha_\pm^{\varepsilon,\tau} - \alpha_\pm^{\tau}, \rho_\pm^{\varepsilon,\tau} - \rho_\pm^{\tau}, \rho^{\varepsilon,\tau} - \rho^{\tau}, P_\pm(\rho_\pm^{\varepsilon,\tau}) - P_\pm(\rho_\pm^{\tau}), P^{\varepsilon,\tau} - P^{\tau}, u^{\varepsilon,\tau} - u^{\tau})$$

and the initial data of δP by

$$\delta P(x,0) = P_0^{\varepsilon,\tau} - P_0^{\tau}, \qquad P_0^{\varepsilon} := \alpha_{+,0}^{\varepsilon,\tau} P_+(\rho_{+,0}^{\varepsilon,\tau}) + \alpha_{-,0}^{\varepsilon,\tau} P_-(\rho_{-,0}^{\varepsilon,\tau}), \quad P_0^{\tau} := P_+(\rho_{+,0}^{\tau}). \tag{5.1}$$

First, to avoid dealing with difficult nonlinearities in the equation of $\delta\alpha_\pm$, we work with the following purely transported variable instead of $\delta\alpha_\pm$:

$$\delta Y := \frac{\alpha_+^{\varepsilon,\tau}\rho_+^{\varepsilon,\tau}}{\rho^{\varepsilon,\tau}} - \frac{\alpha_+^{\tau}\rho_+^{\tau}}{\rho^{\tau}}. \tag{5.2}$$

with the initial data

$$\delta Y(x,0) = Y_0^{\varepsilon,\tau}(x) - Y_0^{\tau}(x), \qquad Y_0^{\varepsilon,\tau} := \frac{\alpha_{+,0}^{\varepsilon,\tau}\rho_{+,0}^{\varepsilon,\tau}}{\alpha_{+,0}^{\varepsilon,\tau}\rho_{+,0}^{\varepsilon,\tau} + \alpha_{-,0}^{\varepsilon,\tau}\rho_{-,0}^{\varepsilon,\tau}}, \quad Y_0^{\tau} := \frac{\alpha_{+,0}^{\tau}\rho_{+,0}^{\tau}}{\alpha_{+,0}^{\tau}\rho_{+,0}^{\tau} + \alpha_{-,0}^{\tau}\rho_{-,0}^{\tau}}. \tag{5.3}$$

Lemma 5.1. *For $d \geq 3$, under the assumption (1.16), δY satisfies the following estimate:*

$$\|\delta Y\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \sqrt{\varepsilon\tau} + o(1)\|\delta u\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})}. \tag{5.4}$$

Proof. Since δY satisfies

$$\partial_t \delta Y + u^{\varepsilon,\tau}\cdot\nabla\delta Y = -\delta u\cdot\nabla\frac{\alpha_+^{\tau}\rho_+^{\tau}}{\rho^{\tau}},$$

Lemma 6.6 and the product law (6.2) gives

$$\|\delta Y\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim e^{\|u^{\varepsilon,\tau}\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}}\Big(\sqrt{\varepsilon\tau} + \|\delta u\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\|\nabla\frac{\alpha_+^{\tau}\rho_+^{\tau}}{\rho^{\tau}}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})}\Big).$$

This together with the uniform estimate (1.9) leads to (5.4). □

We are now ready to estimate $(\delta\alpha_\pm, \delta\rho_\pm, \delta P_\pm, \delta P)$. It is easy to verify that $P^{\varepsilon,\tau}$ satisfies

$$\begin{aligned}\partial_t P^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla P^{\varepsilon,\tau} &= -(\gamma_+\alpha_+^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_-^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau}))\mathrm{div}\, u^{\varepsilon,\tau}\\ &\quad - \alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}\big((\gamma_+-1)P_+(\rho_+^{\varepsilon,\tau}) - (\gamma_--1)P_-(\rho_-^{\varepsilon,\tau})\big)\frac{P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})}{\varepsilon}.\end{aligned} \tag{5.5}$$

And the equation of P^τ reads

$$\partial_t P^\tau + u^\tau\cdot\nabla P^\tau + \frac{\gamma_+\gamma_- P^\tau}{\gamma_+\alpha_-^\tau+\gamma_-\alpha_+^\tau}\operatorname{div} u^\tau = 0. \tag{5.6}$$

However it is not suitable to estimate δP directly from (5.5)-(5.6) as one cannot obtain a $\mathcal{O}(\varepsilon)$ for $\frac{1}{\varepsilon}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)$ in view of (1.9). To overcome this difficulty, we define the effective unknown $Q^{\varepsilon,\tau} := P^{\varepsilon,\tau} - \Gamma_1^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)$ which verifies

$$\begin{aligned}
&\partial_t Q^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla Q^{\varepsilon,\tau} + \Gamma_2^{\varepsilon,\tau}\operatorname{div} u^{\varepsilon,\tau}\\
&\quad = -\Gamma_3^{\varepsilon,\tau}(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))\operatorname{div} u^{\varepsilon,\tau} + (\partial_t\Gamma_1^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla\Gamma_1^{\varepsilon,\tau})\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)
\end{aligned} \tag{5.7}$$

with

$$\begin{cases}
\Gamma_1^{\varepsilon,\tau} := \dfrac{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}\big((\gamma_+-1)P_+(\rho_+^{\varepsilon,\tau}) - (\gamma_--1)P_-(\rho_-^{\varepsilon,\tau})\big)}{\gamma_+\alpha_-^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})},\\[2mm]
\Gamma_2^{\varepsilon,\tau} := \dfrac{\gamma_+\gamma_-P_+(\rho_+^{\varepsilon,\tau})P_-(\rho_-^{\varepsilon,\tau})}{\gamma_+\alpha_-P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})},\\[2mm]
\Gamma_3^{\varepsilon,\tau} := \dfrac{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}\big(\gamma_+P_+(\rho_+^{\varepsilon,\tau}) - \gamma_-P_-(\rho_-^{\varepsilon,\tau})\big)}{\gamma_+\alpha_-^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})}.
\end{cases}$$

With this formulation, it will be possible to derive the $\mathcal{O}(\varepsilon)$ bounds for the last term on the right-hand side of (5.7). Define

$$\delta Q := P^{\varepsilon,\tau} - P^\tau - \Gamma_1^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big). \tag{5.8}$$

The next lemma implies that to estimate $(\delta\alpha_\pm, \delta\rho_\pm, \delta P_\pm, \delta P)$, it is sufficient to control $(\delta Y, \delta Q, P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))$.

