Partially Dissipative Hyperbolic Systems in Critical Homogeneous Besov Spaces[1]

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[1] Joint work with Raphaël Danchin

Introduction

General n-component systems of balance laws in $\mathbb{R}^d$ read:

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(w)}{\partial x_j} = Q(w). \tag{1}$$

The unknown $w = w(t,x)$ with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$ is valued in an open convex subset $\mathcal{O}_w$ of $\mathbb{R}^n$ and $Q, F_j : \mathbb{R}^n \to \mathcal{O}_w$ are given n-vector valued smooth functions on $\mathcal{O}_w$.

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- In the case $Q(w) = 0$, it is well known that (1) supplemented with smooth data admits local-in-time strong solutions that may develop singularities (shock waves) in finite time even if the initial data are small perturbations of a constant solution (A. Majda, D. Serre).

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- A sufficient condition for global existence for small perturbations of a constant solution $\bar{w}$ of (1) is the *total dissipation hypothesis*, namely the damping (or dissipation) term $Q(w)$ acts directly on each component of the system, making the whole solution to tend to $\bar{w}$ exponentially fast.

Partial dissipation

A more reasonable assumption is that dissipation acts only on some components of the system. After suitable change of coordinates, we may write:

$$Q(w) = \begin{pmatrix} 0_{\mathbb{R}^{n_1}} \\ q(w) \end{pmatrix} \text{ where } q(w) \in \mathbb{R}^{n_2}, n_1, n_2 \in \mathbb{N} \text{ and } n_1 + n_2 = n. \tag{2}$$

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- This so-called *partial dissipation hypothesis* arises in many applications such as gas dynamics or numerical simulation of conservation laws by relaxation scheme.
- Now the question is: When does the partial dissipation prevent the formation of singularities?
- A well known example is the isentropic compressible Euler system with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \lambda \rho u = 0, \end{cases} \qquad (E)$$

For this system it was pointed out that the dissipative mechanism, albeit only present in the velocity equation, can prevent the formation of singularities (W. Wang and T. Yang '01 and T. Sideris, B. Thomases and D. Wang '03).

- In the eighties, Shizuta and Kawashima developed a rather explicit linear stability criterion: the (SK) condition. Roughly speaking, it ensures that the partial damping acts on all the components of the solution, although indirectly, so that all the solutions of (1) emanating from small perturbations of a constant state $\bar{w}$ eventually tend to $\bar{w}$.

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- Many results with theses two conditions concerning global existence of small solutions and decay of the the solution. The most recent one being the results of S. Kawashima and J. Xu ('14-'15) in the framework of inhomogeneous critical Besov spaces.
- Here we aim at generalizing their results to the homogeneous setting.
    - $\rightarrow$ More precise estimates concerning the solutions, especially the low frequencies.
    - $\rightarrow$ Useful to treat relaxation limit problems and obtain explicit convergence rate.

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$$f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ s.t. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}$$

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- The main interest of such a decomposition is the following Bernstein inequality:

$$c2^{jk}\|\dot{\Delta}_j f\|_{L^p} \leq \|D^k \dot{\Delta}_j f\|_{L^p} \leq C2^{jk}\|\dot{\Delta}_j f\|_{L^p}$$

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- Besov semi-norms: $\|f\|_{\dot{B}^s_{2,1}} \triangleq \sum_{j \in \mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^2}$.

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- A spectral analysis of the linearized system tells us that:
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- One must control ∇u in $L^1_T(L^\infty)$ to close the a priori estimates.
- $\dot{B}^{d/2}_{2,1} \hookrightarrow L^\infty$. Therefore, one might be tempted to work with $s = \frac{d}{2} - 1$ and $s' = \frac{d}{2} + 1$ i.e.

$$\rho_0^\ell, u_0^\ell \in \dot{B}^{\frac{d}{2}-1}_{2,1} \quad \text{and} \quad \rho_0^h, u_0^h \in \dot{B}^{\frac{d}{2}+1}_{2,1}.$$

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- However, in this framework one cannot treat the case $d = 1$.
- Therefore we will work with $s = \frac{d}{2}$ and $s' = \frac{d}{2} + 1$ and recover the necessary regularity for u. It turns out that in this framework the dependencies with respect to the damping parameter will be more suitable to treat relaxation problems.

