GLOBAL CONVERGENCE RATES IN THE RELAXATION LIMIT FOR THE COMPRESSIBLE EULER AND EULER-MAXWELL SYSTEMS IN SOBOLEV SPACES

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Abstract. We study two relaxation problems in the class of partially dissipative hyperbolic systems: the compressible Euler system with damping and the compressible Euler-Maxwell system. Within a Sobolev framework, we derive a global convergence rate between the strong solutions of the relaxed Euler system and the porous media equation in $\mathbb{R}^d$ ($d \geq 1$) for ill-prepared initial data. In a well-prepared setting, we derive a rate of $\mathcal{O}(\varepsilon^2)$ between the solutions of the compressible Euler system and its first-order asymptotic approximation. Concerning the Euler-Maxwell system, we prove the global strong convergence of its solution to the drift-diffusion model in $\mathbb{R}^3$. These results are achieved by developing an asymptotic expansion approach that, together with stream functions, ensures global-in-time error estimates.

1. Introduction

Relaxation phenomena occur in a wide variety of physical situations, such as modeling blood flow with friction forces, non-equilibrium gas dynamics, kinetic theory, traffic flows, and more (see [3, 44, 47]). They emerge whenever a "stable" equilibrium state of a physical system is perturbed. In this case, systems are described by a set of equations where the source terms contain a large coefficient ε^{-1} with $\varepsilon > 0$ representing a time-relaxation parameter that is related to physical quantities. As ε is small, the solution exhibits a rapid relaxation towards equilibrium.

In this paper, we consider two classical hyperbolic systems with relaxation in the whole space: the damped compressible Euler system and the damped compressible Euler-Maxwell system. In both cases, global-in-time asymptotic stability as the time-relaxation parameter ε goes to zero is established in Sobolev space.

1.1. The compressible Euler system with damping.

First, we consider the Euler equations with relaxation in several space variables:

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho v) = 0,\\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = -\dfrac{\rho v}{\varepsilon},\end{cases} \tag{1.1}$$

with the initial condition

$$(\rho, v)(0, x) = \big(\rho_0^\varepsilon, v_0^\varepsilon\big)(x), \tag{1.2}$$

for the time $t > 0$ and the position $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$. Here, $\varepsilon \in (0, 1]$ is the relaxation time, $\rho = \rho(t, x) > 0$, $u = (u^1, u^2, \cdots, u^d)(t, x)$ and $p = p(t, x)$ are the density, the velocity and the pressure function, respectively. We are interested in strong solutions that are small perturbations of the constant equilibrium

$$(\bar\rho, \bar u) = (\bar\rho, 0)$$

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where $\bar\rho$ is a given positive constant. Additionally, we assume that the pressure is a function of the density ρ and

$$p \in C^\infty_{\rm loc}(\mathbb{R}^+), \quad p'(\bar\rho) > 0. \tag{1.3}$$

Without loss of generality, we assume that $\bar\rho = 1$ in the following. Under the diffusion scaling

$$(\rho^\varepsilon, u^\varepsilon)(t,x) := \left(\rho, \frac{1}{\varepsilon}v\right)\left(\frac{t}{\varepsilon}, x\right),$$

the system (1.1) and the initial condition (1.2) become

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \varepsilon^2 \partial_t(\rho^\varepsilon u^\varepsilon) + \varepsilon^2 \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) = -\rho^\varepsilon u^\varepsilon, \end{cases} \tag{1.4}$$

and

$$(\rho^\varepsilon, u^\varepsilon)(0,x) = (\rho_0^\varepsilon, u_0^\varepsilon)(x) \quad \text{with} \quad u_0^\varepsilon := \frac{1}{\varepsilon} v_0^\varepsilon. \tag{1.5}$$

Formally, let us denote

$$(\rho^*, u^*) := \lim_{\varepsilon \to 0} (\rho^\varepsilon, u^\varepsilon).$$

As $\varepsilon \to 0$, we expect that

$$\begin{cases} \partial_t \rho^* + \operatorname{div}(\rho^* u^*) = 0, \\ \nabla p(\rho^*) = -\rho^* u^* \quad \text{(Darcy's law)}, \end{cases} \tag{1.6}$$

which yields a filtration equation for the density

$$\partial_t \rho^* - \Delta p(\rho^*) = 0, \tag{1.7}$$

subject to the initial condition

$$\rho^*(0,x) = \rho_0^*(x). \tag{1.8}$$

In particular, for the γ-law $p(\rho) = a^2\rho^\gamma$ with $a > 0$ and $\gamma > 1$, we recover the porous medium equation.

1.2. **The compressible Euler-Maxwell system.** We also consider the three-dimensional compressible Euler-Maxwell system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = -\rho\big(E + v \times B\big) - \dfrac{\rho v}{\varepsilon}, \\ \partial_t E - \nabla \times B = \rho^\varepsilon v, \\ \partial_t B + \nabla \times E = 0, \\ \operatorname{div} E = 1 - \rho, \qquad \operatorname{div} B = 0, \end{cases} \tag{1.9}$$

where the unknowns $\rho = \rho(t,x) > 0$, $u = (u^1, u^2, u^3)(t,x)$, $E = (E^1, E^2, E^3)(t,x)$ and $B = (B^1, B^2, B^3)(t,x)$ stand for the density, the velocity, the electric field and the magnetic field, respectively, at the time $t > 0$ and the position $x = (x_1, x_2, x_3)$. The term $\rho(E + u \times B)$ corresponds to the Lorentz force, and $p = p(\rho)$ denotes the pressure function that satisfies (1.3). We supplement system (1.9) with the initial conditions

$$(\rho, v, E, B)(0,x) = (\rho_0^\varepsilon, u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x), \tag{1.10}$$

and we assume that

$$\operatorname{div} E_0^\varepsilon = 1 - \rho_0^\varepsilon, \qquad \operatorname{div} B_0^\varepsilon = 0. \tag{1.11}$$

Note that under (1.11), the constraint relaxations in $(1.9)_5$ hold true for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$.

We are interested in strong solutions of (1.9) that are small perturbations of the constant equilibrium

$$(\bar\rho, \bar u, \bar E, \bar B) = (1, 0, 0, B^e)$$

where $B^e \in \mathbb{R}^3$ is a constant vector. For these solutions, we will investigate their diffusion limit corresponding to the scaling

$$\big(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon\big)(t,x) := \left(\rho, \frac{1}{\varepsilon}v, E, B\right)\left(\frac{t}{\varepsilon}, x\right).$$

The diffusively rescaled Euler-Maxwell system reads:

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \varepsilon^2\partial_t(\rho^\varepsilon u^\varepsilon) + \varepsilon^2 \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) = -\rho^\varepsilon\big(E^\varepsilon + \varepsilon u^\varepsilon \times B^\varepsilon\big) - \rho^\varepsilon u^\varepsilon, \\ \varepsilon\partial_t E^\varepsilon - \nabla\times B^\varepsilon = \varepsilon\rho^\varepsilon u^\varepsilon, \\ \varepsilon\partial_t B^\varepsilon + \nabla\times E^\varepsilon = 0, \\ \operatorname{div} E^\varepsilon = 1-\rho^\varepsilon, \qquad \operatorname{div} B^\varepsilon = 0, \end{cases} \tag{1.12}$$

supplemented with the initial data

$$\big(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon\big)(0,x) = (\rho_0^\varepsilon, u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x) \quad \text{with} \quad u_0^\varepsilon := \frac{1}{\varepsilon}v_0^\varepsilon. \tag{1.13}$$

Setting

$$\big(\rho^*, u^*, E^*, B^*\big) := \lim_{\varepsilon\to 0}\big(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon\big),$$

formally, the dynamics of (1.12) as $\varepsilon \to 0$ is expected to be governed by

$$\begin{cases} \partial_t\rho^* + \operatorname{div}(\rho^* u^*) = 0, \\ \rho^* u^* = -\nabla p(\rho^*) - \rho^* E^*, \\ \nabla\times B^* = 0, \\ \nabla\times E^* = 0, \\ \operatorname{div} E^* = 1-\rho^*, \qquad \operatorname{div} B^* = 0. \end{cases} \tag{1.14}$$

The divergence-free and curl-free conditions imply that B^* is a constant vector. Moreover, setting $\Lambda^{-2} = (-\Delta)^{-1}$, there exists a potential function $\phi^* = \Lambda^{-2}(\rho^*-1)$ such that

$$E^* = \nabla\phi^* = \Lambda^{-2}\nabla\rho^*.$$

It follows that (ρ^*, ϕ^*) solves the drift-diffusion system

$$\begin{cases} \partial_t\rho^* - \operatorname{div}\big(\nabla p(\rho^*) + \rho^*\nabla\phi^*\big) = 0, \\ \Delta\phi^* = 1-\rho^*, \end{cases} \tag{1.15}$$

subject to the initial data

$$\rho^*(0,x) = \rho_0^*(x). \tag{1.16}$$

1.3. **Literature on damped Euler equations and partially dissipative hyperbolic systems.** Before stating the paper's findings, we recall recent efforts devoted to studying partially dissipative hyperbolic systems of the type:

$$\partial_t V + \sum_{j=1}^d A^j(V)\partial_{x_j} V = \frac{H(V)}{\varepsilon}, \tag{1.17}$$

where the unknown $V = V(t,x)$ is a N-vector valued function depending on the time variable $t \in \mathbb{R}^+$ and on the space variable $x \in \mathbb{R}^d (d \geq 1)$. The $A^j(V)$ $(j = 1,..,d)$ and H are given smooth functions on a state space $\mathcal{O}_V \in \mathbb{R}^N$. Note that in the absence of source term $H(V)$, (1.17) reduces to a system of conservation laws. In that case, it is well-known that classical solutions may develop singularities (*e.g.*, shock waves) in finite time, even if initial data are sufficiently smooth and small (see Dafermos [11] and Lax [23]).

A typical example is the following isentropic compressible Euler equations with damping (1.1). For initial data being small perturbations in Sobolev spaces $H^s(\mathbb{R}^d)$ $(s \geq [\frac{d}{2}]+2)$, the global well-posedness and asymptotic behaviors of classical solutions for (1.1) have been studied in numerous works. In Sobolev spaces, Sideris, Thomases and Wang [40] and Wang and Tang [45] studied the optimal time-decay rates of solutions. In the case $\varepsilon = 1$, a natural question arises: what conditions can be imposed on $H(V)$ so it prevents the finite-time blowup of classical solutions? Chen, Levermore and Liu [5] first formulated a notion of the entropy for (1.17). Imposing a technical requirement on the entropy, Yong [53] proved the global existence of classical solutions in a neighbourhood of constant equilibrium $\bar{V} \in \mathbb{R}^N$ satisfying $H(\bar{V}) = 0$ under the Shizuta–Kawashima condition [39]. We also mention that Hanouzet and Natalini [17] obtained a similar global existence result for the one-dimensional problem before the work [53]. Subsequently, Kawashima and Yong [22] removed the technical requirement on the dissipative entropy used in [17, 53] and gave a perfect definition of the entropy notion, which leads to the global existence in regular Sobolev spaces. Then, Bianchini, Hanouzet and Natalini [2] showed that smooth solutions approach the constant equilibrium state $\bar{V}$ in the L^p-norm at the rate $O(t^{-\frac{d}{2}(1-\frac{1}{p})})$, as $t \to \infty$, for $p \in [\min\{d, 2\}, \infty]$. Recently, Beauchard and Zuazua [1] established the equivalence of the Shizuta-Kawashima condition and the Kalman rank condition from control theory. Then, extensions to larger critical spaces were obtained in non-homogeneous settings in s [49, 50] and in some hybrid homogeneous settings in [7–9].

Regarding the relaxation limit as $\varepsilon \to 0$ in systems of the type (1.17), the first justification is due to Marcati, Milani and Secchi [31] in a one-dimensional setting. The limiting procedure was carried out by using the theory of compensated compactness. Then, Liu [28] proved, using the approach based on the theory of nonlinear waves, the relaxation to parabolic equations for genuinely nonlinear hyperbolic systems. Marcati and Milani [30] considered the diffusive scaling for the one-dimensional compressible Euler flow (1.1) and derived Darcy's law in the limit $\varepsilon \to 0$, which is analogous to the one derived in [31]. Later, Marcati and Rubino [32] developed a complete hyperbolic to parabolic relaxation theory for 2×2 genuinely nonlinear hyperbolic balance laws. Junca and Rascle [20] established the relaxation convergence from the isothermal equation (1.1) to the heat equation for arbitrarily large initial data in $BV(\mathbb{R})$ that are bounded away from vacuum.