Lemma 5.2. *For $d\geq 3$, under the assumption (1.16), the following estimates follow:*

$$\begin{cases}
\|(\delta\alpha_\pm,\delta\rho_\pm,\delta\rho)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \|(\delta Y,\delta Q, P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})},\\
\|\delta\rho_\pm\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \|(\delta Q, P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}.
\end{cases} \tag{5.9}$$

Proof. Due to (5.2) and

$$\delta\rho = (\rho_+^{\varepsilon,\tau} - \rho_-^{\varepsilon,\tau})\delta\alpha_+ + \alpha_+^\tau\delta\rho_+ + \alpha_-^\tau\delta\rho_-, \tag{5.10}$$

it holds that

$$\begin{aligned}
\delta Y &= \frac{1}{\rho^{\varepsilon,\tau}\rho^\tau}\big(\rho_+^{\varepsilon,\tau}\rho^\tau\delta\alpha_+^{\varepsilon,\tau} + \alpha_+^\tau\rho^\tau\delta\rho_+ - \alpha_+^\tau\rho_+^\tau\delta\rho\big)\\
&= \frac{1}{\rho^{\varepsilon,\tau}\rho^\tau}\big((\alpha_-^\tau\rho_+^{\varepsilon,\tau}\rho_-^\tau + \alpha_+^\tau\rho_+^\tau\rho_-^{\varepsilon,\tau})\delta\alpha_+ + \alpha_+^\tau\alpha_-^\tau\rho_-^\tau\delta\rho_+ - \alpha_+^\tau\alpha_-^\tau\rho_+^\tau\delta\rho_-\big).
\end{aligned}$$

This implies

$$\delta\alpha_+ = \frac{1}{\alpha_-^\tau\rho_+^{\varepsilon,\tau}\rho_-^\tau + \alpha_+^\tau\rho_+^\tau\rho_-^{\varepsilon,\tau}}\big(\rho^{\varepsilon,\tau}\rho^\tau\delta Y - \alpha_+^\tau\alpha_-^\tau\rho_-^\tau\delta\rho_+ + \alpha_+^\tau\alpha_-^\tau\rho_+^\tau\delta\rho_-\big). \tag{5.11}$$

Inserting (5.11) into (5.10), we have

$$\delta\rho = \frac{\rho_+^{\varepsilon,\tau} - \rho_-^{\varepsilon,\tau}}{\alpha_-^\tau\rho_+^{\varepsilon,\tau}\rho_-^\tau + \alpha_+^\tau\rho_+^\tau\rho_-^{\varepsilon,\tau}}\big(\rho^{\varepsilon,\tau}\rho^\tau\delta Y - \alpha_+^\tau\alpha_-^\tau\rho_-^\tau\delta\rho_+ + \alpha_+^\tau\alpha_-^\tau\rho_+^\tau\delta\rho_-\big) + \alpha_+^\tau\delta\rho_+ + \alpha_-^\tau\delta\rho_-. \tag{5.12}$$

Moreover, we have

$$\delta P_{\pm} = \delta\rho_{\pm}\int_0^1 P'_{\pm}(\theta\rho_{\pm}^{\varepsilon,\tau} + (1-\theta)\rho_{\pm}^{\tau})d\theta \quad \text{and} \quad \delta P = \alpha_{+}^{\varepsilon,\tau}(P_{+}^{\varepsilon,\tau} - P_{-}^{\varepsilon,\tau}) + \delta P_{-}. \tag{5.13}$$

Using the previous uniform estimates (1.9) and (4.5), the product laws (6.1)-(6.2) and the composition estimates (6.4), for some constant states $\bar{\Gamma}_i > 0$ $(i = 1, 2, 3)$, we have

$$\sum_{i=1}^{3}(\|\Gamma_i - \bar{\Gamma}_i\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1})} + \|\partial_t\Gamma_i\|_{L^1_t(\dot{B}^{\frac{d}{2}})}) << 1. \tag{5.14}$$

Therefore, (5.9) follows from (5.8), (5.11)-(5.14), the product laws (6.1)-(6.2) and the fact $\delta\alpha_{+} = -\delta\alpha_{+}$. □

The next lemma pertains to $\mathcal{O}(\sqrt{\varepsilon\tau})$ bounds for $P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})$.

Lemma 5.3. *For $d \geq 3$, under the assumption (1.16), the following estimate is valid:*

$$\|P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \sqrt{\varepsilon\tau}. \tag{5.15}$$

Proof. It is easy to verify from (BN) that $P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})$ satisfies the damped equation

$$\begin{aligned}
&\partial_t\big(P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\big) + u^{\varepsilon,\tau}\cdot\nabla\big(P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\big) + \frac{c_*}{\varepsilon}\big(P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\\
&\quad = ((\gamma_{+}\alpha_{-}^{\varepsilon,\tau}P_{+}(\rho_{+}^{\varepsilon,\tau}) + \gamma_{-}\alpha_{+}^{\varepsilon,\tau}P_{-}(\rho_{-}^{\varepsilon,\tau})) - c_*)\frac{1}{\varepsilon}\big(P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\\
&\qquad - \big(\gamma_{+}P_{+}(\rho_{+}^{\varepsilon,\tau}) - \gamma_{-}P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\operatorname{div} u^{\varepsilon,\tau}.
\end{aligned} \tag{5.16}$$

with $c_* := (\gamma_{+}\bar{\alpha}_{-} + \gamma_{-}\bar{\alpha}_{+})\bar{P}$. The L^2-time type estimates in Lemma 6.6 for the damped transport equation (5.16) lead to

$$\begin{aligned}
&\|P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}\\
&\quad\lesssim e^{\|u^{\varepsilon,\tau}\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}}\Big(\sqrt{\varepsilon\tau} + \sqrt{\varepsilon}\|(\gamma_{+}\alpha_{-}^{\varepsilon,\tau}P_{+}(\rho_{+}^{\varepsilon,\tau}) + \gamma_{-}\alpha_{+}^{\varepsilon,\tau}P_{-}(\rho_{-}^{\varepsilon,\tau}) - c_*)\frac{1}{\varepsilon}\big(P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}\\
&\qquad + \sqrt{\varepsilon}\|\big(\gamma_{+}P_{+}(\rho_{+}^{\varepsilon,\tau}) - \gamma_{-}P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\operatorname{div} u^{\varepsilon,\tau}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}\Big).
\end{aligned}$$

By (1.9) and (6.2), there holds

$$\begin{aligned}
&\sqrt{\varepsilon}\|(\gamma_{+}\alpha_{-}^{\varepsilon,\tau}P_{+}(\rho_{+}^{\varepsilon,\tau}) + \gamma_{-}\alpha_{+}^{\varepsilon,\tau}P_{-}(\rho_{-}^{\varepsilon,\tau}) - c_*)\frac{1}{\varepsilon}\big(P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}\\
&\quad\lesssim \|(\alpha_{\pm}^{\varepsilon,\tau} - \bar{\alpha}_{\pm}, \rho_{\pm}^{\varepsilon,\tau} - \bar{\rho}_{\pm})\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\frac{1}{\sqrt{\varepsilon}}\|P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}\\
&\quad\lesssim o(1)\frac{1}{\sqrt{\varepsilon}}\|P_{+}(\rho_{+}^{\varepsilon,\tau}) - P_{-}(\rho_{-}^{\varepsilon,\tau})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}.
\end{aligned}$$

and

$$\sqrt{\varepsilon}\|\big(\gamma_{+}P_{+}(\rho_{+}^{\varepsilon,\tau}) - \gamma_{-}P_{-}(\rho_{-}^{\varepsilon,\tau})\big)\operatorname{div} u^{\varepsilon,\tau}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \sqrt{\varepsilon}\|u^{\varepsilon,\tau}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \sqrt{\varepsilon\tau}.$$