A basic explanation

Let us take a quick look at the one-dimensional linearized compressible Euler equations:

$$\begin{cases}\partial_t a + \partial_x u = 0\\ \partial_t u + \partial_x a + u = 0,\end{cases} \quad \overset{\longrightarrow}{\dot\Delta_j} \quad \begin{cases}\partial_t \dot\Delta_j a + \partial_x \dot\Delta_j u = 0\\ \partial_t \dot\Delta_j u + \partial_x \dot\Delta_j a + \dot\Delta_j u = 0,\end{cases}$$

Energy estimates gives

$$\frac{1}{2}\frac{d}{dt}\|(\dot\Delta_j a, \dot\Delta_j u)\|_{L^2}^2 + \|\dot\Delta_j u\|_{L^2}^2 = 0 \to \quad \text{lack of coercivity}$$

To compensate this, we will differentiate in time the following quantity

$$\frac{d}{dt}\left(\int_{\mathbb{R}} \dot\Delta_j u \cdot \partial_x \dot\Delta_j a\right) + \|\partial_x \dot\Delta_j a\|_{L^2}^2 - \|\partial_x \dot\Delta_j u\|_{L^2}^2 + \int_{\mathbb{R}} u_j \cdot \partial_x \dot\Delta_j a = 0.$$

Denoting $\mathcal{L}_j^2 = \|(\dot\Delta_j a, \dot\Delta_j u)\|_{L^2}^2 + \varepsilon \min(1, 2^{-2j}) \int_{\mathbb{R}} \dot\Delta_j u \cdot \partial_x \dot\Delta_j a$ where $\varepsilon > 0$ can be as small as necessary, we have

$$\frac{d}{dt}\mathcal{L}_j^2 + \min(1, 2^{2j})\|(\dot\Delta_j a, \dot\Delta_j u)\|_{L^2}^2 \leq 0 \quad \text{as} \quad \|\partial_x \dot\Delta_j a\|_{L^2}^2 \sim 2^{2j}\|\dot\Delta_j a\|_{L^2}^2.$$

Choosing ε small enough such that $\mathcal{L}_j^2 \sim \|(\dot{\Delta}_j a, \dot{\Delta}_j u)\|_{L^2}^2$ and using a Gronwall-type lemma, one obtain

$$\|(\dot{\Delta}_j a, \dot{\Delta}_j u)\|_{L^2} + \min(1, 2^{2j}) \int_0^T \|(\dot{\Delta}_j a, \dot{\Delta}_j u)\|_{L^2} \leq \|(\dot{\Delta}_j a_0, \dot{\Delta}_j u_0)\|_{L^2}.$$

Then, setting $J_0 = 0$, multiplying by $2^{j\frac{d}{2}}$ and summing on $j \leq 0$.

$$\|(a,u)\|^{\ell}_{L^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(a,u)\|^{\ell}_{L^1_T(\dot{B}^{\frac{d}{2}+2}_{2,1})} \leq \|(a_0,u_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}},$$

And for the high frequencies

$$\|(a,u)\|^{h}_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(a,u)\|^{h}_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \leq \|(a_0,u_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$$

- To control the non-linear term of the general system with the regularity indexes we chose, we must control $\|\nabla u\|_{L^1_T(L^\infty)}$:

$$\|\nabla u\|_{L^1_T(L^\infty)} \leq C\|u\|_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \leq C\|u\|^{h}_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} + C\|u\|^{\ell}_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \quad \text{not under control.}$$

Indeed if $J_0 \leq 0$ then, Bernstein inequality implies: $\|f\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}} \leq C\|f\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$.

Low frequencies regularity enhancement for the velocity

- A more precise spectral analysis in the low frequencies regime shows that only one of the eigenvalues has a parabolic behaviour while all the other ones are damped.

  Looking at the equation of $\partial_x u$ alone we have

$$\partial_t \partial_x u + \partial_x u = -\partial_{xx}^2 a.$$

  The classical procedure leads to

$$\|u\|^{\ell}_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|u\|^{\ell}_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \leq \|u_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|a\|^{\ell}_{L^1_T(\dot{B}^{\frac{d}{2}+2}_{2,1})}$$

  Therefore, multiplying this estimate by a constant small enough and adding it to the previous one, we obtain

$$\|(a,u)\|^{\ell}_{L^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} + \|a\|^{\ell}_{L^1_T(\dot{B}^{\frac{d}{2}+2}_{2,1})} + \|u\|^{\ell}_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \leq C\|(a_0,u_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}}$$

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- We now have all the necessary ingredients to consider the quasi-linear system in the L^2 setting.