As for (1.17) in several dimensions, Coulombel, Goudon and Lin [6, 26] employed the classical energy approach and constructed uniform-in-ε smooth solutions to the isentropic Euler equations (1.1) and then they justified the weak relaxation limit in the Sobolev spaces $H^s(\mathbb{R}^d)$ $(s > 1 + d/2,\ s \in \mathbb{Z})$. Xu and Wang [52] improved their works to the setting of critical Besov space $B^{\frac{d}{2}+1}_{2,1}$. More precisely, it is shown that the density converges towards the solution of the porous medium equation, as $\varepsilon \to 0$. Peng and Wasiolek [37] proposed structural stability conditions and constructed an approximate solution using a formal asymptotic expansion with initial layer corrections. It allowed to establish the uniform local existence with respect to ε and the convergence of (1.17) to parabolic-type equations as $\varepsilon \to 0$. Subsequently, under the Shizuta–Kawashima stability condition, they [38] established the uniform global existence and the global-in-time weak convergence from (1.17) to second-order nonlinear parabolic systems by using Aubin-Lions compactness arguments. In the spirit of the stream function approach of [20], Li, Peng and Zhao [24] obtained explicit convergence rates for this relaxation process for $d = 1$. Recently, the first author and Danchin [9] justified the strong relaxation limit of diffusively rescaled solutions of (1.17) globally in time in homogeneous critical Besov spaces with the explicit convergence rate. Finally, we also mention the work of Peng on this limit for large smooth solutions of the isothermal Euler equations in Sobolev spaces [35].

1.4. **Literature on the Euler-Maxwell system.** So far there are several results concerning the global existence, large-time behaviour and asymptotic limit for the isentropic Euler-Maxwell system (1.9). Chen, Jerome and Wang [4] constructed global weak solutions to the initial boundary value problem for arbitrarily large initial data in 1D. In the multidimensional case, the question of global weak solutions is quite open and mainly smooth solutions have been studied. Jerome [19] established the local well-posedness of smooth solutions to the Cauchy problem (1.9)-(1.10) in the framework of Sobolev spaces $H^s(\mathbb{R}^d)$ with $s > \frac{5}{2}$ according to the standard theory for symmetrizable hyperbolic systems. The existence of global smooth solutions near constant equilibrium states has been obtained independently by Peng, Wang & Gu [36] and Duan [13] in Sobolev spaces and by Xu [48] in the inhomogeneous critical Besov space. Ueda, Wang and Kawashima [41] pointed out that the system (1.9) was of regularity-loss type and time-decay estimates were derived in [13, 43]. Later, Ueda, Duan and Kawashima [42] formulated a new structural condition to analyze the weak dissipative mechanism for general hyperbolic systems with non-symmetric relaxation (including the Euler-Maxwell system (1.9)). Xu, Mori and Kawashima [51] developed a general time-decay inequality with the minimal regularity assumption of initial data. Concerning the stability of steady-states, we refer to those works [27, 33, 34]. In the absence of a damping term in (1.9), there are many important works on global or large-time existence using dispersive properties; see [12, 14, 15, 18].

Concerning the relaxation from (1.9) to (1.15), Hajjej and Peng [16] carried out an asymptotic expansion and obtained convergence rates for the relaxation procedure in the case of local-in-time solutions for both well-prepared data and ill-prepared data. Wasiolek [46] obtained the uniform estimates of global solutions with small perturbations in the Sobolev spaces $H^s(\Omega)$ with $s \geq 3$ and $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, and then proved the global weak convergence of the relaxation process from (1.9) to (1.15) via the compactness argument. Recently, Li, Peng and Zhao [25] studied the relaxation limit for global smooth solutions in $\mathbb{T}^3$ and obtained error estimates of smooth periodic solutions between (1.9) and (1.15) by stream function techniques and Poincaré's inequality. Very recently, by developing a new characterization of the dissipation structure and using Fourier analysis tools, the authors of [10] provided a rigorous justification of the global-in-time strong convergence of the relaxation process in $\mathbb{R}^3$ with an explicit convergence rate.

1.5. **Aims of the paper.** The relaxation limit problems for both the compressible Euler system (1.4) and the compressible Euler-Maxwell system (1.12) in the whole space have been studied in numerous references. The strong relaxation limit was justified on the real line $\mathbb{R}$ in [20, 24] based on stream function techniques. In [9, 10], relaxation limits in higher dimensions were established in a strong sense in critical Besov spaces, using a frequency partition of the solution and the Littlewood-Paley theory. A natural question arises: Can the global convergence rates of the high-dimensional relaxation problems be obtained in the classical Sobolev energy framework?

In the present paper, we aim to avoid using a frequency-dependent methodology to broaden the applicability of our approach. Specifically, we expect our method to be suitable for cases where the physical domain is a half-space or a bounded domain, and for establishing such limits at the discrete level. In a Sobolev framework, we establish error estimates for ill-prepared initial data. In particular, we are able to prove the convergence in the natural energy space $L^2(\mathbb{R}^+; L^2)$, which cannot be directly recovered by the results in [9, 10]. Our proofs rely on a high-dimensional version of the stream function techniques and an asymptotic expansion method.

2. Main results

2.1. The Euler system. We first investigate the relaxation problem for the damped Euler equations (1.4). We define the initial energy

$$\mathcal{E}_0^\varepsilon := \|\rho_0^\varepsilon - 1\|_{H^m}^2 + \varepsilon^2 \|u_0^\varepsilon\|_{H^m}^2. \tag{2.1}$$

We also introduce the variables for the momentum

$$q^\varepsilon := \rho^\varepsilon u^\varepsilon, \qquad q^* := \rho^* u^*, \qquad q_0^\varepsilon := \rho_0^\varepsilon u_0^\varepsilon.$$

We recall the classical uniform global existence result for (1.4)-(1.5) from [6, 26], which leads to the existence of a global-in-time solution.

Proposition 2.1. ([6, 26]) *Let $d \geq 1$, $m \geq [\frac{d}{2}] + 2$ and (1.3) hold. There is a positive constant δ, independent of ε, such that, if $\mathcal{E}_0^\varepsilon \leq \delta$, then the Cauchy problem (1.4)-(1.5) admits a unique global smooth solution $(\rho^\varepsilon, u^\varepsilon)$ with $q^\varepsilon = \rho^\varepsilon u^\varepsilon$ which satisfies $(\rho^\varepsilon - 1, q^\varepsilon) \in C(\mathbb{R}^+; H^m)$ and the uniform estimate*

$$\sup_{t\in\mathbb{R}^+} \left(\|\rho^\varepsilon(t) - 1\|_{H^m}^2 + \varepsilon^2 \|u^\varepsilon(t)\|_{H^m}^2 + \varepsilon^2 \|q^\varepsilon(t)\|_{H^m}^2 \right) + \int_0^{+\infty} \left(\|\nabla \rho^\varepsilon(t)\|_{H^{m-1}}^2 + \|u^\varepsilon(t)\|_{H^m}^2 + \|q^\varepsilon(t)\|_{H^m}^2 \right) dt \leq C_1 \mathcal{E}_0^\varepsilon, \tag{2.2}$$

where $C_1 > 0$ is a generic constant.

Moreover, for any given $T > 0$, as $\varepsilon \to 0$, up to subsequences,

$$\begin{cases} \rho^\varepsilon \to \rho^* & \text{strongly in } C([0,T], H^{m-1}_{loc}), \\ q_\varepsilon \rightharpoonup q^* & \text{weakly in } L^2(\mathbb{R}^+, H^m), \end{cases} \tag{2.3}$$

where (ρ^*, u^*) with $u^* = q^*/\rho^*$ being the unique solution of (1.6)-(1.8), and ρ_0^* denotes the weak limit of $(\rho_0^\varepsilon)_{\varepsilon>0}$ in H^m.

For the filtration equation (1.7), one has the following existence result.

Proposition 2.2. *Let $d \geq 1$, $m \geq [\frac{d}{2}] + 2$ and (1.3) hold. There is a positive constant δ^* such that, if $\|\rho_0^* - 1\|_{H^m}^2 \leq \delta^*$, then the Cauchy problem (1.7)-(1.8) admits a unique global classical solution ρ^* which satisfies*

$$\sup_{t\in\mathbb{R}^+} \|\rho^*(t) - 1\|_{H^m}^2 + \int_0^{+\infty} \|\nabla \rho^*(t)\|_{H^m}^2 \, dt \leq C_2 \|\rho_0^* - 1\|_{H^m}^2, \tag{2.4}$$

where $C_2 > 0$ is a generic constant.

Furthermore, let u^* be given by Darcy's law $(1.6)_2$ and set $q^* = \rho^* u^*$. It holds that

$$\sup_{t\in\mathbb{R}^+} \|q^*(t)\|_{H^{m-1}}^2 + \int_0^{+\infty} \|q^*(t)\|_{H^m}^2 \, dt \leq C_2 \|\rho_0^* - 1\|_{H^m}^2. \tag{2.5}$$

Our first main result concerns the global-in-time strong convergence of the Euler equations to the filtration equation and Darcy's law for ill-prepared data.

Theorem 2.1. *Let $d \geq 1$, $m \geq [\frac{d}{2}] + 2$ and (1.3) hold. Let $(\rho^\varepsilon, q^\varepsilon)$ and (ρ^*, q^*) be the solutions obtained in Propositions 2.1 and 2.2 subject to the initial data $(\rho_0^\varepsilon, q_0^\varepsilon)$ and ρ_0^*, respectively. There exists a constant $C > 0$ independent of ε such that*

$$\sup_{t\in\mathbb{R}^+} \|(\rho^\varepsilon - \rho^*)(t)\|_{H^{m-1}}^2 + \int_0^{+\infty} \|\nabla(\rho^\varepsilon - \rho^*)(t)\|_{H^{m-1}}^2 \, dt \leq C(\|\rho_0^\varepsilon - \rho_0^*\|_{H^{m-1}}^2 + \varepsilon^2), \tag{2.6}$$

and

$$\int_0^{+\infty} \|(q^\varepsilon - q^* - q_I^\varepsilon)(t)\|_{H^{m-1}}^2 \, dt \leq C(\|\rho_0^\varepsilon - \rho_0^*\|_{H^{m-1}}^2 + \varepsilon^2), \tag{2.7}$$

where q_I^ε is an initial time-layer correction given by

$$q_I^\varepsilon := e^{-\frac{t}{\varepsilon^2}} q_0^\varepsilon. \tag{2.8}$$

If in addition $\rho_0^\varepsilon - \rho_0^* \in \dot{H}^{-1}$, then the convergence also holds in the following sense:

$$\int_0^\infty \|(\rho^\varepsilon - \rho^*)(t)\|_{L^2}^2 \, dt \leq C(\|\rho_0^\varepsilon - \rho_0^*\|_{\dot{H}^{-1}}^2 + \varepsilon^2). \tag{2.9}$$

Remark 2.1. *Our result holds for ill-prepared data. We say that an initial data is well-prepared if the compatibility condition $\big(\rho^\varepsilon u^\varepsilon + \nabla p(\rho^\varepsilon)\big)|_{t=0} = q_0^\varepsilon + \nabla p(\rho_0^\varepsilon) \to 0$ in H^{m-1} as $\varepsilon \to 0$ holds and ill-prepared if this condition does not hold. In Theorem 2.1, q_0^ε fulfills $\|q_0^\varepsilon\|_{H^m} = \mathcal{O}(\varepsilon^{-1})$, which is required in the existence result: Proposition 2.1.*

Remark 2.2. *The error estimate (2.7) in fact leads to the global strong convergence of $(q^\varepsilon)_{\varepsilon>0}$ to q^*. Using (2.7) and the fact that $\|q_I^\varepsilon\|_{L^1(\mathbb{R}_+;H^m)} \leq C\varepsilon(\varepsilon\|q_0^\varepsilon\|_{H^m}) \leq C\varepsilon$, it holds, as $\varepsilon \to 0$, that*

$$q^\varepsilon \to q^* \ \text{ strongly in } \ L^2(\mathbb{R}^+, H^{m-1}) + L^1(\mathbb{R}_+; H^m),$$

where $X + Y$ denotes the sum space of X and Y.

Remark 2.3. *The $L^2(\mathbb{R}_+; L^2(\mathbb{R}^d))$ error estimate (2.9) is a higher-dimensional version of the estimate that can be found in [20]. If $\rho_0^\varepsilon = \rho_0^*$, no $\dot{H}^{-1}$-type assumptions are required, and we obtain the $C\varepsilon$-bound of the error $\rho^\varepsilon - \rho^*$ in $L^2(\mathbb{R}^+; H^m)$.*

Our next aim is to provide a more precise description of the relaxation approximation in terms of one-order asymptotic expansion. More precisely, we show faster convergence rates between the solution of (1.12) and its first-order approximation. Defining the asymptotic expansion

$$\rho_a^\varepsilon := \rho^* + \varepsilon\rho_1, \quad q_a^\varepsilon := q^* + \varepsilon q_1, \tag{2.10}$$

the pair $(\rho_a^\varepsilon, q_a^\varepsilon)$ will be used to approximate the solution $(\rho^\varepsilon, q^\varepsilon)$ of (1.4) in a suitable sense. Observe that

$$p(\rho_a^\varepsilon) = p(\rho^* + \varepsilon\rho_1) = p(\rho^*) + \varepsilon p'(\rho^*)\rho_1 + O(\varepsilon^2). \tag{2.11}$$

Substituting (2.10) and (2.11) into (1.4) and identifying the coefficients in terms of the power of $\mathcal{O}(\varepsilon)$, we can obtain (ρ_1, q_1) by solving the following linear equations

$$\begin{cases} \partial_t \rho_1 + \operatorname{div} q_1 = 0, \\ q_1 = -\nabla\big(p'(\rho^*)\rho_1\big), \end{cases} \tag{2.12}$$

subject to the initial datum

$$\rho_1|_{t=0} = \rho_{1,0}. \tag{2.13}$$

From (2.12), one sees that ρ_1 solves the linear filtration equation

$$\partial_t \rho_1 - \Delta\big(p'(\rho^*)\rho_1\big) = 0. \tag{2.14}$$

Our next result proves the strong convergence between ρ^ε and its asymptotic expansion ρ_a^ε for well-prepared data and with faster rates.