Therefore, we gain (5.15). □

We are going to estimate $(\delta Q, \delta u)$. By virtue of (BN), (K) and (5.6)-(5.7), $(\delta Q, \delta u)$ satisfies the following equations of damped Euler type:

$$
\begin{cases}
\partial_t \delta Q + u^{\varepsilon,\tau}\cdot\nabla\delta Q + \Gamma_2^{\varepsilon,\tau}\mathrm{div}\,\delta u = \delta F_1,\\
\partial_t \delta u + u^{\varepsilon,\tau}\cdot\nabla\delta u + \dfrac{1}{\bar\rho}\nabla\delta Q + (\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\rho^{\tau}})\nabla P^{\tau} + \dfrac{\delta u}{\tau} = \delta F_2,
\end{cases}
\tag{5.17}
$$

with the nonlinear terms

$$
\begin{cases}
\delta F_1 = -\delta u\cdot\nabla P^{\tau} - (\Gamma_2^{\varepsilon,\tau} - \dfrac{\gamma_+\gamma_- P^{\tau}}{\gamma_+\alpha_-^{\tau} + \gamma_-\alpha_+^{\tau}})\mathrm{div}\,u^{\tau}\\
\qquad\quad - \Gamma_3^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)\mathrm{div}\,u^{\varepsilon,\tau} + (\partial_t\Gamma_1^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla\Gamma_1^{\varepsilon,\tau})\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big),\\
\delta F_2 := -\delta u\cdot\nabla u^{\tau} - \dfrac{1}{\rho^{\varepsilon,\tau}}\nabla\Big(\Gamma_1^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)\Big).
\end{cases}
$$

In order to establish the uniform-in-τ convergence estimates, we follow the ideas in Section 3 to overcome the issue caused by the overdamping phenomenon.

Lemma 5.4. *Let $d \geq 3$, $0 < \varepsilon \leq \tau \leq 1$, and the threshold J_τ be given by (2.1). Then under the assumption (1.16), there holds*

$$
\begin{aligned}
&\|(\delta Q,\delta u)\|_{L_t^\infty(\dot B^{\frac d2-2}\cap\dot B^{\frac d2-1})} + \tau\|\delta Q\|^{\ell}_{L_t^1(\dot B^{\frac d2})} + \|\delta Q\|^{h}_{L_t^1(\dot B^{\frac d2-1})} + \sqrt{\tau}\|\delta Q\|_{\widetilde L_t^2(\dot B^{\frac d2-1})}\\
&\quad + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L_t^2(\dot B^{\frac d2-2}\cap\dot B^{\frac d2-1})} + \|\delta u\|_{L_t^1(\dot B^{\frac d2-1})}\\
&\quad \lesssim \sqrt{\varepsilon\tau} + o(1)\|\delta Y\|_{L_t^\infty(\dot B^{\frac d2-2}\cap\dot B^{\frac d2-1})}.
\end{aligned}
\tag{5.18}
$$

Proof. As in Section 3, we split the proof into three parts:

- **Step 1: $\dot B^{\frac d2-2}$-estimates in low frequencies**

We introduce the new damped mode

$$
\delta z := \delta u + \frac{\tau}{\rho^{\varepsilon,\tau}}\nabla\delta Q + \tau(\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^{\tau}})\nabla P^{\tau},
$$

so (5.17) rewrites as

$$
\begin{cases}
\partial_t\delta Q - \dfrac{\bar\Gamma_2\tau}{\bar\rho}\Delta\delta Q = -\bar\Gamma_2\mathrm{div}\,z + \delta F_3,\\
\partial_t\delta z + \dfrac{\delta z}{\tau} = \dfrac{\tau}{\bar\rho}\nabla(\dfrac{\bar\Gamma_2\tau}{\bar\rho}\Delta\delta Q - \bar\Gamma_2\mathrm{div}\,z) + \delta F_4.
\end{cases}
\tag{5.19}
$$

where $\bar\Gamma_2 > 0$ is the constant state of $\Gamma_2^{\varepsilon,\tau}$, and δF_i $(i = 3, 4)$ is defined by

$$
\begin{cases}
\delta F_3 := -u^{\varepsilon,\tau}\cdot\nabla\delta Q - (\Gamma_2^{\varepsilon,\tau} - \bar\Gamma_2)\mathrm{div}\,\delta u + \bar\Gamma_2\tau\mathrm{div}\Big((\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\bar\rho})\nabla\delta Q + (\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\rho^{\tau}})\nabla P^{\tau}\Big) + \delta F_1,\\
\delta F_4 := -u^{\varepsilon,\tau}\cdot\nabla\delta u + \dfrac{\tau}{\bar\rho}\nabla\delta F_3 + \tau(\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\bar\rho})\nabla\partial_t\delta Q - \tau\partial_t(\dfrac{1}{\rho^{\varepsilon,\tau}})\nabla\delta Q + \tau\partial_t\big((\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\rho^{\tau}})\nabla P^{\tau}\big) + \delta F_2.
\end{cases}
$$

Then by similar arguments used to get (3.14)-(3.17), we deduce from (5.19) and the choice (2.1) of the threshold J_τ that

$$
\begin{aligned}
&\|(\delta Q,\delta z)\|^{\ell}_{\widetilde L_t^\infty(\dot B^{\frac d2-2})} + \tau\|\delta Q\|^{\ell}_{L_t^1(\dot B^{\frac d2})} + \sqrt\tau\|\delta Q\|^{\ell}_{\widetilde L_t^2(\dot B^{\frac d2-1})} + \frac{1}{\tau}\|\delta z\|^{\ell}_{L_t^1(\dot B^{\frac d2-2})}\\
&\quad \lesssim \sqrt{\varepsilon\tau} + \|(\delta F_3,\delta F_4)\|^{\ell}_{L_t^1(\dot B^{\frac d2-2})}.
\end{aligned}
\tag{5.20}
$$

We first estimate δF_3. From (1.9), (5.14) and the product map $\dot B^{\frac d2-2}\times\dot B^{\frac d2}\to\dot B^{\frac d2-2}$ for $d\geq 3$, one obtains

$$
\begin{aligned}
&\|u^{\varepsilon,\tau}\cdot\nabla\delta Q+(\Gamma_2^{\varepsilon,\tau}-\bar\Gamma_2)\mathrm{div}\,\delta u\|_{L^1_t(\dot B^{\frac d2-2})}\\
&\lesssim\frac{1}{\sqrt\tau}\|u^{\varepsilon,\tau}\|_{\widetilde L^2_t(\dot B^{\frac d2})}\sqrt\tau\|\delta Q\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\|\Gamma_2^{\varepsilon,\tau}-\bar\Gamma_2\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim o(1)(\sqrt\tau\|\delta Q\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}).
\end{aligned}
$$