Back to Euler system

- We consider (E) supplemented with initial data (ρ_0, v_0) that is a perturbation of some constant state $(\bar{\rho}, \bar{v}) = (1, 0)$, $\lambda > 0$ and P a (smooth) pressure law satisfying

$$P'(\rho) > 0 \quad \text{for} \quad \rho \quad \text{close to} \quad 1 \quad \text{and} \quad P'(1) = 1. \tag{P}$$

Considering the unknown $n(\rho) = \displaystyle\int_1^\rho \frac{P'(s)}{s}\, ds$, we can rewrite (E) under the form

$$\begin{cases} \partial_t n + v \cdot \nabla n + \operatorname{div} v + G(n) \operatorname{div} v = 0, \\ \partial_t v + v \cdot \nabla v + \nabla n + \lambda v = 0, \end{cases} \tag{3}$$

where $G(n)$ is defined by the relation $G(n(\rho)) = P'(\rho) - 1$.

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where $G(n)$ is defined by the relation $G(n(\rho)) = P'(\rho) - 1$.

- The rescaling $(n, v)(t, x) \triangleq (\widetilde{n}, \widetilde{v})(\lambda t, \lambda x)$ reduces the proof to $\lambda = 1$ and the inverse scaling will give the desired dependency with respect to λ (using the homogeneity property of the norms considered).

Lyapunov Functional

Inspired by the work of R. Danchin on the compressible Navier-Stokes system, we consider the following functionals:

$$\mathcal{L}_j^2 = \int_{\mathbb{R}^d} |(\dot{\Delta}_j n, \dot{\Delta}_j v)|^2 + \varepsilon \int_{\mathbb{R}^d} \dot{\Delta}_j v \cdot \nabla \dot{\Delta}_j n \quad \text{if } j < 0$$

$$\mathcal{L}_j^2 = \int_{\mathbb{R}^d} (|\dot{\Delta}_j n|^2 + (1 + G(n))|\dot{\Delta}_j v|^2) + \varepsilon 2^{-2j} \int_{\mathbb{R}^d} \dot{\Delta}_j v \cdot \nabla \dot{\Delta}_j n \quad \text{if } j \geq 0$$

where $\varepsilon > 0$ is a fixed constant such that for $j \in \mathbb{Z}$

$$|\mathcal{L}_j(t)|^2 \sim \|(\dot{\Delta}_j n, \dot{\Delta}_j v)\|_{L^2}^2. \tag{4}$$

Simple computations leads to (with some simplifications)

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + c2^{2j}\mathcal{L}_j^2 \leq Cc_j 2^{-j\frac{d}{2}} \|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|(n,v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathcal{L}_j \quad \text{for } j < 0.$$

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + c\mathcal{L}_j^2 \leq Cc_j 2^{-j(\frac{d}{2}+1)} \|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|(n,v)\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathcal{L}_j + \int_0^t |\dot{\Delta}_j v|^2 \partial_t G(n) \quad \text{for } j \geq 0.$$

Closure of the estimates : a first attempt

Concerning $\partial_t G(n)$, we have

$$\|\partial_t G(n)\|_{L^\infty} \lesssim \|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}\|\nabla n\|_{L^\infty}. \tag{5}$$

Introducing the following Lyapunov functional:

$$\mathcal{L} = \sum_{j\leq 0} 2^{j\frac{d}{2}}\sqrt{\mathcal{L}_j^2} + \sum_{j>s0} 2^{j(\frac{d}{2}+1)}\sqrt{\mathcal{L}_j^2} \sim \|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(n,v)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$$

eventually leads to

$$\mathcal{L}(t) + \int_0^t \|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}} + \int_0^t \|(n,v)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{L}(0) + C\int_0^t \|v\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathcal{L}. \tag{6}$$

To close the estimates we need to recover the control of v^ℓ in $L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})$.
We are going to look at the equation on v as a damped transport equation:

$$\partial_t v + v\cdot\nabla v + v = \nabla n.$$

Standard computations leads to

$$\|v(t)\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|v\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \|v_0\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|a\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}} + \int_0^t \|v\|^{2}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}.$$

Multiplying this equation by a small enough constant, we can absorb the linear term $\int_0^t \|a\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}}$ by the left hand side of (6). Finally we have

$$\mathcal{L}(t) + \int_0^t \|n\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}} + \int_0^t \|v\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|(n,v)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{L}(0) + C\int_0^t \|v\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathcal{L}.$$

Setting

$$X_2(t) \triangleq \|(n,v)\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(n,v)\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|n\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} + \|(n,v)\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}$$

we easily reach

$$X_2(t) \leq X_0 + X_2^2(t)$$

and we can conclude from a standard bootstrap argument the existence of global small solution. Scaling back, we obtain the following theorem.