Theorem 2.2. *Let $d \geq 1$, $m \geq [\frac{d}{2}] + 2$, (1.3) and the assumptions in Propositions 2.1 and 2.2 hold. Let $(\rho^\varepsilon, q^\varepsilon)$ and (ρ^*, q^*) be the solutions obtained in Propositions 2.1 and 2.2 subject to the initial data $(\rho_0^\varepsilon, q_0^\varepsilon)$ and ρ_0^*, respectively. In addition, assume $\rho_{1,0} \in H^m$, $q_0^* = \lim_{\varepsilon\to 0} q_0^\varepsilon = -\nabla p(\rho_0^*)$ and*

$$\|\rho_0^\varepsilon - \rho_0^* - \varepsilon\rho_{1,0}\|_{H^{m-2}} \leq \varepsilon^2 \quad \text{and} \quad \|q_0^\varepsilon - q_0^*\|_{H^{m-1}} \leq \varepsilon. \tag{2.15}$$

There exists a constant C independent of ε such that

$$\sup_{t\in\mathbb{R}^+}\|(\rho^\varepsilon-\rho^*-\varepsilon\rho_1)(t)\|_{H^{m-2}}^2+\int_0^{+\infty}\|\nabla(\rho^\varepsilon-\rho^*-\varepsilon\rho_1)(t)\|_{H^{m-2}}^2\,dt\le C\varepsilon^4, \tag{2.16}$$

and

$$\begin{cases}\displaystyle\int_0^{+\infty}\|(q^\varepsilon-q^*)(t)\|_{H^{m-2}}^2\,dt\le C\varepsilon^2,\\ \displaystyle\sup_{t\in\mathbb{R}^+}\|(q^\varepsilon-q^*-\varepsilon q_1)(t)\|_{H^{m-2}}^2\le C\varepsilon^2,\\ \displaystyle\int_0^{+\infty}\|(q^\varepsilon-q^*-\varepsilon q_1)(t)\|_{H^{m-2}}^2\,dt\le C\varepsilon^4.\end{cases} \tag{2.17}$$

2.2. **Strategy of proof for the Euler system.** We now provide some comments on the proofs of Theorems 2.1 and 2.2. To justify the strong convergence in the energy space $L^2(\mathbb{R}^+;L^2)$, we adapt the stream function technique—originally developed in one-dimensional settings (see [20,24])—to the multi-dimensional framework. More precisely, we introduce the *stream function* associated with the equation of $\rho^\varepsilon-\rho^*$:

$$N^\varepsilon(t,x):=-\int_0^t(q^\varepsilon-q^*)(t',x)\,dt'+\Lambda^{-2}\nabla(\rho_0^\varepsilon-\rho_0^*)(x),$$

which satisfies

$$\partial_t N^\varepsilon=-(q^\varepsilon-q^*),\qquad \operatorname{div}N^\varepsilon=\tilde\rho^\varepsilon.$$

The key idea is to take the L^2 inner product of the equation satisfied by N^ε. Under the additional assumption that $\rho_0^\varepsilon-\rho_0^*\in\dot H^{-1}$, this yields an $L^2(\mathbb{R}^+;L^2)$ estimate for $\rho^\varepsilon-\rho^*$ (see Lemma 4.2).

We further derive higher-order error estimates without having to rely on a $\dot H^{-1}$ assumption on the initial data. To this end, we perform hyperbolic energy estimates on the error variable $(\tilde\rho^\varepsilon,\tilde q^\varepsilon)=(\rho^\varepsilon-\rho^\cdot q^\varepsilon-q^*-q_I^\varepsilon)$, which satisfies

$$\begin{cases}\partial_t\tilde\rho^\varepsilon+\operatorname{div}\tilde q^\varepsilon=-\operatorname{div}q_I^\varepsilon,\\ \varepsilon^2\partial_t\tilde q^\varepsilon+\nabla\big(p(\rho^\varepsilon)-p(\rho^*)\big)+\tilde q^\varepsilon=-\varepsilon^2\partial_t q^*+\mathcal{R}^\varepsilon,\end{cases} \tag{2.18}$$

with $\mathcal{R}^\varepsilon=\varepsilon^2\operatorname{div}\big(q^\varepsilon\otimes q^\varepsilon/\rho^\varepsilon\big)$. The initial layer correction q_I^ε ensures that $\tilde q^\varepsilon|_{t=0}=-\nabla p(\rho_0^*)$, thus avoiding the singularity at time $t=0$. To control the right-hand side of (2.18), we shall make full use of the following bounds:

$$\|\operatorname{div}q_I^\varepsilon\|_{L^1(\mathbb{R}_+;H^{m-1})}=\mathcal{O}(\varepsilon),\quad \varepsilon^2\|\partial_t q^*\|_{L^2(\mathbb{R}_+;H^{m-2})}=\mathcal{O}(\varepsilon^2),\quad \|\mathcal{R}^\varepsilon\|_{L^2(\mathbb{R}_+;H^{m-1})}=\mathcal{O}(\varepsilon).$$

A key challenge arises due to the limited spatial regularity of $\partial_t q^*$. To overcome this, since the second term above is bounded by $\mathcal{O}(\varepsilon^2)$, we perform estimates in H^{m-1} and use the bounds on q^ε and q^* from the existence results. These considerations lead to an $L^\infty(\mathbb{R}+;H^{m-1})$-$\mathcal{O}(\varepsilon)$-bound for $\tilde\rho^\varepsilon$ and an $L^\infty(\mathbb{R}+;H^{m-1})\cap L^2(\mathbb{R}+;H^{m-1})$-$\mathcal{O}(\varepsilon)$-bound for $\tilde q^\varepsilon$. Moreover, exploiting the dissipative structure induced by the pressure term, we also obtain the higher-order control of $\nabla\tilde\rho^\varepsilon$ in $L^2(\mathbb{R}+;H^{m-1})$ (see Lemmas 4.3 and 4.4).

Concerning the faster convergence rate $\mathcal{O}(\varepsilon^2)$ for the error terms $(\tilde\rho_a^\varepsilon,\tilde q_a^\varepsilon):=(\rho^\varepsilon-\rho_a^\varepsilon,q^\varepsilon-q_a^\varepsilon)$, we notice that thanks to the well-prepared assumption (2.15), the right-hand side terms of (2.18) are indeed of order $\mathcal{O}(\varepsilon^2)$. Then our strategy is to reformulate the system as a parabolic equation with $\mathcal{O}(\varepsilon^2)$ source terms and perform refined energy estimates on $\tilde\rho_a^\varepsilon$. By reverting to the damped formulation for $\tilde q_a^\varepsilon$, we obtain the desired convergence rate of q^ε toward q_a^ε (cf. Lemmas 5.1–5.5).

2.3. **The Euler-Maxwell system.** We now present our results for the Euler-Maxwell system (1.12). Let the initial energy $\mathcal{X}_0^\varepsilon$ be defined by

$$\mathcal{X}_0^\varepsilon := \|\rho_0^\varepsilon - 1\|_{H^m}^2 + \varepsilon^2\|u_0^\varepsilon\|_{H^m}^2 + \|E_0^\varepsilon\|_{H^m}^2 + \|B_0^\varepsilon - B^e\|_{H^m}^2. \tag{2.19}$$

We define $\phi^* := \Lambda^{-2}(\rho^* - 1)$ and $E^* := \Lambda^{-2}\nabla\rho^*$. We introduce the momentum variables:

$$q^\varepsilon := \rho^\varepsilon u^\varepsilon, \quad q^* := \rho^* u^*, \quad q_0^\varepsilon := \rho_0^\varepsilon u_0^\varepsilon.$$

We recall the uniform global existence result for the scaled Euler-Maxwell system (1.12) (see [46]), which leads to the global weak convergence toward the drift-diffusion system (1.14).

Proposition 2.3 ([46]). *Let $d = 3$, $m \geq 3$, and (1.3) hold. There exists a constant $\delta_1 > 0$ independent of ε such that, if*

$$\mathcal{X}_0^\varepsilon \leq \delta_1, \tag{2.20}$$

the Cauchy problem (1.12)-(1.13) admits a unique global smooth solution $(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon)$ which satisfies $(\rho^\varepsilon - 1, q^\varepsilon, E^\varepsilon, B^\varepsilon - B^e) \in C(\mathbb{R}+; H^m)$ and the uniform estimate

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \left(\|\rho^\varepsilon(t) - 1\|_{H^m}^2 + \varepsilon^2\|u^\varepsilon(t)\|_{H^m}^2 + \varepsilon^2\|q^\varepsilon(t)\|_{H^m}^2\right)\\
&\quad + \sup_{t\in\mathbb{R}^+} \left(\|E^\varepsilon(t)\|_{H^m}^2 + \|B^\varepsilon(t) - B^e\|_{H^m}^2\right)\\
&\quad + \int_0^{+\infty} \left(\|\rho^\varepsilon(t)\|_{H^m}^2 + \|u^\varepsilon(t)\|_{H^m}^2 + \|q^\varepsilon(t)\|_{H^m}^2\right) dt\\
&\quad + \int_0^{+\infty} \left(\|E^\varepsilon(t)\|_{H^{m-1}}^2 + \|\nabla B^\varepsilon(t)\|_{H^{m-2}}^2\right) dt \leq C\mathcal{X}_0^\varepsilon,
\end{aligned} \tag{2.21}$$

where $C > 0$ is a generic constant independent of ε and the time. Moreover, for any finite time $T > 0$, as $\varepsilon \to 0$, we have, up to subsequences,

$$\begin{cases}
\rho^\varepsilon \longrightarrow \rho^* & \text{strongly in } C([0,T], H_{loc}^{m-1}),\\
q_\varepsilon \longrightarrow q^* & \text{weakly in } L^2(\mathbb{R}^+, H^m),\\
E_\varepsilon \longrightarrow E^* & \text{strongly in } C([0,T], H_{loc}^{m-1}),\\
B_\varepsilon \longrightarrow \bar{B} & \text{strongly in } C([0,T]; H_{loc}^{m-1}),
\end{cases} \tag{2.22}$$

where $(\rho^*, u^*, E^*, \bar{B})$ with $u^* = q^*/\rho^*$ is the unique solution of (1.14)-(1.16) with the initial data ρ_0^* being the weak limit of $(\rho_0^\varepsilon)_{\varepsilon>0}$ in $H^m(\mathbb{R})$.

We have the following result for the limit system (1.14)-(1.16).

Proposition 2.4. *Let $m \geq 3$ and (1.3) hold. There is a positive constant δ_1^* such that if $\|\rho_0^* - 1\|_{H^m}^2 \leq \delta_1^*$, then the Cauchy problem (1.15)-(1.16) admits a unique global classical solution (ρ^*, ϕ^*). Define $E^* = \nabla\phi^*$ and $q^* = -\nabla p(\rho^*) - \rho^* E^*$. Then, (ρ^*, q^*, E^*) satisfies*

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+} (\|\rho^*(t) - 1\|_{H^m}^2 + \|\nabla q^*(t)\|_{H^{m-2}}^2 + \|\nabla E^*(t)\|_{H^{m-1}}^2)\\
&\quad + \int_0^{+\infty} (\|\nabla\rho^*(t)\|_{H^m}^2 + \|\nabla q^*(t)\|_{H^{m-1}}^2 + \|\nabla E^*(t)\|_{H^m}^2)\, dt \leq C\|\rho_0^* - 1\|_{H^m}^2,
\end{aligned} \tag{2.23}$$

where $C > 0$ is a generic constant.

Our result about the global strong convergence of the relaxation limit from (1.12) to (1.14) is stated as follows.