By virtue of (1.9), (2.2), (5.9), (5.15) and (6.2), we also have

$$
\begin{aligned}
\tau\|\mathrm{div}\,\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)\|^\ell_{L^1_t(\dot B^{\frac d2-2})}&\lesssim\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}\tau\|P^\tau-\bar P\|_{L^1_t(\dot B^{\frac d2+1})}\\
&\lesssim o(1)\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}+\sqrt{\varepsilon\tau}.
\end{aligned}
$$

As in the previous analysis (3.20), the tricky nonlinear term can be estimated as

$$
\begin{aligned}
\tau\|\mathrm{div}\,\Big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\delta Q\Big)\|^\ell_{L^1_t(\dot B^{\frac d2-2})}&\lesssim\tau\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\delta Q^\ell\|^\ell_{L^1_t(\dot B^{\frac d2-1})}+\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\delta Q^h\|^\ell_{L^1_t(\dot B^{\frac d2-2})}\\
&\lesssim o(1)(\tau\|\delta Q\|^\ell_{L^1_t(\dot B^{\frac d2})}+\|\delta Q\|^h_{L^1_t(\dot B^{\frac d2-1})}).
\end{aligned}
$$

Similarly, one can show

$$
\begin{aligned}
\|\delta F_1\|_{L^1_t(\dot B^{\frac d2-2})}\lesssim&\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}\sqrt\tau\|Q^\tau\|_{\widetilde L^2_t(\dot B^{\frac d2})}+\|(\Gamma_2^{\varepsilon,\tau}-\frac{\gamma_+\gamma_-P^\tau}{\gamma_+\alpha_-^\tau+\gamma_-\alpha_+^\tau})\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}\|u^\tau\|_{L^1_t(\dot B^{\frac d2+1})}\\
&+\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2+1})}\\
&+(\|\partial_t\Gamma_1^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}+\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}\|\nabla\Gamma_1^{\varepsilon,\tau}\|_{\widetilde L^\infty_t(\dot B^{\frac d2})})\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}\\
\lesssim&\, o(1)(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac d2-2})})+\sqrt{\varepsilon\tau}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|\delta F_3\|_{L^1_t(\dot B^{\frac d2-2})}\lesssim&\, o(1)(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}+\|\delta Q\|_{L^1_t(\dot B^{\frac d2})}+\sqrt\tau\|\delta Q\|_{\widetilde L^2_t(\dot B^{\frac d2-1}_{2,1})}\\
&+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}.
\end{aligned}
\tag{5.21}
$$

We turn to the estimate of δF_4. Similar calculations give

$$
\|-u^{\varepsilon,\tau}\cdot\nabla\delta u+\frac{\tau}{\rho^{\varepsilon,\tau}}\nabla\delta F_3\|_{L^1_t(\dot B^{\frac d2-2})}\lesssim o(1)\|\delta u\|_{L^1_t(\dot B^{\frac d2-1}_{2,1})}+\|\delta F_3\|_{L^1_t(\dot B^{\frac d2-2})}.
$$

and

$$
\|\tau\partial_t(\frac{1}{\rho^{\varepsilon,\tau}})\nabla\delta Q\|_{L^1_t(\dot B^{\frac d2-2})}\lesssim\|\partial_t\rho^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}\|\delta Q\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}\lesssim o(1)\|\delta Q\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}.
$$

Using the equation $(5.17)_1$, one has

$$
\begin{aligned}
&\|\tau(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\partial_tQ\|_{L^1_t(\dot B^{\frac d2-2})}\\
&\lesssim\|\rho^{\varepsilon,\tau}-\bar\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}(\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}\|\delta Q\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}+\|\delta F_1\|_{L^1_t(\dot B^{\frac d2-2})})\\
&\lesssim o(1)(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}.
\end{aligned}
$$

Similarly, the term δF_2 can be estimated easily as follows:

$$
\begin{aligned}
\|\delta F_2\|_{L^1_t(\dot B^{\frac d2-2})} &\lesssim \|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}\|u^\tau\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}+\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim o(1)\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}+\sqrt{\varepsilon\tau}+\varepsilon.
\end{aligned}
$$

To bound the term $\tau\partial_t\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)$, since $\partial_t\delta\rho=-\mathrm{div}\,\big(\delta\rho u^{\varepsilon,\tau}+\rho^\tau\delta u\big)$ follows, we use (1.9), (5.9), (5.15) and (6.4) that

$$
\begin{aligned}
\|\tau\partial_t\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)\|_{L^1_t(\dot B^{\frac d2-2})} &\lesssim \|\partial_t\delta\rho\|_{L^1_t(\dot B^{\frac d2-2})}\|\nabla P^\tau\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}+\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}\tau\|\nabla P^\tau\|_{L^1_t(\dot B^{\frac d2})}\\
&\lesssim o(1)(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2}\cap\dot B^{\frac d2-1})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}.
\end{aligned}
$$

We thence get

$$
\begin{aligned}
\|\delta F_3\|_{L^1_t(\dot B^{\frac d2-2})} &\lesssim o(1)(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}+\|\delta Q\|_{L^1_t(\dot B^{\frac d2})}+\sqrt\tau\|\delta Q\|_{\widetilde L^2_t(\dot B^{\frac d2-1}_{2,1})}\\
&\quad+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}.
\end{aligned}
\tag{5.22}
$$

Substituting the above estimates (5.21)-(5.22) into (5.20) and taking the advantage of $\delta u=\delta z-\frac{\tau}{\rho^{\varepsilon,\tau}}\nabla\delta Q-\tau(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau$, we obtain

$$
\begin{aligned}
&\|(\delta Q,\delta z)\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}+\tau\|\delta Q\|^\ell_{L^1_t(\dot B^{\frac d2})}+\sqrt\tau\|\delta Q\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\frac1\tau\|\delta z\|^\ell_{L^1_t(\dot B^{\frac d2-2})}\\
&\quad+\|\delta u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2-2})}+\frac{1}{\sqrt\tau}\|\delta u\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\|\delta u\|^\ell_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim\sqrt{\varepsilon\tau}+o(1)(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-2}\cap\dot B^{\frac d2-1})}+\tau\|\delta Q\|^\ell_{L^1_t(\dot B^{\frac d2})}+\|\delta Q\|^h_{L^1_t(\dot B^{\frac d2-1})}+\sqrt\tau\|\delta Q\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\\
&\quad+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}).
\end{aligned}
\tag{5.23}
$$

- **Step 2: $\dot B^{\frac d2-2}$-estimates of $(\delta Q,\delta u)$ in high frequencies**