L^2 main result

Theorem (T. C-B. and R. Danchin '21)

Let $d \geq 1$. There exists $c_0 = c_0(p) > 0$ such that, if we set the threshold between low and high frequencies at $J_0 \triangleq \lfloor \log_2 \lambda \rfloor$, then, whenever the initial data (n_0, v_0) satisfies

$$\|(n_0, v_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \lambda^{-1} \|(n_0, v_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

System (19)-(P) admits a unique global solution (n, v) satisfying

$$\|(n, v)\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \lambda^{-1} \|(n, v)\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \lambda^{-1} \|n\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} + \|(n, v)\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}$$
$$+ \|v\|_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \lambda^{1/2} \|v\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} \lesssim \|(n_0, v_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \lambda^{-1} \|(n_0, v_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}.$$

Moreover there exists a Lyapunov functional such that

$$\mathcal{L} \sim \|(n, v)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \lambda^{-1} \|(n, v)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$$

L^p approach

- In low frequencies, the matrix of the system corresponding to frequency ξ has two real eigenvalues and in high frequencies, two complex conjugated eigenvalues.

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- An opposite dichotomy was used to treat the compressible Navier-Stokes system.

- Main idea: introduce an "effective velocity" in the spirit of D. Hoff '06 and B. Haspot '11 that may be seen as an approximate dissipative eigenmode of the system in the low frequency regime.

Low frequencies analysis: Damped mode in the simplest case

Let us consider again the one-dimensional linearized compressible Euler equations:

$$\begin{cases} \partial_t a + \partial_x u = 0 \\ \partial_t u + \partial_x a + u = 0, \end{cases}$$

Define $z = u + \partial_x a$, we can rewrite the system in the following way

$$\begin{cases} \partial_t a - \partial_{xx}^2 a = -\partial_x z \\ \partial_t z + z = -\partial_{xx}^2 u. \end{cases}$$

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- The parabolic effect for the density directly appears in the equations of a.
- In low frequencies, the linear terms of the right-hand side will be negligible.
- We will look at the first equation as a heat equation with a convection term, and at the second one as a damped transport equation.

Low frequency analysis: Damped mode for the quasi-linear system

We consider the damped mode $z = v + \nabla n + v \cdot \nabla v$, the couple (n, z) satisfies

$$\begin{cases} \partial_t n - \Delta n = -v \cdot \nabla n - \operatorname{div} z - G(n)\operatorname{div} v - \operatorname{div}(V \cdot \nabla V) \\ \partial_t z + z = -\nabla \operatorname{div}\, v - \partial_t(v \cdot \nabla v) + -\nabla(v \cdot \nabla n) + \nabla(G(n)\operatorname{div} v) \end{cases} \tag{7}$$

Simple computations lead to (omitting some terms for simplicity):

$$\|n(t)\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} + c_p \int_0^t \|n\|^{\ell}_{\dot B^{\frac{d}{p}+2}_{p,1}} \lesssim \|n_0\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} + \int_0^t \|z\|^{\ell}_{\dot B^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|v\|_{\dot B^{\frac{d}{p}+1}_{p,1}} \|n\|_{\dot B^{\frac{d}{p}}_{p,1}}$$

$$\|z(t)\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} + \int_0^t \|z\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} \le \|z_0\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} + \int_0^t \|v\|^{\ell}_{\dot B^{\frac{d}{p}+2}_{p,1}} + X_p(T)^2.$$

This time we cannot absorb the linear right-hand side terms as previously. We need to fix a threshold between the low and high frequencies J_0 small enough. Indeed, owing to Bernstein inequality, there exists an absolute constant C such that for any couple $(\sigma, \sigma') \in \mathbb{R}^2$ with $\sigma \le \sigma'$, we have

$$\|f\|^{\ell}_{\dot B^{\sigma'}_{p,1}} \le C 2^{J_0(\sigma' - \sigma)} \|f\|^{\ell}_{\dot B^{\sigma}_{p,1}}. \tag{8}$$

Low frequencies analysis: Recovering regularity for v

Since $v = z - \nabla n - v \cdot \nabla v$, we have

$$\|v\|^{\ell}_{L^1_T(\dot B^{\frac{d}{p}+1}_{p,1})} \leq \|z\|^{\ell}_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})} + \|\nabla n\|^{\ell}_{L^1_T(\dot B^{\frac{d}{p}+1}_{p,1})} + C\|\nabla v\|_{L^1_T(\dot B^{\frac{d}{p}+1}_{p,1})}\|v\|_{L^\infty_T(\dot B^{\frac{d}{p}}_{p,1})},$$

$$\|v\|^{\ell}_{L^2_T(\dot B^{\frac{d}{p}}_{p,1})} \leq \|z\|^{\ell}_{L^2_T(\dot B^{\frac{d}{p}}_{p,1})} + \|\nabla n\|^{\ell}_{L^2_T(\dot B^{\frac{d}{p}}_{p,1})} + C\|\nabla v\|_{L^2_T(\dot B^{\frac{d}{p}}_{p,1})}\|v\|_{L^\infty_T(\dot B^{\frac{d}{p}}_{p,1})}.$$