Theorem 2.3. *Let the assumptions in Propositions 2.3 and 2.4 hold, and let* $(\rho^\varepsilon, q^\varepsilon, E^\varepsilon, B^\varepsilon)$ *and* (ρ^*, q^*, E^*) *be the solutions obtained in these two statements, respectively. Suppose additionally that* $\rho_0^* - 1 \in \dot{H}^{-1}$*. Then, it holds that*

$$\sup_{t\in\mathbb{R}^+}\Big(\|(\rho^\varepsilon-\rho^*)(t)\|_{H^{m-1}}^2+\|(E^\varepsilon-E^*)(t)\|_{H^{m-1}}^2+\|B^\varepsilon(t)-B^e\|_{H^{m-1}}^2\Big)$$
$$+\int_0^{+\infty}\Big(\|(\rho^\varepsilon-\rho^*)(t)\|_{H^m}^2+\|(E^\varepsilon-E^*)(t)\|_{H^{m-1}}^2+\|\nabla B^\varepsilon(t)\|_{H^{m-2}}^2\Big)\,dt$$
$$\leq C\big(\|\rho_0^\varepsilon-\rho_0^*\|_{H^{m-1}}^2+\|E_0^\varepsilon-E_0^*\|_{H^{m-1}}^2+\|B_0^\varepsilon-B^e\|_{H^{m-1}}^2+\varepsilon^2\big), \tag{2.24}$$

and

$$\int_0^{+\infty}\|(q^\varepsilon-q^*-q_I^\varepsilon)(t)\|_{H^{m-2}}^2\,dt$$
$$\leq C(\|\rho_0^\varepsilon-\rho_0^*\|_{H^{m-1}}^2+\|E_0^\varepsilon-E_0^*\|_{H^{m-1}}^2+\|B_0^\varepsilon-B^e\|_{H^{m-1}}^2+\varepsilon^2), \tag{2.25}$$

where C *is a constant independent of* ε, $E_0^* = \nabla\Lambda^{-2}\rho_0^*$, *and* q_I^ε *is an initial layer correction given by*

$$q_I^\varepsilon(t,x) := q_0^\varepsilon(x)e^{-\frac{t}{\varepsilon^2}}. \tag{2.26}$$

Remark 2.4. *The condition* $\rho_0^* - 1 \in \dot{H}^{-1}$ *ensures the* L^2*-regularity of* E^* *and* q^**, which is needed to establish error estimates in* L^2*. As observed in [25] for the periodic case, the electric field error* $E^\varepsilon - E^*$ *can be viewed as a stream-type function for the Euler-Maxwell system, leading to the* $L^2(\mathbb{R}^+; L^2)$*-error estimate of* $\rho^\varepsilon - \rho^*$*.*

Remark 2.5. *The initial data considered in Theorem 2.3 are ill-prepared as the compatibility condition* $\big(q^\varepsilon + \nabla p(\rho^\varepsilon) + \rho^\varepsilon E^\varepsilon\big)|_{t=0} = q_0^\varepsilon + \nabla p(\rho_0^\varepsilon) + \rho_0^\varepsilon E_0^\varepsilon \to 0$ *in* H^{m-1} *as* $\varepsilon \to 0$ *may not hold.*

2.4. **Strategy of proof for the Euler-Maxwell system.** The proof of Theorem 2.3 builds upon strategies developed for the compressible Euler equations. When analyzing the error $(\tilde\rho^\varepsilon, \tilde q^\varepsilon, \tilde E^\varepsilon, \tilde B^\varepsilon) = (\rho^\varepsilon - \rho^*, q^\varepsilon - q^*, E^\varepsilon - E^*, B^\varepsilon - B^e)$, the main difficulty arises from the presence of the zero-order source term $\partial_t E_*$ in the equation for $\tilde E^\varepsilon$ (see (6.1)). To address this, we consider the following first-order asymptotic expansion:

$$\big(\rho_a^\varepsilon, q_a^\varepsilon, E_a^\varepsilon, B_a^\varepsilon\big) := \big(\rho^*, q^*, E^*, \bar B\big) + \varepsilon\big(\rho_1, q_1, E_1, B_1\big).$$

We observe that the system for $(\tilde\rho_a^\varepsilon, \tilde q_a^\varepsilon, \tilde E_a^\varepsilon, \tilde B_a^\varepsilon) = (\rho^\varepsilon - \rho_a^\varepsilon, q^\varepsilon - q^*, E^\varepsilon - E^*, B^\varepsilon - B^e)$ shares a similar dissipative structure to that of the linearized Euler–Maxwell system, modulo source terms that are uniformly bounded by $\mathcal{O}(\varepsilon)$. This enables us to show $\mathcal{O}(\varepsilon)$-bounds for $(\tilde\rho_a^\varepsilon, \tilde q_a^\varepsilon, \tilde E_a^\varepsilon, \tilde B_a^\varepsilon)$. Then, since the profile $\big(\rho_1, q_1, E_1, B_1\big)$ is globally defined, one can recover the desired $\mathcal{O}(\varepsilon)$-bounds for the original error unknown $(\tilde\rho^\varepsilon, \tilde q^\varepsilon, \tilde E^\varepsilon, \tilde B^\varepsilon)$ by combining refined error estimates with the asymptotic expansion.

3. Preliminary and toolbox

Throughout this paper, $C > 0$ denotes a harmless constant independent of t and ε. The notation $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ stand for the Fourier transform and the inverse transform of the function f and we write

$$\Lambda^\sigma := (-\Delta)^{\frac{\sigma}{2}} = \mathcal{F}^{-1}\Big(|\xi|^\sigma \mathcal{F}(\cdot)\Big), \quad \sigma \in \mathbb{R}.$$

In the case $\sigma = 2$, we have $-\Delta = \Lambda^2$. Let $s \in \mathbb{R}^d$. We denote by H^s the Sobolev space of exponent s with the standard norm

$$\|f\|_{H^s} := \Big(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi\Big)^{\frac{1}{2}} < \infty.$$

In particular, when s is a positive integer, we have

$$\|f\|_{H^s} \sim \sum_{0\le|\alpha|\le s} \|\partial^\alpha f\|_{L^2}^2.$$

Furthermore, we denote by $\dot{H}^s(\mathbb{R}^d)$ the homogeneous Sobolev space endowed with the norm

$$\|f\|_{\dot{H}^s} := \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2\, d\xi\Big)^{\frac{1}{2}} \sim \|\Lambda^s f\|_{L^2}.$$

The following Sobolev and Moser-type inequalities can be found in [21, 29]. They will be repeatedly used in the proof.

Lemma 3.1 ([21, 29]). *The following estimates hold.*

- *Let $s > \frac{d}{2}$. The embedding from H^s to $L^\infty \cap C^0$ is continuous and for $u \in H^s$,*

$$\|u\|_{L^\infty} \le C\|u\|_{H^s}. \tag{3.1}$$

- *Let $-\frac{d}{2} < s < 0$ and $q = \frac{2d}{d-2s}$. The embedding from L^q to H^s is continuous and for $u \in L^q$,*

$$\|u\|_{H^s} \le C\|u\|_{L^q}. \tag{3.2}$$

- *Let $s_1, s_2 \in \mathbb{R}$. The operator Λ^{s_1} is an isomorphism from H^{s_2} to $H^{s_2-s_1}$.*
- *For $u \in H^{m-1}$ and $v \in H^s$ with $s > 0$,*

$$\|uv\|_{\dot{H}^s} \le C\|u\|_{L^\infty}\|v\|_{\dot{H}^s} + C\|u\|_{\dot{H}^s}\|v\|_{L^\infty}. \tag{3.3}$$

  Consequently, the Sobolev space H^s with $s > \frac{d}{2}$ is an algebra and we have

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}. \tag{3.4}$$

- *For $u \in H^{s_1}$ and $v \in \dot{H}^s$ with $s_1 > \frac{d}{2}$ and $-\frac{d}{2} < s_2 < \frac{d}{2}$, we have $uv \in \dot{H}^s$ and*

$$\|uv\|_{\dot{H}^s} \le C\|u\|_{H^{s_1}}\|v\|_{\dot{H}^s}. \tag{3.5}$$

- *Let g be a smooth function on a compact set $D \subset \mathbb{R}^n$ and let $k \ge 1$ be an integer. For $u \in H^k \cap L^\infty$ satisfying $u(x) \in D$,*

$$\sum_{|\alpha|=k} \|\partial^\alpha g(u)\|_{L^2} \le C\|g\|_{C^k(D)}\|u\|_{L^\infty}^{k-1}\|u\|_{\dot{H}^k}. \tag{3.6}$$

We recall the Gagliardo-Nirenberg inequality.

Lemma 3.2. *Assume that q, r satisfy $1 \le q, r \le \infty$ and $j, m \in \mathbb{Z}^+$ satisfy $0 \le j < m$. For any $f \in C_0^\infty(\mathbb{R}^d)$, we have*

$$\|D^j f\|_{L^p} \le C\|D^m f\|_{L^r}^\alpha \|f\|_{L^q}^{1-\alpha}, \tag{3.7}$$

where

$$\frac{1}{p} - \frac{j}{d} = \alpha(\frac{1}{r} - \frac{m}{d}) + (1-\alpha)\frac{1}{q}, \quad \frac{j}{m} \le \alpha \le 1,$$

and C depends on m, d, j, q, r, α.

4. The Euler system (I): Proof of Theorem 2.1

4.1. The error equations.

In what follows, we will always denote by $(\rho^\varepsilon, q^\varepsilon)$ and (ρ^*, q^*) the solutions to the Euler equations and the filtration equation obtained in Propositions 2.1 and 2.2, respectively.

We introduce the error unknowns

$$\tilde{\rho}^\varepsilon := \rho^\varepsilon - \rho^*, \qquad \tilde{q}^\varepsilon := q^\varepsilon - q^* - q_I^\varepsilon \tag{4.1}$$

where the initial layer correction q_I^ε given by (2.8) allows us to avoid the time-singularity of $\tilde{q}^\varepsilon$ at $t=0$. Note that q_I^ε satisfies

$$\varepsilon^2\partial_t q_I^\varepsilon + q_I^\varepsilon = 0, \qquad q_I^\varepsilon|_{t=0} = q_0^\varepsilon = \frac{1}{\varepsilon}\rho_0^\varepsilon u_0^\varepsilon. \tag{4.2}$$

With these notations, the error system reads

$$\begin{cases} \partial_t\tilde{\rho}^\varepsilon + \operatorname{div}\tilde{q}^\varepsilon = \mathcal{R}_1^\varepsilon, \\ \varepsilon^2\partial_t\tilde{q}^\varepsilon + p'(1)\nabla\tilde{\rho}^\varepsilon + \tilde{q}^\varepsilon = \mathcal{R}_0^\varepsilon + \mathcal{R}_2^\varepsilon + \mathcal{R}_3^\varepsilon, \\ (\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon)|_{t=0} = (\rho^\varepsilon - \rho_0^*, \nabla p(\rho_0^*)), \end{cases} \tag{4.3}$$

where

$$\begin{cases} \mathcal{R}_0^\varepsilon = -\big(p'(\rho^\varepsilon) - p'(\rho^*)\big)\nabla\rho^* - \big(p'(\rho^\varepsilon) - p'(1)\big)\nabla\tilde{\rho}^\varepsilon, \\ \mathcal{R}_1^\varepsilon = -\operatorname{div} q_I^\varepsilon, \\ \mathcal{R}_2^\varepsilon = -\varepsilon^2\operatorname{div}\Big(\dfrac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon}\Big), \\ \mathcal{R}_3^\varepsilon = -\varepsilon^2\partial_t q^*. \end{cases} \tag{4.4}$$

The term $\mathcal{R}_0^\varepsilon$ will be controlled using the bounds for the error $\tilde{\rho}^\varepsilon$ and the smallness of $\rho^\varepsilon - 1$ and $\rho^* - 1$. Concerning the remainder terms $\mathcal{R}_i^\varepsilon$ $(i = 1,2,3)$, we will use the following lemma.

Lemma 4.1. *Let $\mathcal{R}_i$ $(i=1,2,3)$ be given by (4.4). We have*

$$\int_0^\infty \|\mathcal{R}_1^\varepsilon(t)\|_{H^{m-1}}^2\,dt \le C, \qquad \int_0^\infty \|\mathcal{R}_1^\varepsilon(t)\|_{H^{m-1}}\,dt \le C\varepsilon, \tag{4.5}$$

$$\int_0^\infty \|\mathcal{R}_2^\varepsilon(t)\|_{H^{m-1}}^2\,dt \le C\varepsilon^2, \tag{4.6}$$

$$\int_0^\infty \|\mathcal{R}_3^\varepsilon(t)\|_{H^{m-2}}^2\,dt \le C\varepsilon^4, \tag{4.7}$$

where $C>0$ is a constant independent of ε.