Applying Δ_j to (5.17), one gets

$$
\begin{cases}
\partial_t\dot\Delta_j\delta Q+u^{\varepsilon,\tau}\cdot\nabla\dot\Delta_j\delta Q+\Gamma_2^{\varepsilon,\tau}\mathrm{div}\,\dot\Delta_j\delta u=\dot\Delta_j\delta F_1+\delta R_{1,j},\\
\partial_t\dot\Delta_j\delta u+u^{\varepsilon,\tau}\cdot\nabla\dot\Delta_j\delta u+\dfrac{1}{\rho^{\varepsilon,\tau}}\nabla\dot\Delta_j\delta Q+\dfrac{\dot\Delta_j\delta u}{\tau}=-\dot\Delta_j\big((\dfrac{1}{\rho^{\varepsilon,\tau}}-\dfrac{1}{\rho^\tau})\nabla P^\tau\big)+\dot\Delta_j\delta F_2+\delta R_{2,j}+\delta R_{3,j},
\end{cases}
$$

with $\delta R_{1,j}:=[u^{\varepsilon,\tau},\dot\Delta_j]\nabla\delta Q+[\Gamma_2^{\varepsilon,\tau},\dot\Delta_j]\mathrm{div}\,\dot\Delta_j\delta u$, $\delta R_{2,j}:=[u^{\varepsilon,\tau},\dot\Delta_j]\nabla\delta u$ and $\delta R_{3,j}:=[\frac{1}{\rho^{\varepsilon,\tau}},\dot\Delta_j]\nabla\delta Q$.

Similarly to the high-frequency analysis in Subsection 3.2, one gains

$$
\begin{aligned}
&\frac{d}{dt}\int(\frac{1}{\rho^{\varepsilon,\tau}}|\dot\Delta_j\delta Q|^2+\Gamma_2^{\varepsilon,\tau}|\dot\Delta_j\delta u|^2)dx+\frac1\tau\|\dot\Delta_j\delta u\|^2_{L^2}\\
&\lesssim\big(\|\mathrm{div}\,u^{\varepsilon,\tau}\|_{L^\infty}+\|\nabla\Gamma_2^{\varepsilon,\tau}\|_{L^\infty}+\|\nabla\frac{1}{\rho^{\varepsilon,\tau}}\|_{L^\infty}+\|\partial_t\frac{1}{\rho^{\varepsilon,\tau}}\|_{L^\infty}+\|\partial_t\Gamma_2^{\varepsilon,\tau}\|_{L^\infty}\big)\|\dot\Delta_j\delta Q\|_{L^2}\|\dot\Delta_j\delta u\|_{L^2}\\
&\quad+\|\dot\Delta_j(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\|_{L^2}\|\dot\Delta_j\delta u\|_{L^2}+\|\dot\Delta_j(\delta F_1,\delta F_2)\|_{L^2}\|\dot\Delta_j(\delta Q,\delta u)\|_{L^2}\\
&\quad+\|\delta R_{1,j}\|_{L^2}\|\dot\Delta_j\delta Q\|_{L^2}+\|(\delta R_{2,j},\delta R_{3,j})\|_{L^2}\|\dot\Delta_j\delta u\|_{L^2},
\end{aligned}
\tag{5.24}
$$

and the cross term

$$
\begin{aligned}
&\frac{d}{dt}\int \dot\Delta_j\delta u\cdot\nabla\dot\Delta_j\nabla\delta P dx+\int\big(\frac{1}{\rho^{\varepsilon,\tau}}|\nabla\dot\Delta_j\delta Q|^2-\Gamma_2^{\varepsilon,\tau}|\mathrm{div}\,\dot\Delta_j\delta u_j|^2+\frac{1}{\tau}\dot\Delta_j\delta u\cdot\nabla\dot\Delta_j\nabla\delta P\big)dx\\
&\quad\lesssim (\|u^{\varepsilon,\tau}\|_{L^\infty}\|\nabla\dot\Delta_j\delta u\|_{L^2}+\|\dot\Delta_j\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)\|_{L^2}+\|(\dot\Delta_j\delta F_2,\delta R_{2,j},\delta R_{3,j})\|_{L^2})\|\nabla\dot\Delta_j\nabla\delta P\|_{L^2}\\
&\quad+(\|u^{\varepsilon,\tau}\|_{L^\infty}\|\nabla\dot\Delta_j\delta Q\|_{L^2}+\|\dot\Delta_j\delta F_1\|_{L^2}+\|\delta R_{1,j}\|_{L^2})\|\delta u\|_{L^2}. \qquad (5.25)
\end{aligned}
$$

For all $j\geq J_\tau$, multiplying (5.25) by a suitable small constant and adding the resulting inequality and (5.24) together, we can derive the Lyapunov inequality similar to (3.35)-(3.42) and then show the following L^1-time type estimates:

$$
\begin{aligned}
&\tau\|(\delta Q,\delta u)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|(\delta Q,\delta u)\|^h_{L^1_t(\dot B^{\frac d2-1})}\\
&\quad\lesssim\sqrt{\tau\varepsilon}+(\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})}+\|(\partial_t\rho^{\varepsilon,\tau},\partial_t\Gamma_2^{\varepsilon,\tau})\|_{L^1_t(\dot B^{\frac d2})})\tau\|(\delta Q,\delta u)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\\
&\qquad+\|(\rho^{\varepsilon,\tau}-\bar\rho,\Gamma_2^{\varepsilon,\tau}-\bar\Gamma_2)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2+1})}\|(\delta Q,\delta u)\|^h_{L^1_t(\dot B^{\frac d2-1})}+\tau\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\|^h_{L^1_t(\dot B^{\frac d2-1})}\\
&\qquad+\tau\sum_{j\geq J_\tau-1}2^{(\frac d2-1)j}\|(\delta R_{1,j},\delta R_{2,j},\delta R_{3,j})\|_{L^2}+\tau\|(\delta F_1,\delta F_2)\|^h_{L^1_t(\dot B^{\frac d2-1})}.
\end{aligned}
$$

By (1.9), (5.9), (5.15), the product laws (6.2) and the commutator estimates (6.3), it is easy to show

$$
\tau\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\|^h_{L^1_t(\dot B^{\frac d2-1})}\lesssim\|\delta\rho\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\tau\|P^\tau-\bar P\|_{L^1_t(\dot B^{\frac d2+1})}\lesssim o(1)\|(\delta Y,\delta Q)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\sqrt{\varepsilon\tau},
$$

and

$$
\tau\sum_{j\geq J_\tau-1}2^{(\frac d2-1)j}\|(\delta R_{1,j},\delta R_{2,j})\|_{L^2}\lesssim\|\nabla u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}\|(\delta Q,\delta u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\lesssim o(1)\|(\delta Q,\delta u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}.
$$