Setting

$$X_p(t) = \|(n,v)\|^{\ell}_{L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} + \|(n,v)\|^{h}_{L^\infty_t(\dot B^{\frac{d}{2}+1}_{2,1})} + \|n\|^{\ell}_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})} + \|v\|^{\ell}_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}$$
$$+ \|(n,v)\|^{h}_{L^1_t(\dot B^{\frac{d}{2}+1}_{2,1})} + \|z\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})} + \|v\|_{L^2_t(\dot B^{\frac{d}{p}}_{p,1})}$$

Using the embedding $\dot B^{\frac{d}{2}+\alpha}_{2,1} \hookrightarrow \dot B^{\frac{d}{p}+\alpha}_{p,1}$, we conclude that

$$\|(n,v)(t)\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} + \int_0^t \|n\|^{\ell}_{\dot B^{\frac{d}{p}+2}_{p,1}} + \int_0^t \|v\|^{\ell}_{\dot B^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|z\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} \lesssim \|(n_0,v_0)\|^{\ell}_{\dot B^{\frac{d}{p}}_{p,1}} + X_p(T)^2.$$

High frequencies analysis: Issues

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- Although the functional framework for high frequencies is the same as in the first part of this talk, one cannot repeat exactly the same computations.
- The non-linear terms contain a little amount of low frequencies of n and v. But they only belong to spaces of the type $\dot{B}^s_{p,1}$ for some $p > 2$ (and thus not in some $\dot{B}^{s'}_{2,1}$).
- To overcome the difficulty, we have to study more carefully the commutators in our 'hybrid' functional framework.
  The usual commutator estimate would give

$$2^{js} \left\| [v, \dot{\Delta}_j] \nabla v \right\|_{L^2} \leq C c_j \|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|v\|_{\dot{B}^s_{2,1}} \quad \text{with} \quad \sum_{j \in \mathbb{Z}} c_j = 1$$

$$\|v\|_{\dot{B}^s_{2,1}} \leq \|v\|^h_{\dot{B}^s_{2,1}} + \|v\|^\ell_{\dot{B}^s_{2,1}} \quad \text{not under control for any s}$$

A more general commutator estimate

Commutator estimate

Let $p \in [2, \min(4, 2d/(d-2)]$ and define p^* by the relation $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}\cdot$ For all $j \in \mathbb{Z}$, denote $\mathcal{R}_j \triangleq \dot{S}_{j-1} w \, \nabla \dot{\Delta}_j f - \dot{\Delta}_j (w \, \nabla f)$.
There exists a constant C depending only on the threshold number J_0 between low and high frequencies and on s, such that

$$\sum_{j \geq J_0} \left(2^{j(\frac{d}{2}+1)} \|\mathcal{R}_j\|_{L^2} \right) \leq C \Big(\|\nabla w\|_{L^\infty} \|f\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|\nabla f\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|w\|^\ell_{\dot{B}^{2+\frac{d}{p^*}}_{p^*,1}} + \|\nabla f\|_{L^\infty} \|w\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|\nabla f\|^\ell_{\dot{B}^{d-\frac{d}{p}}_{p,1}} \|\nabla w\|^\ell_{\dot{B}^{-\frac{d}{p^*}}_{p^*,1}} \Big)\cdot$$

Moreover, if $2 \leq p \leq 2d/d-1$, we have

$$\sum_{j \geq J_0} \left(2^{j(\frac{d}{2}+1)} \|\mathcal{R}_j\|_{L^2} \right) \leq C \Big(\|\nabla w\|_{L^\infty} \|f\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|\nabla f\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|\nabla w\|^\ell_{\dot{B}^{\frac{d}{p}+1}_{p,1}} + \|\nabla f\|_{L^\infty} \|\nabla w\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|\nabla f\|^\ell_{\dot{B}^{\frac{d}{p}+1}_{p,1}} \|\nabla w\|^\ell_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \Big)\cdot$$

High frequencies analysis: a priori estimates

With this lemma in hand, one just need to differentiate in time the same functional as previously:

$$\mathcal{L}_j^2 = \int_{\mathbb{R}^d} (|\dot{\Delta}_j n|^2 + (1+G(n))|\dot{\Delta}_j v|^2) + \varepsilon 2^{-2j} \int_{\mathbb{R}^d} \dot{\Delta}_j v \cdot \nabla \dot{\Delta}_j n \quad \text{if } j \geq 0$$

and use the above commutator estimate. The problematic term is

$$\int_0^t \|\nabla v\|_{L^\infty} \|G(n)\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \|v\|_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \|G(n)\|^h_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})}.$$