Proof. For the term $\mathcal{R}_1^\varepsilon$, using the product law (3.4), we have

$$\|q_0^\varepsilon\|_{H^m} \le \frac{1}{\varepsilon}\|u_0^\varepsilon\|_{H^m} + \frac{1}{\varepsilon}\|(\rho_0^\varepsilon - 1)u_0^\varepsilon\|_{H^m} \le \frac{C}{\varepsilon}(1 + \|\rho_0^\varepsilon - 1\|_{H^m})\|u_0^\varepsilon\|_{H^m}.$$

Together with the definition $\mathcal{R}_1^\varepsilon = -\operatorname{div} q_I^\varepsilon = -e^{-\frac{t}{\varepsilon^2}}\operatorname{div} q_0^\varepsilon$ and the assumptions of ρ_0^ε, u_0^ε and ρ_0^*, this implies that

$$\int_0^\infty \|\mathcal{R}_1^\varepsilon(t)\|_{H^{m-1}}^2\,dt \le C\int_0^\infty e^{-\frac{2t}{\varepsilon^2}}\,dt\,\|\operatorname{div} q_0^\varepsilon\|_{H^{m-1}}^2 \le C\varepsilon^2\|q_0^\varepsilon\|_{H^m}^2 \le C.$$

Similarly, we have

$$\int_0^\infty \|\mathcal{R}_1^\varepsilon(t)\|_{H^{m-1}}\,dt = \int_0^\infty e^{-\frac{t}{\varepsilon^2}}\,dt\,\|q_0^\varepsilon\|_{H^m} \le C\varepsilon^2\|q_0^\varepsilon\|_{H^m} \le C\varepsilon.$$

Thus, (4.5) follows.

Concerning $\mathcal{R}_2^\varepsilon$, it holds by (2.2), (3.4) and (3.6) that

$$\begin{aligned}&\int_0^\infty \|\mathcal{R}_2^\varepsilon(t)\|_{H^{m-1}}^2\,dt\\ \le\quad & C\varepsilon^4\int_0^\infty \Big(1+\Big\|\frac{1}{\rho^\varepsilon}(t)-1\Big\|_{H^m}^2\Big)\varepsilon^2\|q^\varepsilon(t)\|_{H^m}^4\,dt\\ \le\quad & C\varepsilon^2\big(1+\sup_{t\in\mathbb{R}^+}\|\rho^\varepsilon(t)-1\|_{H^m}^2\big)\sup_{t\in\mathbb{R}^+}\varepsilon^2\|q^\varepsilon(t)\|_{H^m}^2\int_0^\infty \|q^\varepsilon(t)\|_{H^m}^2\,dt\le C\varepsilon^2.\end{aligned}$$

Finally, for $\mathcal{R}_3^\varepsilon$, observing that

$$\partial_t q^* = \nabla(p'(\rho^*)\partial_t\rho^*) = -\nabla(p'(\rho^*)\,\mathrm{div}\,q^*),$$

we have

$$\int_0^\infty \|\partial_t q^*(t)\|_{H^{m-2}}^2\,dt \le \big(1+\sup_{t\in\mathbb{R}^+}\|\rho^*(t)-1\|_{H^m}^2\big)\int_0^\infty \|q^*(t)\|_{H^m}^2\,dt\le C, \tag{4.8}$$

where we used (2.4) and (3.4). This completes the proof of Lemma 4.1. □

4.2. **Error estimates in $L^2(\mathbb{R}_+;L^2)$.** We are now in a position to establish estimates for $(\tilde\rho^\varepsilon,\tilde q^\varepsilon)=(\rho^\varepsilon-\rho^*,q^\varepsilon-q^*-q_I^\varepsilon)$. First, we obtain a rate of convergence for $\tilde\rho^\varepsilon$ in $L^2(\mathbb{R}^+;L^2)$.

Lemma 4.2. *Assuming $\rho_0^\varepsilon-\rho_0^*\in\dot H^{-1}$, we have*

$$\int_0^\infty \|\tilde\rho^\varepsilon(t)\|_{L^2}^2\,dt\le C\|\rho_0^\varepsilon-\rho_0^*\|_{\dot H^{-1}}^2+C\varepsilon^2. \tag{4.9}$$

Proof. We define the *stream function*

$$N^\varepsilon(t,x):=-\int_0^t (q^\varepsilon-q^*)(t',x)\,dt'+N_0^\varepsilon(x), \tag{4.10}$$

with $N_0^\varepsilon(x):=\Lambda^{-2}\nabla(\rho_0^\varepsilon-\rho_0^*)(x)$ such that

$$\partial_t N^\varepsilon=-(q^\varepsilon-q^*),\qquad \mathrm{div}\,N^\varepsilon=\tilde\rho^\varepsilon. \tag{4.11}$$

Note that (4.3) can be rewritten as

$$\begin{cases}\partial_t\tilde\rho^\varepsilon+\mathrm{div}(q^\varepsilon-q^*)=0,\\ q^\varepsilon-q^*=-\nabla\big(p(\rho^\varepsilon)-p(\rho^*)\big)-\varepsilon^2\partial_t q^\varepsilon-\mathcal{R}_2^\varepsilon.\end{cases} \tag{4.12}$$

Now we establish the desired estimate (4.9) using (4.11) and (4.12). Taking the inner product of $(4.12)_1$ by N^ε and using (4.11) and $(4.12)_2$, we have

$$\begin{aligned}\frac{d}{dt}\|N^\varepsilon\|_{L^2}^2&=-2\langle q^\varepsilon-q^*,N^\varepsilon\rangle\\ &=2\langle\nabla\big(p(\rho^\varepsilon)-p(\rho^*)\big),N^\varepsilon\rangle+2\varepsilon^2\langle\partial_t q^\varepsilon,N^\varepsilon\rangle+2\langle\mathcal{R}_2^\varepsilon,N^\varepsilon\rangle.\end{aligned} \tag{4.13}$$

In what follows, we estimate each term on the right-hand side of the above equality. First, since the pressure satisfies (1.3) and ρ^ε is close to 1 uniformly, there exists a uniform constant $p_1>0$ such that

$$2\langle\nabla\big(p(\rho^\varepsilon)-p(\rho^*)\big),N^\varepsilon\rangle=-2\langle p(\rho^\varepsilon)-p(\rho^*),\rho^\varepsilon-\rho^*\rangle\le-2p_1\|\tilde\rho^\varepsilon\|_{L^2}^2. \tag{4.14}$$

Next, according to (4.11), it holds that

$$\begin{aligned}2\varepsilon^2\langle\partial_t q^\varepsilon,N^\varepsilon\rangle&=2\varepsilon^2\frac{d}{dt}\langle q^\varepsilon,N^\varepsilon\rangle-2\varepsilon^2\langle q^\varepsilon,\partial_t N^\varepsilon\rangle\\ &=2\varepsilon^2\frac{d}{dt}\langle q^\varepsilon,N^\varepsilon\rangle+2\varepsilon^2\langle q^\varepsilon,q^\varepsilon-q^*\rangle.\end{aligned}$$

As $q^* = -\nabla p(\rho^*)$, the second term on the right-hand side of the above equality can be analyzed by

$$2\varepsilon^2\langle q^\varepsilon, q^\varepsilon - q^*\rangle \le C\varepsilon^2\big(\|q^\varepsilon\|_{L^2}^2 + \|\nabla\rho^*\|_{L^2}^2\big).$$

Consequently, we have

$$2\varepsilon^2\langle \partial_t q^\varepsilon, N^\varepsilon\rangle \le 2\varepsilon^2\frac{d}{dt}\langle q^\varepsilon, N^\varepsilon\rangle + C\varepsilon^2\big(\|q^\varepsilon\|_{L^2}^2 + \|\nabla\rho^*\|_{L^2}^2\big). \tag{4.15}$$

Moreover, from (2.2) and (4.4), it is clear that

$$\begin{aligned}2\langle \mathcal{R}_2^\varepsilon, N^\varepsilon\rangle &\le 2\|\mathcal{R}_2^\varepsilon\|_{L^2}\|N^\varepsilon\|_{L^2}\\ &\le C\varepsilon^2\|q^\varepsilon\|_{H^m}^2\|N^\varepsilon\|_{L^2}\\ &\le C\varepsilon^4\|q^\varepsilon\|_{H^m}^2 + C\|q^\varepsilon\|_{H^m}^2\|N^\varepsilon\|_{L^2}^2. \end{aligned}\tag{4.16}$$

Since $\varepsilon \in (0,1]$, combining (4.13)-(4.16) together with (2.2), we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\|N^\varepsilon\|_{L^2}^2 - 2\varepsilon^2\langle q^\varepsilon, N^\varepsilon\rangle\big) + 2p_1\|\tilde\rho^\varepsilon\|_{L^2}^2\\ &\le C\varepsilon^2\big(\|q^\varepsilon\|_{H^m}^2 + \|\nabla\rho^*\|_{H^m}^2\big) + C\|q^\varepsilon\|_{H^m}^2\|N^\varepsilon\|_{L^2}^2,\end{aligned}\tag{4.17}$$

which implies that, for all $t \ge 0$,

$$\begin{aligned}&\|N^\varepsilon\|_{L^2}^2 + 2p_1\int_0^t \|\tilde\rho^\varepsilon(t')\|_{L^2}^2\,dt'\\ &\quad\le \|N_0^\varepsilon\|_{L^2}^2 + 2\varepsilon^2\langle q^\varepsilon, N^\varepsilon\rangle\Big|_0^t + C\varepsilon^2\int_0^t \big(\|q^\varepsilon(t')\|_{H^m}^2 + \|\nabla\rho^*(t')\|_{H^m}^2\big)\,dt\\ &\qquad + C\int_0^t \|q^\varepsilon(t')\|_{H^m}^2\|N^\varepsilon(t')\|_{L^2}^2\,dt'.\end{aligned}\tag{4.18}$$

Here, one has

$$2\varepsilon^2\langle q^\varepsilon, N^\varepsilon\rangle\Big|_0^t \le \frac{1}{2}\|N^\varepsilon\|_{L^2}^2 + C\varepsilon^4\|q^\varepsilon\|_{L^2}^2 + C\|N_0^\varepsilon\|_{L^2}^2 + C\varepsilon^4\|q_0^\varepsilon\|_{L^2}^2.$$

Therefore, applying Grönwall's lemma to (4.18), (2.2) and the fact that $\varepsilon^2\|q_0^\varepsilon\|_{L^2}^2 \le C$ leads to

$$\sup_{t\in\mathbb{R}^+}\|N^\varepsilon(t)\|_{L^2}^2 + \int_0^\infty \|\tilde\rho^\varepsilon(t)\|_{L^2}^2\,dt \le C\|N_0^\varepsilon\|_{L^2}^2 + C\varepsilon^2.$$

Since $\Lambda^{-2}\nabla$ maps $\dot H^{-1}$ to L^2, we have $\|N_0^\varepsilon\|_{L^2} \le C\|\rho_0^\varepsilon - \rho_0^*\|_{\dot H^{-1}}$. Therefore, we arrive at (4.9). □

4.3. **Error estimates in $L^\infty(\mathbb{R}^+;L^2)\cap L^2(\mathbb{R}^+;\dot H^1)$.** Without assuming that the initial error is in $\dot H^{-1}$, we have the following lemma.

Lemma 4.3. *It holds*

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\big(\|\tilde\rho^\varepsilon(t)\|_{L^2}^2 + \varepsilon^2\|\tilde q^\varepsilon(t)\|_{L^2}^2\big) + \int_0^\infty \big(\|\nabla\tilde\rho^\varepsilon(t)\|_{L^2}^2 + \|\tilde q^\varepsilon(t)\|_{L^2}^2\big)\,dt\\ &\quad\le C\|\rho_0^\varepsilon - \rho_0^*\|_{L^2}^2 + C\varepsilon^2,\end{aligned}\tag{4.19}$$

where $C > 0$ is a constant independent of ε.