For the tricky commutator term $R_{3,j}$, we have

$$
\begin{aligned}
\tau\sum_{j\geq J_\tau-1}2^{(\frac d2-1)j}\|R_{3,j}\|_{L^1_t(L^2)}&\lesssim\tau\sum_{j\geq J_\tau-1}2^{\frac d2 j}\|[\frac{1}{\rho^{\varepsilon,\tau}},\dot\Delta_j]\nabla\delta Q^\ell\|_{L^1_t(L^2)}+\sum_{j\geq J_\tau-1}2^{(\frac d2-1)j}\|[\frac{1}{\rho^{\varepsilon,\tau}},\dot\Delta_j]\nabla\delta Q^h\|_{L^1_t(L^2)}\\
&\lesssim\|\nabla\rho^{\varepsilon,\tau}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}(\tau\|\delta Q^\ell\|_{L^1_t(\dot B^{\frac d2})}+\|\delta Q^h\|_{L^1_t(\dot B^{\frac d2-1})})\\
&\lesssim o(1)(\tau\|\delta Q\|^\ell_{L^1_t(\dot B^{\frac d2})}+\|\delta Q\|^h_{L^1_t(\dot B^{\frac d2-1})}).
\end{aligned}
$$

For δF_1 and δF_2, similar computations give rise to

$$
\begin{aligned}
&\|(\delta F_1,\delta F_2)\|_{L^1_t(\dot B^{\frac d2-1})}\\
&\quad\lesssim\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}\|(P^\tau-\bar P,u^\tau)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2+1})}+\|\Gamma_2^{\varepsilon,\tau}-\frac{\gamma_+\gamma_-P^\tau}{\gamma_+\alpha_-^\tau+\gamma_-\alpha_+^\tau}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\|u^\tau\|_{L^1_t(\dot B^{\frac d2+1})}\\
&\quad+(\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2+1})}+\|\partial_t\Gamma_1^{\varepsilon,\tau}+u^{\varepsilon,\tau}\cdot\nabla\Gamma_1^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})})\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})} \qquad (5.26)\\
&\qquad+\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{L^1_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}\\
&\quad\lesssim o(1)(\|(\delta Y,\delta Q)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}+\varepsilon.
\end{aligned}
$$

We thus get

$$
\begin{aligned}
&\|(\delta Q,\delta u)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac d2-2})}+\tau\|(\delta Q,\delta u)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\\
&\quad+\|(\delta Q,\delta u)\|^h_{L^1_t(\dot B^{\frac d2-1})}+\sqrt{\tau}\|\delta Q\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|u\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2-2})}\\
&\lesssim\sqrt{\varepsilon\tau}+o(1)(\|(\delta Q,\delta u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\tau\|\delta Q\|^\ell_{L^1_t(\dot B^{\frac d2})}+\tau\|\delta Q\|^h_{L^1_t(\dot B^{\frac d2-1})}+\tau\|\delta u\|_{L^1_t(\dot B^{\frac d2-1})}).
\end{aligned}
\tag{5.27}
$$

- **Step 3: $\dot B^{\frac d2-1}$-estimates of $(\delta Q,\delta u)$ in all frequencies**

We need to further establish the uniform $\dot B^{\frac d2-1}$-bounds. To this end, owing to (5.24), we obtain the L^2-in-time estimates

$$
\begin{aligned}
&\|(\delta Q,\delta u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\\
&\lesssim\sqrt{\varepsilon\tau}+\big(\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}+\|(\partial_t\rho^{\varepsilon,\tau},\partial_t\Gamma_2^{\varepsilon,\tau})\|_{L^1_t(\dot B^{\frac d2})}\big)^{\frac12}\|(\delta Q,\delta u)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\\
&\quad+\|(\rho^{\varepsilon,\tau}-\bar\rho,\Gamma_2^{\varepsilon,\tau}-\bar\Gamma_2)\|^{\frac12}_{\widetilde{L}^\infty_t(\dot B^{\frac d2+1})}\big(\sqrt{\tau}\|\delta Q\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\big)^{\frac12}\big(\frac{1}{\sqrt{\tau}}\|\delta u\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\big)^{\frac12}\\
&\quad+\|(\delta F_1,\delta F_2)\|^{\frac12}_{L^1_t(\dot B^{\frac d2-1})}\|(\delta Q,\delta u)\|^{\frac12}_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\|^{\frac12}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\|\delta u\|^{\frac12}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\\
&\quad+\sum_{j\in\mathbb{Z}}2^{(\frac d2-1)j}\Big(\int_0^t\big(\|\delta R_{1,j}\|_{L^2}\|\dot\Delta_j\delta Q\|_{L^2}+\|(\delta R_{2,j},\delta R_{3,j})\|_{L^2}\|\dot\Delta_j u\|_{L^2}\big)d\tau\Big)^{\frac12}.
\end{aligned}
$$

One has

$$
\begin{aligned}
&\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\|^{\frac12}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\|\delta u\|^{\frac12}_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\\
&\lesssim\|\delta\rho\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\big(\sqrt{\tau}\|P^\tau-\bar P\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}\big)^{\frac12}\big(\frac{1}{\sqrt{\tau}}\|\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\big)^{\frac12}\\
&\lesssim o(1)(\|(\delta Y,\delta Q)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}.
\end{aligned}
$$

Concerning the commutator terms, we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}2^{(\frac d2-1)j}\Big(\int_0^t\big(\|\delta R_{1,j}\|_{L^2}\|\dot\Delta_j\delta Q\|_{L^2}+\|\delta R_{2,j}\|_{L^2}\|\dot\Delta_j u\|_{L^2}\big)d\tau\Big)^{\frac12}\\
&\lesssim\Big(\frac{1}{\sqrt{\tau}}\|\nabla u^{\varepsilon,\tau}\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\|\delta Q\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\sqrt{\tau}\|\delta Q\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\Big)^{\frac12}\\
&\quad+\Big(\frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\|\nabla\Gamma_2^{\varepsilon,\tau}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2+1})}\sqrt{\tau}\|\delta Q\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\Big)^{\frac12}+\big(\|\nabla u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac d2})}\|\delta u\|^2_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\big)^{\frac12}\\
&\quad+\big(\|\nabla\rho^{\varepsilon,\tau}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}\sqrt{\tau}\|\delta Q\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\big)^{\frac12}\\
&\lesssim o(1)(\|(\delta Q,\delta u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\sqrt{\tau}\|\delta Q\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}).
\end{aligned}
$$

Gathering (5.26) and the above three estimates, we have

$$
\begin{aligned}
&\|(\delta Q,\delta u)\|_{L^\infty_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\\
&\quad\lesssim\sqrt{\varepsilon\tau}+o(1)(\|(\delta Y,\delta Q,\delta u)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta Q\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})})+\sqrt{\varepsilon\tau}.
\end{aligned}
\tag{5.28}
$$

Finally, combining (5.4), (5.23), (5.27) and (5.28), we end up with (5.18) which completes the proof of Lemma 5.4. □