Indeed, the usual composition lemma ($G(0)=0$) yields

$$\begin{aligned}
\|G(n)\|^h_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} &\leq C\|n\|_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} && \text{we lost the frequency regime restriction} \\
&\leq C\|n\|^h_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} + C\|n\|^\ell_{L^\infty_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} && \text{not under control}
\end{aligned}$$

Two solutions:

- In the case of a gamma law $P(\rho) = \rho^\gamma/\gamma$ with $\gamma > 0$, $G(n) = (\gamma - 1)n$.
- In the case of a general pressure, one must work out a composition lemma *preserving* the frequency restriction with respect to the Lebesgue index. Result in progress following an idea from Chen-Miao-Zheng '11.

Gathering everything, we end up with

$$X_p(t) \leq C\left(X_{p,0} + X_p^2(t)\right)$$

and again we conclude with a bootstrap argument.
Scaling back we proved the following theorem:

Theorem (T. C-B. and R. Danchin '21)

Let $d \geq 1$ and $2 \leq p \leq \min\left(4, \frac{2d}{d-1}\right)$. There exist $k = k(p) \in \mathbb{Z}$ and $c_0 = c_0(p) > 0$ such that, if we set the threshold between low and high frequencies at $J_0 \triangleq \lfloor \log_2 \lambda \rfloor + k$, then, whenever

$$\|(n_0, v_0)\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \lambda^{-1} \|(n_0, v_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

System (E)-(P) admits a unique global solution (n, v) satisfying

$$\|(n,v)\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \lambda^{-1} \|(n,v)\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \lambda^{-1} \|n\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|(n,v)\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}$$
$$+ \|\lambda v + \nabla n\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \lambda^{1/2} \|v\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|(n_0, v_0)\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \lambda^{-1} \|(n_0, v_0)\|^{h,}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}.$$

Decay estimates in the L^2 framework

- Decay is essentially dicted by the linear analysis of the low frequencies. To derive a decay rate one must assume a stronger assumption on the low frequencies: $(n_0, u_0) \in \dot{B}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$.
- In the case $\sigma_1 = d/2$ this is reminiscent of the usual L^1 assumption as $L^1 \hookrightarrow \dot{B}^{-\frac{d}{2}}_{2,\infty}$.
- Here we aim at obtaining decay estimates without any additional smallness condition. This will be done following an idea developed by Z. Xin and J. Xu '21 concerning the compressible Navier-Stokes system.

- Step 1: Preservation of the extra negative regularity for low frequencies through the time evolution: $\|(n,u)(t)\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \leq \|(n_0,u_0)\|_{\dot{B}^{-\sigma_1}_{2,\infty}}$. Essentially this can be proved using a classical procedure and handling the non-linear terms in Besov space with a negative regularity index.
- Step 2: Lyapunov functional: With the previous computations we have

$$\mathcal{L}(t) + c' \int_0^t \mathcal{H} \leq \mathcal{L}(0).$$

Clearly, one can start the proof from any time $t_0 \geq 0$ and get in a similar way:

$$\mathcal{L}(t_0+h) + c' \int_{t_0}^{t_0+h} \mathcal{H} \leq \mathcal{L}(t_0), \quad h \geq 0.$$

This ensures that $\mathcal{L}$ is non-increasing on $\mathbb{R}^+$ (hence differentiable almost everywhere) and that for all $t_0 \geq 0$ and $h > 0$,

$$\frac{\mathcal{L}(t_0+h) - \mathcal{L}(t_0)}{h} + c' \frac{1}{h} \int_{t_0}^{t_0+h} \mathcal{H} \leq 0.$$

Consequently, passing to the limit $h \to 0$ gives

$$\frac{d}{dt}\mathcal{L} + c'\mathcal{H} \leq 0 \quad \text{a. e. on} \quad \mathbb{R}^+ \tag{9}$$

with

$$\mathcal{L} \sim \|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(n,v)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \text{ and } \mathcal{H} \sim \|v\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|n\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}} + \|n\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$$

- Step 3: lower bound for $\mathcal{H}$ by interpolation arguments

$$\|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} \lesssim \left(\|(n,v)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}}\right)^{\theta}\left(\|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}}\right)^{1-\theta} \quad \text{with} \quad \theta = \frac{2}{\frac{d}{2}+2+\sigma_1}$$