Proof. Taking the L^2 inner product of (4.3)$_1$ and (4.3)$_2$ with $p'(1)\tilde{\rho}^\varepsilon$ and $\tilde{q}^\varepsilon$, respectively, we arrive at

$$\begin{aligned}&\frac{d}{dt}\Big(p'(1)\|\tilde{\rho}^\varepsilon\|_{L^2}^2+\varepsilon^2\|\tilde{q}^\varepsilon\|_{L^2}^2\Big)+2\|\tilde{q}^\varepsilon\|_{L^2}^2\\ &=2p'(1)\langle\mathcal{R}_1^\varepsilon,\tilde{\rho}^\varepsilon\rangle+2\langle\mathcal{R}_0^\varepsilon+\mathcal{R}_2^\varepsilon+\mathcal{R}_3^\varepsilon,\tilde{q}^\varepsilon\rangle\\ &\le 2p'(1)\|\mathcal{R}_1^\varepsilon\|_{L^2}\|\tilde{\rho}^\varepsilon\|_{L^2}+C(\|\mathcal{R}_0^\varepsilon\|_{L^2}^2+\|\mathcal{R}_2^\varepsilon\|_{L^2}^2+\|\mathcal{R}_3^\varepsilon\|_{L^2}^2)+\|\tilde{q}^\varepsilon\|_{L^2}^2,\end{aligned}\tag{4.20}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of $L^2(\mathbb{R}^d)$. To bound $\mathcal{R}_0^\varepsilon$, we use the fact that

$$\mathcal{R}_0^\varepsilon=\pi(\rho^\varepsilon,\rho^*)\tilde{\rho}^\varepsilon\nabla\rho^*+\pi(\rho^\varepsilon,1)(\rho^\varepsilon-1)\nabla\tilde{\rho}^\varepsilon.$$

where

$$p'(s_1)-p'(s_2)=\pi(s_1,s_2)(s_1-s_2),\qquad \pi(s_1,s_1)=\int_0^1 p''(\theta s_1+(1-\theta)s_2)d\theta.$$

Due to (2.2) and (2.4), the densities ρ^ε and ρ^* have uniform upper bounds, so $\pi(\rho^\varepsilon,\rho^*)$ and $\pi(\rho^\varepsilon,1)$ are also bounded from above. In the case $d\ge 3$, the Cauchy-Schwarz inequality, together with Sobolev's embeddings $\dot{H}^1\hookrightarrow L^{\frac{2d}{d-2}}$ and $H^{m-1}\hookrightarrow L^2\cap L^\infty\hookrightarrow L^d\cap L^\infty$, leads to

$$\begin{aligned}\|\mathcal{R}_0^\varepsilon\|_{L^2}^2&\le\|\pi(\rho^\varepsilon,\rho^*)\|_{L^\infty}^2\|\tilde{\rho}^\varepsilon\|_{L^{\frac{2d}{d-2}}}^2\|\nabla\rho^*\|_{L^d}^2+\|\pi(\rho^\varepsilon,1)\|_{L^\infty}^2\|\rho^\varepsilon-1\|_{L^\infty}^2\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2\\ &\le C\|\rho^\varepsilon-1\|_{H^{m-1}}^2\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2\\ &\le C(\delta+\delta^*)\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2.\end{aligned}$$

In the case $d=2$, we take advantage of the Gagliardo-Nirenberg-Sobolev inequality in Lemma 3.2 and (2.4) to derive

$$\begin{aligned}\|\mathcal{R}_0^\varepsilon\|_{L^2}^2&\le\|\pi(\rho^\varepsilon,\rho^*)\|_{L^\infty}^2\|\tilde{\rho}^\varepsilon\|_{L^4}^2\|\nabla\rho^*\|_{L^4}^2+\|\pi(\rho^\varepsilon,1)\|_{L^\infty}^2\|\rho^\varepsilon-1\|_{L^\infty}^2\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2\\ &\le C\|\tilde{\rho}^\varepsilon\|_{L^2}\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}\|\nabla\rho^*\|_{L^2}\|\nabla^2\rho^*\|_{L^2}+C\|\rho^\varepsilon-1\|_{H^{m-1}}^2\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2\\ &\le C(\delta+\delta^*)\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2+C\|\nabla^2\rho^*\|_{L^2}^2\|\tilde{\rho}^\varepsilon\|_{L^2}^2.\end{aligned}$$

Finally, the case $d=1$ can be addressed similarly:

$$\begin{aligned}\|\mathcal{R}_0^\varepsilon\|_{L^2}^2&\le\|\pi(\rho^\varepsilon,\rho^*)\|_{L^\infty}^2\|\tilde{\rho}^\varepsilon\|_{L^\infty}^2\|\nabla\rho^*\|_{L^2}^2+\|\pi(\rho^\varepsilon,1)\|_{L^\infty}^2\|\rho^\varepsilon-1\|_{L^\infty}^2\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2\\ &\le C\|\tilde{\rho}^\varepsilon\|_{L^2}\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}\|\nabla\rho^*\|_{L^2}^2+C\|\rho^\varepsilon-1\|_{H^{m-1}}^2\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2\\ &\le C(\delta+\delta^*)\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2+C\|\nabla\rho^*\|_{L^2}^2\|\tilde{\rho}\|_{L^2}^2.\end{aligned}$$

Consequently, for all $d\ge 1$, we have

$$\begin{aligned}\int_0^t\|\mathcal{R}_0^\varepsilon(t')\|_{L^2}^2\,dt'&\le C(\delta+\delta^*)\int_0^t\|\nabla\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt'\\ &\qquad+C\sup_{t'\in[0,t]}\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\int_0^t\|\nabla\rho^*(t')\|_{H^1}^2\,dt'.\end{aligned}\tag{4.21}$$

Noting that $\varepsilon^2\|\tilde{q}^\varepsilon(0)\|_{L^2}^2=\varepsilon^2\|q^*(0)\|_{L^2}^2=\varepsilon^2\|\nabla p(\rho^*)(0)\|_{L^2}\leq C\varepsilon^2$, and integrating (4.20) over $[0,t']$ with any $0\leq t'\leq t<\infty$, using (4.21) and then taking the supremum over $[0,t]$, we have

$$\begin{aligned}
&\sup_{t'\in[0,t]}\big(\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2+\varepsilon^2\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\big)+\int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\leq C\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+C\varepsilon^2+C(\delta+\delta^*)\int_0^t\|\nabla\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt'+C\sup_{t'\in[0,t]}\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\int_0^t\|\nabla\rho^*(t')\|_{H^1}^2\,dt'\\
&\quad+C\sup_{t'\in[0,t]}\|\tilde{\rho}^\varepsilon(t')\|_{L^2}\int_0^t\|\mathcal{R}_1^\varepsilon(t')\|_{L^2}\,dt'+C\int_0^t\big(\|\mathcal{R}_2^\varepsilon(t')\|_{L^2}^2+\|\mathcal{R}_3^\varepsilon(t')\|_{L^2}^2\big)\,dt'\\
&\leq C\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+C\varepsilon^2+\Big(\frac{1}{2}+C\delta_*\Big)\sup_{t'\in[0,t]}\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2+C(\delta+\delta_*)\int_0^t\|\nabla\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt',
\end{aligned}$$

where we have used (4.5)-(4.7) and the fact that

$$C\sup_{t'\in[0,t]}\|\tilde{\rho}^\varepsilon(t')\|_{L^2}\int_0^t\|\mathcal{R}_1^\varepsilon(t')\|_{L^2}\,dt'\leq\frac{1}{2}\sup_{t'\in[0,t]}\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2+C\Big(\int_0^t\|\mathcal{R}_1^\varepsilon(t')\|_{L^2}\,dt'\Big)^2.$$

Since δ_* can be small enough, we end up with

$$\begin{aligned}
&\sup_{t'\in[0,t]}\big(\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2+\varepsilon^2\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\big)+\int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\leq C\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+C\varepsilon^2+C(\delta+\delta_*)\int_0^t\|\nabla\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt'.
\end{aligned}\tag{4.22}$$

In order to address the last term in (4.22), we shall establish the $L^2(0,t;L^2)$ estimate of $\nabla\tilde{\rho}^\varepsilon$. Taking the inner product of $(4.3)_2$ with $\nabla\tilde{\rho}^\varepsilon$ leads to

$$\begin{aligned}
p'(1)\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2&=\varepsilon^2\langle\partial_t\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle-\langle\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle+\langle\mathcal{R}_0^\varepsilon+\mathcal{R}_2^\varepsilon+\mathcal{R}_3^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle\\
&\leq\varepsilon^2\langle\partial_t\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle+\frac{1}{2}p'(1)\|\nabla\tilde{\rho}^\varepsilon\|_{L^2}^2+C\|\tilde{q}^\varepsilon\|_{L^2}^2\\
&\quad+C(\|\mathcal{R}_0^\varepsilon\|_{L^2}^2+\|\mathcal{R}_2^\varepsilon\|_{L^2}^2+\|\mathcal{R}_3^\varepsilon\|_{L^2}^2).
\end{aligned}\tag{4.23}$$

Here, using $(4.3)_1$ yields

$$\begin{aligned}
\varepsilon^2\langle\partial_t\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle&=\varepsilon^2\frac{d}{dt}\langle\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle+\varepsilon^2\|\operatorname{div}\tilde{q}^\varepsilon\|_{L^2}^2+\varepsilon^2\langle\operatorname{div}\tilde{q}^\varepsilon,\mathcal{R}_1^\varepsilon\rangle\\
&\leq\varepsilon^2\frac{d}{dt}\langle\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle+C\varepsilon^2\|\operatorname{div}\tilde{q}^\varepsilon\|_{L^2}^2+C\varepsilon^2\|\mathcal{R}_1^\varepsilon\|_{L^2}^2.
\end{aligned}\tag{4.24}$$

Substituting (4.24) into (4.23) and integrating the resulting inequality over $[0,t]$, we have

$$\begin{aligned}
&\frac{1}{2}p'(1)\int_0^t\|\nabla\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\leq\varepsilon^2\langle\tilde{q}^\varepsilon,\nabla\tilde{\rho}^\varepsilon\rangle\Big|_0^t+C\int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'+C\varepsilon^2\int_0^t\|\operatorname{div}q^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\quad+C\int_0^t\big(\|\mathcal{R}_0^\varepsilon(t')\|_{L^2}^2+\|\mathcal{R}_2^\varepsilon(t')\|_{L^2}^2+\|\mathcal{R}_3^\varepsilon(t')\|_{L^2}^2+\varepsilon^2\|\mathcal{R}_1^\varepsilon(t')\|_{L^2}^2\big)\,dt'.
\end{aligned}\tag{4.25}$$

To analyze the first term on the right-hand side of (4.25), we note that, due to (2.2), (2.4), and $\tilde{q}^\varepsilon|_{t=0} = -\nabla p(\rho_0^*)$,

$$\begin{aligned}\varepsilon^2\langle \tilde{q}^\varepsilon, \nabla\tilde{\rho}^\varepsilon\rangle\Big|_0^t &\le \varepsilon^2\|\tilde{q}^\varepsilon\|_{L^2}^2 + C\varepsilon^2\big(\|\rho^\varepsilon - 1\|_{H^1}^2 + \|\rho^* - 1\|_{H^1}^2\big)\\ &\quad + \varepsilon^2\|\nabla p(\rho_0^*)\|_{L^2}\big(\|\nabla\rho_0^\varepsilon\|_{L^2} + \|\nabla\rho_0^*\|_{L^2}\big)\\ &\le \varepsilon^2\|\tilde{q}^\varepsilon\|_{L^2}^2 + C\varepsilon^2.\end{aligned}$$

Moreover, one has

$$\begin{aligned}&\int_0^t \|\operatorname{div}\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'\\ &\le C\int_0^t \big(\|\operatorname{div} q^\varepsilon(t')\|_{L^2}^2 + \|\operatorname{div} q^*(t')\|_{L^2}^2 + \|\operatorname{div} q_I^\varepsilon(t')\|_{L^2}^2\big)\,dt' \le C.\end{aligned}\tag{4.26}$$

In view of (4.5)-(4.7), (4.21), (4.25) and (4.26), it holds that

$$\int_0^t \|\nabla\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt' \le C\varepsilon^2\sup_{t'\in[0,t]}\|\tilde{q}^\varepsilon(t')\|_{L^2}^2 + C\int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt' + C\varepsilon^2.\tag{4.27}$$

Plugging (4.27) into (4.22) gives rise to

$$\begin{aligned}&\sup_{t'\in[0,t]}\Big(\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2 + \varepsilon^2\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\Big) + \int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'\\ &\quad\le C\|\rho_0^\varepsilon - \rho_0^*\|_{L^2}^2 + C\varepsilon^2\\ &\qquad + C(\delta+\delta_*)\Big(\varepsilon^2\sup_{t'\in[0,t]}\|\tilde{q}^\varepsilon(t')\|_{L^2}^2 + \int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt' + \varepsilon^2\Big).\end{aligned}$$

Since δ and δ_* are suitably small, we derive

$$\sup_{t'\in[0,t]}\big(\|\tilde{\rho}^\varepsilon(t')\|_{L^2}^2 + \varepsilon^2\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\big) + \int_0^t\|\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt' \le C\|\rho_0^\varepsilon - \rho_0^*\|_{L^2}^2 + C\varepsilon^2,$$

which, together with (4.27), yields (4.19) and finishes the proof of Lemma 4.3. □

4.4. **Higher-order error estimates.** We have the following lemma.

Lemma 4.4. *It holds*

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde{\rho}^\varepsilon(t)\|_{H^{m-2}}^2 + \varepsilon^2\|\nabla\tilde{q}^\varepsilon(t)\|_{H^{m-2}}^2\big)\\ &\quad + \int_0^\infty\big(\|\nabla^2\tilde{\rho}^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde{q}^\varepsilon(t)\|_{H^{m-2}}^2\big)\,dt\\ &\le C\|\nabla(\rho_0^\varepsilon - \rho_0^*)\|_{H^{m-2}}^2 + C\varepsilon^2,\end{aligned}\tag{4.28}$$

where $C>0$ is a constant independent of ε.