5.2 Strong convergence of System (K$_\tau$) to System (PM)

This subsection is devoted to the proof of (1.19) in Theorem 1.4. Define the error variables

$$(\delta\beta_\pm,\delta\varrho_\pm,\delta\varrho,\delta\Pi,\delta v):=(\beta_\pm^\tau-\beta_\pm,\varrho_\pm^\tau-\varrho_\pm,\varrho^\tau-\varrho,\Pi^\tau-\Pi,v^\tau-v).$$

First, similarly to (5.1), we need to estimate the variable $\delta Z:=\frac{\beta_+^\tau\varrho_+^\tau}{\varrho^\tau}-\frac{\beta_+\varrho_+}{\varrho}$ instead of $\delta\beta$, where the initial data of δz is

$$\delta Z(x,0)=Z_0^\tau(x)-Z_0(x),\quad Z_0^\tau:=\frac{\alpha_{+,0}^\tau\rho_{+,0}^\tau}{\alpha_{+,0}^\tau\rho_{+,0}^\tau+\alpha_{-,0}^\tau\rho_{-,0}^\tau},\quad Z_0:=\frac{\beta_{+,0}\varrho_{+,0}}{\beta_{+,0}\varrho_{+,0}+\beta_{-,0}\varrho_{-,0}}.\tag{5.29}$$

Indeed, arguing similarly as in Lemma 5.2, we obtain from (K$_\tau$) and (PM) that

$$\begin{cases}\delta\beta_+=\dfrac{1}{\beta_-^\tau\varrho_+^\tau\varrho_-+\beta_+\varrho_+\varrho_-^\tau}\big(\varrho^\tau\varrho\delta Z-\beta_+\beta_-\varrho_-\delta\varrho_++\beta_+^\tau\beta_-\varrho_+\delta\varrho_-\big),\\ \delta\varrho^\tau=\dfrac{\varrho_+^\tau-\varrho_-^\tau}{\beta_-\varrho_+^\tau\varrho_-+\beta_+\varrho_+\varrho_-^\tau}\big(\varrho^\tau\varrho\delta Z-\beta_+\beta_-\varrho_-\delta\varrho_++\beta_+\beta_-\varrho_+\delta\varrho_-\big)+\beta_+\delta\varrho_++\beta_-\delta\varrho_-^\tau,\\ \delta\Pi=\delta\varrho_+\displaystyle\int_0^1P_+'(\theta\varrho_+^\tau+(1-\theta)\varrho_+)d\theta=\delta\varrho_-\int_0^1P_-'(\theta\varrho_-^\tau+(1-\theta)\varrho_-)d\theta,\end{cases}\tag{5.30}$$

which leads to

$$\begin{cases}\|\delta\varrho_\pm\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta\varrho_\pm\|_{L^1_t(\dot B^{\frac d2+1})}\sim\|\delta\Pi\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta\Pi\|_{L^1_t(\dot B^{\frac d2+1})},\\ \|\delta\beta_\pm\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}\lesssim\|(\delta Z,\delta\Pi)\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}.\end{cases}\tag{5.31}$$

It is therefore sufficient to estimate $(\delta\Pi,\delta v,\delta Z)$ to recover information on all the error unknowns.

Next, note that δZ satisfies the transport equation

$$\partial_t\delta Z+v^\tau\cdot\nabla\delta Z=-\delta v\cdot\nabla\frac{\beta_+\varrho_+}{\varrho}.\tag{5.32}$$

Using Lemma 6.6, (4.5) and the product law (6.2), we get

$$\|\delta Z\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}\lesssim e^{\|v^\tau\|_{L^1_t(\dot B^{\frac d2+1})}}\|\delta v\|_{L^1_t(\dot B^{\frac d2})}\|\nabla\frac{\beta_+\varrho_+}{\varrho}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2}\cap\dot B^{\frac d2+1})}\lesssim o(1)\|\delta v\|_{L^1_t(\dot B^{\frac d2})}.\tag{5.33}$$

Then, we perform the key estimates of $\delta\Pi$. From (K$_\tau$) and (PM), it is easy to see

$$\partial_t\Pi^\tau+v^\tau\cdot\nabla\Pi^\tau=\frac{\gamma_+\gamma_-\Pi^\tau}{\gamma_+\beta_-^\tau+\gamma_-\beta_+^\tau}\mathrm{div}\,(\frac{\nabla\Pi^\tau}{\varrho^\tau})-\frac{\gamma_+\gamma_-\Pi^\tau}{\gamma_+\beta_-^\tau+\gamma_-\beta_+^\tau}\mathrm{div}\,z^\tau,\qquad z^\tau:=v^\tau+\frac{\nabla\Pi^\tau}{\varrho^\tau}.$$

Thence the equation of $\delta\Pi$ reads

$$\begin{aligned}\partial_t\delta\Pi-\bar c\Delta\delta\Pi=&-v^\tau\cdot\nabla\delta\Pi-\delta v\cdot\nabla\Pi+\Big(\frac{\gamma_+\gamma_-\Pi^\tau}{\gamma_+\beta_+^\tau+\gamma_-\beta_-^\tau}-\frac{\gamma_+\gamma_-\Pi}{\gamma_+\beta_++\gamma_-\beta_-}\Big)\mathrm{div}\,(\frac{1}{\varrho^\tau}\nabla\Pi^\tau)\\&+\frac{\gamma_+\gamma_-\Pi}{\gamma_+\beta_++\gamma_-\beta_-}\mathrm{div}\,\big((\frac{1}{\varrho^\tau}-\frac{1}{\varrho})\nabla\Pi^\tau\big)-\frac{\gamma_+\gamma_-\Pi^\tau}{\gamma_+\beta_-^\tau+\gamma_-\beta_+^\tau}\mathrm{div}\,z^\tau,\end{aligned}\tag{5.34}$$

with the constant $\bar c:=\frac{\gamma_+\gamma_-P_+(\bar\rho_+)}{(\gamma_+\bar\alpha_-+\gamma_-\bar\alpha_+)\bar\rho}>0$. We mention that the convergence rate τ is $\mathcal{O}(\tau)$ bound comes from the uniform estimates (4.5) of the effective unknown z^τ. Indeed, by Lemma 6.5, the uniform estimates (4.5), the smallness of the initial data (1.7), the product laws (6.1) and the composition estimates