Therefore

$$\|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}+2}_{2,1}} \geq c\, C_0^{-\frac{\theta}{1-\theta}} \left(\|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)^{\frac{1}{1-\theta}}$$

and

$$\|(n,v)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \geq c\, C_0^{-\frac{\theta}{1-\theta}} \left(\|(n,v)\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\right)^{\frac{1}{1-\theta}}.$$

- Step 4: differential inequality

This leads to

$$\frac{d}{dt}\mathcal{L} + c_0\mathcal{L}^{\frac{1}{1-\theta}} \leq 0$$

where c_0 only depends on the initial data. Then

$$\mathcal{L} \leq (1 + c_0' t)^{1-\frac{1}{\theta}}\mathcal{L}(0).$$

Using that $\mathcal{L} \sim \|(n,v)(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}+1}_{2,1}}$, we obtain

$$\|(n,v)(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq C(1+t)^{-\alpha_1}\|(n_0,v_0)\|_{\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}+1}_{2,1}} \quad \text{with} \quad \alpha_1 \triangleq \frac{1}{2}\Big(\sigma_1 + \frac{d}{2}\Big).$$

- Step 5: Improvement of the decay rate for the "directly-damped" component in low frequencies.
- Step 6: Improvement of the decay rate of high frequencies.

Decay Theorem

Under the hypotheses of the existence theorem and if $(n_0, v_0) \in \dot{B}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ then, there exists a constant C depending only on σ_1 such that

$$\|(n,v)(t)\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \leq C\,\|(n_0,v_0)\|_{\dot{B}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0. \tag{10}$$

Furthermore, if $\sigma_1 > 1 - d/2$ then, denoting

$$\langle t\rangle \triangleq \sqrt{1+t^2}, \quad \alpha_1 \triangleq \frac{\sigma_1 + \frac{d}{2} - 1}{2}, C_0 \triangleq \|(n_0,v_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}} + \|(n_0,v_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}},$$

we have the following decay estimates:

$$\sup_{t\geq 0} \left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}}(n,v)(t)\right\|^{\ell}_{\dot{B}^{\sigma}_{2,1}} \leq CC_0 \ \text{ if } \ -\sigma_1 < \sigma \leq d/2-1,$$

$$\sup_{t\geq 0} \left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}+\frac{1}{2}}v(t)\right\|^{\ell}_{\dot{B}^{\sigma}_{2,1}} \leq CC_0 \ \text{ if } \ -\sigma_1 < \sigma \leq d/2-2,$$

$$\text{and} \quad \sup_{t\geq 0} \left\|\langle t\rangle^{2\alpha_1}(n,v)(t)\right\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq CC_0.$$

Extensions

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- It can also be used to derive existence result for specific systems not satisfying the (SK) condition: we studied a damped Baer-Nunziato system in collaboration with C. Burtea and J. Tan.
- The damped mode is an useful tool to study relaxation limit problem and to derive explicit convergence rate. Work in progress: compressible Euler system to porous media equations, and hyperbolic-parabolic chemotaxis system to Keller-Segel system.

Presentation of the problem

Here we present briefly the results obtained in collaboration with C. Burtea and J. Tan.
We consider the following damped Baer-Nunziato system:

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \dfrac{\alpha_+ \alpha_-}{2\mu + \lambda} \left(P_+ \left(\rho_+ \right) - P_- \left(\rho_- \right) \right), \\ \partial_t \left(\alpha_\pm \rho_\pm \right) + \operatorname{div} \left(\alpha_\pm \rho_\pm u \right) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = \alpha_+ P_+ \left(\rho_+ \right) + \alpha_- P_- \left(\rho_- \right) \end{cases} \qquad (BN)$$

- Not symmetrizable:
  - The equations of the volume fractions cannot be put in conservative form $\rightarrow$ not a system of conservation law.
  - The entropy that is naturally associated to this system is not positive definite since it is linear with respect to the volume fractions.
- Doesn't satisfy the (SK) condition
  - The associated quasilinear system admits the eigenvalue 0.

Solutions

- It turns out that the situation is not too degenerate in the sense that the eigenspace associated to the eigenvalue 0 is of dimension 1 and that, roughly speaking, the non-degenerate part (i.e. the part associated to non-zero eigenvalues) fulfils the (SK) condition.
- We rewrite the (BN)-system in terms of new variables so as to highlight a subsystem for which the linearized does verify the (SK) condition which is coupled throughout lower-order terms with a transport equation.
- We construct an appropriate weighted energy-functional which allows us to tackle the lack of symmetry of the system, provides decay information and allows us to close the estimate uniformly with respect to the relaxation parameter.