Proof. We perform similar energy estimates as in Lemma 4.3. Compared with the computations in Lemma 4.3, we need to treat $\mathcal{R}_i^\varepsilon$ $(i = 0,1,2,3)$ in a more careful manner. Let $\alpha\in\mathbb{N}^d$ with $1\le|\alpha|\le m-1$. Applying ∂^α to (4.3), we have

$$\begin{cases}\partial_t\partial^\alpha\tilde{\rho}^\varepsilon + \operatorname{div}\partial^\alpha\tilde{q}^\varepsilon = \partial^\alpha\mathcal{R}_1^\varepsilon,\\ \partial_t\partial^\alpha\tilde{q}^\varepsilon + p'(1)\nabla\partial^\alpha\tilde{\rho}^\varepsilon + \partial^\alpha\tilde{q}^\varepsilon = \partial^\alpha\mathcal{R}_0^\varepsilon + \partial^\alpha\mathcal{R}_2^\varepsilon + \partial^\alpha\mathcal{R}_3^\varepsilon.\end{cases}\tag{4.29}$$

This implies the energy equality

$$\frac{d}{dt}\Big(p'(1)\|\partial^\alpha\tilde\rho^\varepsilon\|_{L^2}^2+\varepsilon^2\|\partial^\alpha\tilde q^\varepsilon\|_{L^2}^2\Big)+2\|\partial^\alpha\tilde q^\varepsilon\|_{L^2}^2$$
$$+2\langle\partial^\alpha\mathcal{R}_1^\varepsilon,\partial^\alpha\tilde\rho^\varepsilon\rangle+2\langle\partial^\alpha\mathcal{R}_0^\varepsilon+\mathcal{R}_2^\varepsilon+\partial^\alpha\mathcal{R}_3^\varepsilon,\partial^\alpha\tilde q^\varepsilon\rangle. \tag{4.30}$$

We now handle the terms on the right-hand side of (4.30). First, the term involving $\mathcal{R}_0^\varepsilon$ is analyzed by

$$\|\partial^\alpha\mathcal{R}_0^\varepsilon\|_{L^2}^2\le\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(\rho^*))\nabla\rho^*\big)\big\|_{L^2}^2+\big\|\partial^\alpha\big((p'(\rho^*)-p'(1))\nabla\tilde\rho^\varepsilon\big)\big\|_{L^2}^2.$$

It follows from (2.4) and (3.4) that

$$\begin{aligned}&\big\|\partial^\alpha\big((p'(\rho^*)-p'(1))\nabla\tilde\rho^\varepsilon\big)\big\|_{L^2}^2\\
&\le C\|p'(\rho^*)-p'(1)\|_{L^\infty}^2\|\partial^\alpha\nabla\tilde\rho^\varepsilon\|_{L^2}^2+C\|\partial^\alpha p'(\rho^*)\|_{L^2}^2\|\nabla\tilde\rho^\varepsilon\|_{L^\infty}^2\\
&\le C\|\rho^*-1\|_{H^m}^2\|\nabla\tilde\rho^\varepsilon\|_{H^{m-1}}^2\\
&\le C\delta_*\|\nabla\tilde\rho^\varepsilon\|_{L^2}^2+C\delta_*\|\nabla^2\tilde\rho^\varepsilon\|_{H^{m-2}}^2.\end{aligned}$$

Similarly, we have

$$\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(\rho^*))\nabla\rho^*\big)\|_{L^2}^2\le C\|\tilde\rho^\varepsilon\|_{L^\infty}^2\|\partial^\alpha\nabla\rho^*\|_{L^2}^2+C\|\partial^\alpha\tilde\rho^\varepsilon\|_{L^2}^2\|\nabla\rho^*\|_{L^\infty}^2.$$

In the case $d\ge3$, according to the Gagliardo-Nirenberg inequality (Lemma 3.2), for any $f\in H^{m-1}$ and $\theta\in(0,1)$ such that $0=\theta(\frac d2-1)+(1-\theta)(\frac d2-m+1)$, we discover

$$\|f\|_{L^\infty}\le C\|f\|_{\dot H^1}^\theta\|f\|_{\dot H^{m-1}}^{1-\theta}\le C\|\nabla f\|_{H^{m-2}},$$

from which we infer

$$\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(\rho^*))\nabla\rho^*\big)\big\|_{L^2}^2\le C\|\nabla\rho^*\|_{H^{m-1}}^2\|\nabla\tilde\rho^\varepsilon\|_{H^{m-2}}^2.$$

In the case $d=1,2$, we also have

$$\|f\|_{L^\infty}\le C\|f\|_{L^2}^{1-\frac d4}\|f\|_{\dot H^2}^{\frac d4},$$

so using (2.4), we have

$$\begin{aligned}&\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(\rho^*))\nabla\rho^*\big)\big\|_{L^2}^2\\
&\le C\|\tilde\rho^\varepsilon\|_{L^2}^{2-\frac d2}\|\nabla^2\tilde\rho^\varepsilon\|_{L^2}^{\frac d2}\|\nabla\partial^\alpha\rho^*\|_{L^2}^2+C\|\partial^\alpha\tilde\rho^\varepsilon\|_{L^2}^2\|\nabla\rho^*\|_{H^{m-1}}^2\\
&\le C\|\nabla\rho^*\|_{H^{m-1}}^2(\|\tilde\rho^\varepsilon\|_{L^2}^2+\|\nabla^2\tilde\rho^\varepsilon\|_{L}^2)+C\|\nabla\rho^*\|_{H^{m-1}}^2\|\partial^\alpha\tilde\rho^\varepsilon\|_{L^2}^2\\
&\le C\|\nabla\rho^*\|_{H^{m-1}}^2\|\tilde\rho^\varepsilon\|_{L^2}^2+C\delta_*\|\nabla\tilde\rho^\varepsilon\|_{L^2}^2+C\delta_*\|\nabla^2\tilde\rho^\varepsilon\|_{H^{m-2}}^2.\end{aligned}$$

Hence, combining the above two cases $d\ge3$ and $d=1,2$ and using the error estimate (4.19), we arrive at

$$\begin{aligned}\int_0^t\|\partial^\alpha\mathcal{R}_0^\varepsilon(t')\|_{L^2}^2\,dt'&\le C\sup_{t'\in[0,t]}\|\tilde\rho^\varepsilon(t')\|_{L^2}^2\int_0^t\|\nabla\rho^*(t')\|_{H^{m-1}}^2\,dt'\\
&\quad+C\delta_*\int_0^t\|\nabla\tilde\rho^\varepsilon(t')\|_{L^2}^2\,dt'+C\delta_*\int_0^t\|\nabla^2\tilde\rho^\varepsilon(t')\|_{H^{m-2}}^2\,dt'\\
&\le C\varepsilon^2+C\delta_*\int_0^t\|\nabla^2\tilde\rho^\varepsilon(t')\|_{H^{m-2}}^2\,dt'.\end{aligned}\tag{4.31}$$

Note that the integration associated with $\partial^\alpha \mathcal{R}_i^\varepsilon$ $(i=1,2)$ can be handled similarly as in Lemma 4.3:

$$\begin{aligned} 2\int_0^t \langle \partial^\alpha \mathcal{R}_2^\varepsilon, \partial^\alpha \tilde{q}^\varepsilon\rangle\, dt' &\le \frac{1}{4}\int_0^t \|\partial^\alpha \tilde{q}^\varepsilon(t')\|_{L^2}^2\, dt' + C\int_0^t \|\partial^\alpha \mathcal{R}_2^\varepsilon(t')\|_{L^2}^2\, dt' \\ &\le \frac{1}{4}\int_0^t \|\partial^\alpha \tilde{q}^\varepsilon(t')\|_{L^2}^2\, dt' + C\varepsilon^2, \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} 2\int_0^t \langle \partial^\alpha \mathcal{R}_1^\varepsilon, \partial^\alpha \tilde{\rho}^\varepsilon\rangle\, dt' &\le 2\sup_{t'\in[0,t]} \|\partial^\alpha \tilde{\rho}^\varepsilon(t')\|_{L^2} \int_0^t \|\mathcal{R}_1^\varepsilon(t')\|_{H^{m-1}}\, dt' \\ &\le \frac{1}{4}\sup_{t'\in[0,t]} \|\partial^\alpha \tilde{\rho}^\varepsilon(t')\|_{L^2}^2 + C\Big(\int_0^t \|\mathcal{R}_1^\varepsilon(t')\|_{H^{m-1}}\, dt'\Big)^2 \\ &\le \frac{1}{4}\sup_{t'\in[0,t]} \|\partial^\alpha \tilde{\rho}^\varepsilon(t')\|_{L^2}^2 + C\varepsilon^2. \end{aligned} \tag{4.33}$$

However, the term involving $\partial^\alpha \mathcal{R}_3^\varepsilon = -\varepsilon^2\partial^\alpha\partial_t q^*$ requires a more elaborate analysis. Indeed, $\partial^\alpha \mathcal{R}_3^\varepsilon$ may not be bounded in $L^2(\mathbb{R}^+;L^2)$ for $|\alpha| = m-1$ but obeys a faster rate (see (4.7)). To address this difficulty, we observe that

$$\tilde{q}^\varepsilon = q^\varepsilon - q^* - e^{-\frac{t}{\varepsilon^2}}\frac{1}{\varepsilon}\rho_0^\varepsilon u_0^\varepsilon.$$

Recall the operator Λ defined in Section 3. By integration by parts, it holds

$$\begin{aligned} &\langle \partial^\alpha \mathcal{R}_3^\varepsilon, \partial^\alpha \tilde{q}^\varepsilon\rangle \\ &= \langle \Lambda^{-1}\partial^\alpha \mathcal{R}_3^\varepsilon, \Lambda\partial^\alpha q^\varepsilon\rangle - \frac{\varepsilon^2}{2}\frac{d}{dt}\|\partial^\alpha q^*\|_{L^2}^2 + \varepsilon e^{-\frac{t}{\varepsilon^2}}\langle \Lambda^{-1}\partial^\alpha \mathcal{R}_3^\varepsilon, \Lambda\partial^\alpha(\rho_0^\varepsilon u_0^\varepsilon)\rangle. \end{aligned}$$

This, together with (2.2), (2.4)-(2.5), (4.7) and the regularities of initial data, leads to

$$\begin{aligned} &\int_0^t \langle \partial^\alpha \mathcal{R}_3^\varepsilon, \partial^\alpha \tilde{q}^\varepsilon\rangle\, dt' \\ &\le \frac{\varepsilon^2}{2}\|q^*\|_{H^{m-1}}^2 + \frac{\varepsilon^2}{2}\|\nabla p(\rho_0^*)\|_{H^{m-1}}^2 \\ &\quad + \Big(\int_0^t \|\mathcal{R}_3^\varepsilon(t')\|_{H^{m-2}}^2\, dt'\Big)^{\frac{1}{2}}\Big(\int_0^t \Big(\|q^\varepsilon(t')\|_{H^m}^2 + \varepsilon^2 e^{-\frac{2t'}{\varepsilon^2}}\|\rho_0^\varepsilon u_0^\varepsilon\|_{H^m}^2\Big)\, dt'\Big)^{\frac{1}{2}} \le C\varepsilon^2. \end{aligned} \tag{4.34}$$

Recall that $\varepsilon^2\|\partial^\alpha \tilde{q}^\varepsilon(0)\|_{L^2}^2 = \varepsilon^2\|\partial^\alpha q^*(0)\|_{L^2}^2 \le \varepsilon^2\|\nabla p(\rho^*)(0)\|_{H^{m-1}} \le C\varepsilon^2$. Integrating (4.30) over $[0,t]$ and substituting (4.31), (4.32)-(4.34) into the resulting inequality, we have

$$\begin{aligned} &\sup_{t'\in[0,t]}\Big(\|\nabla\tilde{\rho}^\varepsilon(t)\|_{H^{m-2}}^2 + \varepsilon^2\|\nabla\tilde{q}^\varepsilon(t)\|_{H^{m-2}}^2\Big) + \int_0^t \|\nabla\tilde{q}^\varepsilon(t')\|_{H^{m-2}}^2\, dt' \\ &\le C\|\nabla(\rho_0^\varepsilon - \rho_0^*)\|_{H^{m-2}}^2 + C\varepsilon^2 + C\delta_*\int_0^t \|\nabla^2\tilde{\rho}^\varepsilon(t')\|_{H^{m-2}}^2\, dt'. \end{aligned} \tag{4.35}$$

Let us now turn to capturing the higher-order dissipation of $\tilde{\rho}^\varepsilon$ required in (4.35). We rewrite $(4.29)_2$ by

$$p'(1)\nabla\tilde{\rho}_\alpha^\varepsilon + \partial^\alpha\tilde{q}^\varepsilon = \partial^\alpha\mathcal{R}_0^\varepsilon + \partial^\alpha\mathcal{R}_2^\varepsilon - \partial_t\partial^\alpha q^\varepsilon. \tag{4.36}$$