(6.4), one obtains

$$\begin{aligned}
&\|\delta\Pi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta\Pi\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\\
&\quad\lesssim \tau+\|v^\tau\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\|\delta\Pi\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta v\|_{L^1_t(\dot{B}^{\frac{d}{2}})}\|\Pi-\bar{P}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}\\
&\qquad+\|\delta\Pi\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\|\Pi^\tau-\bar{P}\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}+\|(\frac{1}{\varrho^\tau}-\frac{1}{\varrho})\nabla\Pi^\tau\|_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|z^\tau\|_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\quad\lesssim o(1)(\|\delta Z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}+\|(\delta\varrho_\pm,\delta\Pi)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}+\|(\delta\varrho_\pm,\delta\Pi)\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})})+\tau,
\end{aligned}\tag{5.35}$$

where we have used the key fact

$$\begin{aligned}
\|(\frac{1}{\varrho^\tau}-\frac{1}{\varrho})\nabla\Pi^\tau\|_{L^1_t(\dot{B}^{\frac{d}{2}})}&\lesssim(\|\delta Z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}\|\Pi^\tau-\bar{P}\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}+\|\delta\varrho_\pm\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\|\Pi^\tau-\bar{P}\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\\
&\lesssim o(1)(\|\delta Z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}+\|\delta\varrho_\pm\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})})
\end{aligned}\tag{5.36}$$

derived from (4.5), (5.30) and (6.2). Gathering (5.31) and (5.35) together, we get

$$\|\delta\Pi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta\Pi\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\lesssim o(1)\|\delta Z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}+\|\delta\Pi\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})})+\tau.\tag{5.37}$$

For δv, in view of (4.5), (5.36) and $\delta v=(\frac{1}{\varrho^\tau}-\frac{1}{\varrho})\nabla\Pi^\tau-\frac{1}{\varrho}\nabla\delta\Pi+z^\tau$, the error unknown δv can be bounded by

$$\begin{aligned}
\|\delta v\|_{L^1_t(\dot{B}^{\frac{d}{2}})}&\lesssim\|\delta\Pi\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}+\|(\frac{1}{\varrho^\tau}-\frac{1}{\varrho})\nabla\Pi^\tau\|_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|z^\tau\|_{L^1_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim\|\delta Z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}+\|\delta\Pi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta\Pi\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}+\tau.
\end{aligned}\tag{5.38}$$

The combination of (5.31) and (5.33)-(5.37) gives rise to (1.19), which completes the proof of Theorem 1.4.

6 Appendix

We recall some properties of Besov spaces and related estimates which will be used repeatedly in this paper. The reader can refer to [2, Chapters 2-3] for more details.

The first lemma pertains to the so-called Bernstein inequalities.

Lemma 6.1. *Let $0<r<R$, $1\leq p\leq q\leq\infty$ and $k\in\mathbb{N}$. For any function $u\in L^p$ and $\lambda>0$, it holds*

$$\begin{cases}\text{Supp }\mathcal{F}(u)\subset\{\xi\in\mathbb{R}^d\mid|\xi|\leq\lambda R\}\Rightarrow\|D^k u\|_{L^q}\lesssim\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},\\ \text{Supp }\mathcal{F}(u)\subset\{\xi\in\mathbb{R}^d\mid\lambda r\leq|\xi|\leq\lambda R\}\Rightarrow\|D^k u\|_{L^p}\sim\lambda^k\|u\|_{L^p}.\end{cases}$$

The following Morse-type product estimates in Besov spaces play a fundamental role in our analysis of nonlinear terms.

Lemma 6.2. *The following statements hold:*

- *Let $s>0$. Then $\dot{B}^s\cap L^\infty$ is a algebra and*

$$\|uv\|_{\dot{B}^s}\lesssim\|u\|_{L^\infty}\|v\|_{\dot{B}^s}+\|v\|_{L^\infty}\|u\|_{\dot{B}^s}.\tag{6.1}$$

- *Let s_1, s_2 satisfy $s_1, s_2 \leq \frac{d}{2}$ and $s_1+s_2>0$. Then there holds*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{s_1}}\|v\|_{\dot{B}^{s_2}}. \tag{6.2}$$

The following commutator estimates are used to control some nonlinearities in high frequencies.

Lemma 6.3. *Let $p \in [1,\infty]$ and $-\frac{d}{2}-1 \leq s \leq \frac{d}{2}+1$. Then it holds*

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[v,\dot{\Delta}_j]\partial_i u\|_{L^2} \lesssim \|\nabla v\|_{\dot{B}^{\frac{d}{2}}}\|u\|_{\dot{B}^s}, \quad i=1,...,d, \tag{6.3}$$

for the commutator $[A,B] := AB-BA$.

We recall the classical estimates about the continuity for composition of functions (cf. [34]):

Lemma 6.4. *Let $m \in \mathbb{N}$, $s>0$, and $G \in C^\infty(\mathbb{R}^m)$ satisfy $G(0,...,0)=0$. Then for any $f_i \in \dot{B}^s \cap L^\infty$ $(i=1,...,m)$, there exists a constant $C>0$ depending on $\|(f_1,...,f_m)\|_{L^\infty}$, F, s, m and d such that*

$$\|G(f_1,...,f_m)\|_{\dot{B}^s_{p,r}} \leq C\|(f_1,...,f_m)\|_{\dot{B}^s_{p,r}}. \tag{6.4}$$

We recall the optimal regularity estimates of the heat equation as follows.

Lemma 6.5. *Let $c_1>0$, $s\in\mathbb{R}$ and $1\leq p\leq\infty$. For given time $T>0$, assume $u_0 \in \dot{B}^s$ and $f \in \widetilde{L}^p(0,T;\dot{B}^{s-2+\frac{2}{p}})$. If u solves the problem*

$$\begin{cases} \partial_t u - c_1\Delta u = f, & x\in\mathbb{R}^d, \quad t>0,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^d, \end{cases}$$

then the following estimate is fulfilled:

$$\|u\|_{\widetilde{L}^\infty_t(\dot{B}^s)} + c_1^{\frac{1}{p_1}}\|u\|_{\widetilde{L}^p_t(\dot{B}^{s+\frac{2}{p}})} \leq C(\|u_0\|_{\dot{B}^s} + c_1^{\frac{1}{p}-1}\|f\|_{\widetilde{L}^p_t(\dot{B}^{s-2+\frac{2}{p}})}), \quad t\in(0,T),$$

where $C>0$ is a constant independent of T and c_1.

Finally, we have the optimal regularity estimates of the damped transport equation. Since it can be shown directly by the commutator estimates (6.3) and Grönwall's inequality, we omit the proof for brevity.

Lemma 6.6. *Let $c_2\geq 0$, $-\frac{d}{2}<s\leq\frac{d}{2}+1$ and $1\leq p\leq\infty$. For given time $T>0$, assume $u_0\in\dot{B}^s$, $v\in L^1(0,T;B^{\frac{d}{2}+1})$ and $f\in\widetilde{L}^p(0,T;\dot{B}^s)$. If u solves the problem*

$$\begin{cases} \partial_t u + v\cdot\nabla u + c_2 u = f, & x\in\mathbb{R}^d, \quad t>0,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^d, \end{cases}$$

then it holds that

$$\|u\|_{\widetilde{L}^\infty_t(\dot{B}^s)} + c_2^{\frac{1}{p}}\|u\|_{\widetilde{L}^p_t(\dot{B}^s)} \leq Ce^{C\|v\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}}(\|u_0\|_{\dot{B}^s} + c_2^{\frac{1}{p}-1}\|f\|_{\widetilde{L}^p_t(\dot{B}^s)}), \quad t\in(0,T),$$

where $C>0$ is a constant independent of T and c_2.

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