Elements of proof

System (BN) can be recast into

$$\begin{cases} \partial_t y + u \cdot \nabla y = 0 \\ \partial_t w + u \cdot \nabla w + \big(h_1 + H_1\big)\operatorname{div} u + \big(h_2 + H_2\big)\dfrac{w}{\nu} = S_2, \\ \partial_t r + u \cdot \nabla r + \big(h_3 + H_3\big)\operatorname{div} u = S_3, \\ \partial_t u + u \cdot \nabla u + \eta u + \big(h_5 + H_5\big)\nabla r + \big(h_6 + H_6\big)\nabla w = S_4 \end{cases} \tag{11}$$

To obtain an a priori estimate for the last 3 equations, we derive in time the following functionals:

$$\mathcal{L}_j^2 = \int_{\mathbb{R}^d} \Big(\frac{h_6}{h_1}|w_j|^2 + \frac{h_5}{h_3}|r_j|^2 + |u_j|^2 + 2\varepsilon u_j \cdot \nabla r_j \Big) \quad \text{for } j < 0.$$

$$\mathcal{L}_j^2 = \int_{\mathbb{R}^d} \Big(\frac{h_6 + H_6}{h_1 + H_1}|w_j|^2 + \frac{h_5 + H_5}{h_3 + H_3}|r_j|^2 + |u_j|^2 + 2\varepsilon 2^{-2j} u_j \cdot \nabla r_j \Big) \quad \text{for } j \geq 0.$$

Elements of proof

Then, for $-\frac{d}{2} < s_1 \leq \frac{d}{2} - 1$ and $s_1 + 1 \leq s_2 \leq \frac{d}{2} + 1$, the following estimate holds:

$$\|(w,r,u)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{s_1}_{2,1})} + \|(w,r,u)\|^{h}_{\widetilde{L}^{\infty}_t(\dot{B}^{s_2}_{2,1})} + \kappa\left(\|(w,r,u)\|^{\ell}_{L^1_t(\dot{B}^{s_1+2}_{2,1})} + \|(w,r,u)\|^{h}_{L^1_t(\dot{B}^{s_2}_{2,1})}\right)$$
$$+\int_0^t \|(\frac{w}{\nu},\eta u,\partial_t w,\partial_t r,\partial_t u)\|^{\ell}_{\dot{B}^{s_1+1}_{2,1}} + \int_0^t \|(\frac{w}{\nu},\eta u,\partial_t w,\partial_t r,\partial_t u)\|^{h}_{\dot{B}^{s_2-1}_{2,1}}$$
$$\leq \exp\Big(C\big(H(t)+V(t)\big)\Big)\left(\|(w_0,r_0,u_0)\|_{\dot{B}^{s_1}_{2,1}\cap\dot{B}^{s_2}_{2,1}} + \int_0^t \|(S_2,S_3,S_4)\|_{\dot{B}^{s_1}_{2,1}\cap\dot{B}^{s_2}_{2,1}}\right)$$

where $V(t) := \int_0^t \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}+1}_{2,1}}$, $H(t) := \sum_{i=1}^{6} \|\partial_t H_i(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}$ and $\kappa := \kappa(\varepsilon)$.

- Strength of this method: we can treat the singular relaxation limit problem as $\nu \to 0$.
- We are able to show rigorously that the so-called Kapilla system is obtained as a relaxation limit from the (BN) system and derive the convergence rate of this process.

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_t\left(\alpha_\pm \rho_\pm\right) + \operatorname{div}\left(\alpha_\pm \rho_\pm u\right) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = P_+\left(\rho_+\right) = P_-\left(\rho_-\right). \end{cases} \tag{K}$$

Denoting with an index ν the solution of (BN), we are able to prove that

$$\|(\alpha_\pm^\nu - \alpha_\pm, \rho_\pm^\nu - \rho_\pm, u^\nu - u)\|_{L^\infty(\dot{B}_{2,1}^{\frac{d}{2}-\frac{1}{2}})}$$
$$+ \|\rho_\pm^\nu - \rho_\pm\|_{L^2(\dot{B}_{2,1}^{\frac{d}{2}-\frac{1}{2}})} + \|u^\nu - u\|_{L^1(\dot{B}_{2,1}^{\frac{d}{2}-\frac{1}{2}})} \leq C\sqrt{\nu}.$$

if the initial data of both systems are close enough with respect to $\sqrt{\nu}$.

- By interpolation one can recover a convergence rate of $\nu^{\frac{1}{3}}$ for the above quantities in $\dot{B}_{2,1}^{\frac{d}{2}}$.

Thank you for your attention!