Multiplying (4.36) by $\nabla\partial^\alpha\tilde{\rho}^\varepsilon$ and using $\partial_t\partial^\alpha\tilde{\rho}^\varepsilon = -\partial^\alpha \operatorname{div}(q^\varepsilon - q^*)$ and integration by parts, we deduce that, after direct computations,

$$\begin{aligned}
p'(1)\|\nabla\partial^\alpha\tilde{\rho}^\varepsilon\|_{L^2}^2 &= -\varepsilon^2\frac{d}{dt}\langle\partial^\alpha q^\varepsilon, \nabla\partial^\alpha\tilde{\rho}^\varepsilon\rangle - \varepsilon^2\langle\operatorname{div}\partial^\alpha q^\varepsilon, \partial_t\partial^\alpha\tilde{\rho}^\varepsilon\rangle - \langle\partial^\alpha\tilde{q}^\varepsilon, \nabla\partial^\alpha\tilde{\rho}^\varepsilon\rangle\\
&\quad + \langle\partial^\alpha\mathcal{R}_0^\varepsilon + \partial^\alpha\mathcal{R}_2^\varepsilon, \nabla\partial^\alpha\tilde{\rho}^\varepsilon\rangle\\
&\le -\varepsilon^2\frac{d}{dt}\langle\partial^\alpha q^\varepsilon, \nabla\partial^\alpha\tilde{\rho}^\varepsilon\rangle + \varepsilon^2(\|\partial^\alpha\operatorname{div} q^*\|_{L^2}^2 + \|\partial^\alpha\operatorname{div} q^\varepsilon\|_{L^2}^2)\\
&\quad + \frac{1}{2}p'(1)\|\nabla\partial^\alpha\tilde{\rho}^\varepsilon\|_{L^2}^2 + C\|\partial^\alpha\tilde{q}^\varepsilon\|_{L^2}^2\\
&\quad + C\big(\|\partial^\alpha\mathcal{R}_0^\varepsilon\|_{L^2}^2 + \|\partial^\alpha\mathcal{R}_2^\varepsilon\|_{L^2}^2\big).
\end{aligned}$$

This leads to

$$\begin{aligned}
\frac{1}{2}p'(1)\int_0^t \|\nabla\partial^\alpha\tilde{\rho}^\varepsilon(t')\|_{L^2}^2\,dt' &\le -\varepsilon^2\langle\partial^\alpha q^\varepsilon, \nabla\partial^\alpha\tilde{\rho}^\varepsilon\rangle\Big|_0^t + C\int_0^t \|\partial^\alpha\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\quad + C\varepsilon^2\int_0^t \big(\|q^\varepsilon(t')\|_{H^m}^2 + \|q^*(t')\|_{H^m}^2\big)\,dt'\\
&\quad + C\int_0^t \big(\|\mathcal{R}_0^\varepsilon(t')\|_{H^{m-1}}^2 + \|\mathcal{R}_2^\varepsilon(t')\|_{H^{m-1}}^2\big)\,dt'. \qquad (4.37)
\end{aligned}$$

Due to $\varepsilon\|q^\varepsilon\|_{H^m} + \varepsilon\|q_0^\varepsilon\|_{H^m} \le C$, the first term on the right-hand side of (4.37) is bounded by

$$\begin{aligned}
\varepsilon^2\langle\partial^\alpha q^\varepsilon, \nabla\partial^\alpha\tilde{\rho}^\varepsilon\rangle\Big|_0^t &\le \varepsilon\big(\varepsilon\|q^\varepsilon\|_{H^m}\big)\|\nabla\tilde{\rho}^\varepsilon\|_{H^{m-2}} + \varepsilon\big(\varepsilon\|q_0^\varepsilon\|_{H^m}\big)\|\nabla\tilde{\rho}^\varepsilon(0)\|_{H^{m-2}}\\
&\le C\|\nabla\tilde{\rho}^\varepsilon\|_{H^{m-2}}^2 + C\|\nabla(\rho_0^\varepsilon - \rho_0)\|_{H^{m-2}}^2 + C\varepsilon^2. \qquad (4.38)
\end{aligned}$$

In addition, from the regularities of $(\rho^\varepsilon, q^\varepsilon)$ and (ρ^*, q^*) one has

$$\int_0^t \Big(\|q^\varepsilon(t')\|_{H^m}^2 + \|q^*(t')\|_{H^m}^2\Big)\,dt' \le C, \qquad (4.39)$$

and

$$\int_0^t \|\nabla\tilde{\rho}^\varepsilon(t')\|_{H^m}^2\,dt' \le C\int_0^t (\|\nabla\rho^\varepsilon(t')\|_{H^m}^2 + \|\nabla\rho^*(t')\|_{H^m}^2)\,dt' \le C. \qquad (4.40)$$

Combining (4.5)-(4.6), (4.31), (4.37)-(4.40) and using the smallness of δ_*, we arrive at

$$\int_0^t \|\nabla^2\tilde{\rho}^\varepsilon(t')\|_{H^{m-2}}^2 \le C\|\nabla\tilde{\rho}^\varepsilon\|_{H^{m-1}}^2 + C\varepsilon^2. \qquad (4.41)$$

Plugging (4.41) into (4.35), using the smallness of δ_* and returning to (4.41), we get the desired estimate (4.28) and finish the proof of Lemma 4.4. □

5. The Euler system (II): Proof of Theorem 2.2

5.1. **The error equations.** In this section, we prove Theorem 2.2 and show a faster convergence rate between $(\rho^\varepsilon, q^\varepsilon)$ and its first-order asymptotic expansion $(\rho_a^\varepsilon, q_a^\varepsilon)$ with

$$\rho_a^\varepsilon = \rho^* + \varepsilon\rho_1 \quad\text{and}\quad q_a^\varepsilon = q^* + \varepsilon q_1,$$

where the profiles ρ_1 and q_1 solve the system (2.12). According to (1.6) and (2.12), $(\rho_a^\varepsilon, q_a^\varepsilon)$ satisfies

$$\begin{cases}\partial_t\rho_a^\varepsilon + \operatorname{div} q_a^\varepsilon = 0,\\ q_a^\varepsilon = -\nabla\big(p(\rho^*) + \varepsilon p'(\rho^*)\rho_1\big).\end{cases} \qquad (5.1)$$

A key ingredient is to establish the error estimates of $(\tilde\rho^\varepsilon_a, \tilde q^\varepsilon_a)$ given by

$$\tilde\rho^\varepsilon_a = \rho^\varepsilon - \rho^\varepsilon_a, \qquad \tilde q^\varepsilon_a = q^\varepsilon - q^\varepsilon_a. \tag{5.2}$$

By a direct computation, the equations of $(\tilde\rho^\varepsilon_a, \tilde q^\varepsilon_a)$ read

$$\begin{cases} \partial_t \tilde\rho^\varepsilon_a + \operatorname{div} \tilde q^\varepsilon_a = 0, \\ \tilde q^\varepsilon_a = -\nabla\big(p(\rho^\varepsilon) - p(\rho^\varepsilon_a)\big) + \mathcal{R}^\varepsilon_4 - \varepsilon^2 \partial_t q^\varepsilon, \end{cases} \tag{5.3}$$

where

$$\mathcal{R}^\varepsilon_4 = -\varepsilon^2 \operatorname{div}\Big(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}\Big) - \nabla\big(p(\rho^\varepsilon_a) - p(\rho^*) - \varepsilon p'(\rho^*)\rho_1\big). \tag{5.4}$$

Before analyzing $(\tilde\rho^\varepsilon, \tilde q^\varepsilon)$, we shall derive regularity estimates for (ρ_1, q_1).

5.2. Estimates for (ρ_1, q_1).

Lemma 5.1. *There exists a pair (ρ_1, q_1) solving (2.12)-(2.13) with the initial data $\rho_1|_{t=0} = \rho_{1,0} \in H^m$. In addition, there exists a constant C depending only on ρ^*_0 and $\rho_{1,0}$ such that*

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+} \big(\|\rho_1(t)\|^2_{H^m} + \|q_1(t)\|^2_{H^{m-1}}\big) \\ &\quad + \int_0^\infty \big(\|\nabla\rho_1(t)\|^2_{H^m} + \|q_1(t)\|^2_{H^m} + \|\partial_t q_1(t)\|^2_{H^{m-2}}\big) dt \le C. \end{aligned} \tag{5.5}$$

Proof. Standard theorem ensures that (2.14) with $\rho_1|_{t=0} = \rho_{1,0} \in H^m$ admits a unique global solution $\rho_1 \in C(\mathbb{R}^+; H^m)$. Direct computations on (2.14) yield that, for all $\alpha \in \mathbb{R}^d$ such that $0 \le |\alpha| \le m$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\rho_1\|^2_{L^2} + p'(1)\|\partial^\alpha\nabla\rho_1\|^2_{L^2} \\ &\le \big\|\partial^\alpha\nabla\big((p'(\rho^*) - p'(1))\rho_1\big)\big\|_{L^2}\|\partial^\alpha\nabla\rho_1\|_{L^2} \\ &\le \frac{p'(1)}{2}\|\partial^\alpha\nabla\rho_1\|^2_{L^2} + C\big\|\partial^\alpha\nabla\big((p'(\rho^*) - p'(1))\rho_1\big)\big\|^2_{L^2}. \end{aligned} \tag{5.6}$$

Using (3.1) and (3.6), we have

$$\begin{aligned} &\sum_{0\le|\alpha|\le m} \big\|\nabla\partial^\alpha\big((p'(\rho^*) - p'(1))\rho_1\big)\big\|_{L^2} \\ &\le \|\nabla p'(\rho^*)\rho_1\|_{H^m} + \|(p'(\rho^*) - p'(1))\nabla\rho_1\|_{H^m} \\ &\le C\|\nabla p'(\rho^*)\|_{H^m}\|\rho_1\|_{H^m} + \|p'(\rho^*) - p'(1)\|_{H^m}\|\nabla\rho_1\|_{H^m} \\ &\le C\|\rho_1\|_{H^m}\|\nabla\rho^*\|_{H^m} + C\|\rho^* - 1\|_{H^m}\|\nabla\rho_1\|_{H^m}. \end{aligned}$$

Therefore, integrating (5.6) in time and recalling (2.4), we arrive at

$$\begin{aligned} &\sup_{t'\in[0,t]} \|\rho_1(t')\|^2_{H^m} + \int_0^t \|\nabla\rho_1(t')\|^2_{H^m}\,dt' \\ &\le C\|\rho_{1,0}\|^2_{H^m} + \sup_{t'\in[0,t]}\|\rho_1(t')\|^2_{H^m}\int_0^t \|\nabla\rho^*(t')\|^2_{H^m}\,dt' \\ &\quad + \sup_{t'\in[0,t]}\|\rho^*(t') - 1\|^2_{H^m}\int_0^t \|\nabla\rho_1(t')\|^2_{H^m}\,dt' \\ &\le C\|\rho_{1,0}\|^2_{H^m} + C\delta^*\Big(\sup_{t'\in[0,t]}\|\rho_1(t)\|^2_{H^m} + \int_0^t \|\nabla\rho_1(t')\|^2_{H^m}\,dt'\Big). \end{aligned}$$

Consequently, as the constant $\delta^* > 0$ given by Proposition 2.2 is suitably small, we obtain the desired bounds of ρ_1:

$$\sup_{t\in\mathbb{R}^+} \|\rho_1(t)\|_{H^m}^2 + \int_0^\infty \|\nabla\rho_1(t)\|_{H^m}^2\,dt \leq C\|\rho_{0,1}\|_{H^m}^2 + C\|\rho^* - 1\|_{H^m}^2. \tag{5.7}$$

Next, we establish the estimates of q_1. It follows from (2.13) and (3.1) that

$$\begin{aligned}\|q_1\|_{H^k} &\leq C\|\nabla\rho_1\|_{H^k} + C\|\nabla(p'(\rho^*) - p'(1))\rho_1)\|_{H^k}\\ &\leq C\|\nabla\rho_1\|_{H^k} + \|\nabla p'(\rho^*)\rho_1\|_{H^k} + \|(p'(\rho^*) - p'(1))\nabla\rho_1\|_{H^k}\\ &\leq C\|\nabla\rho_1\|_{H^k} + C\|\nabla\rho^*\|_{H^{m-1}}\|\rho_1\|_{H^k} + C\|\nabla\rho^*\|_{H^k}\|\rho_1\|_{H^{m-1}}\\ &\quad + \|\rho^* - 1\|_{H^{m-1}}\|\nabla\rho_1\|_{H^k} + \|\rho^* - 1\|_{H^k}\|\nabla\rho_1\|_{H^{m-1}},\end{aligned}$$

for $k = 1, ..., m$. Hence, owing to (2.4) and (5.7), q_1 satisfies

$$\sup_{t\in\mathbb{R}^+} \|q_1(t)\|_{H^{m-1}} \leq \sup_{t\in\mathbb{R}^+} \big(1 + \|\rho^*(t) - 1\|_{H^m}\big)\|\rho_1(t)\|_{H^m} \leq C,$$

and

$$\begin{aligned}\int_0^\infty \|q^*(t)\|_{H^m}^2\,dt \leq& \big(1 + \sup_{t\in\mathbb{R}^+} \|\rho^*(t) - 1\|_{H^m}^2\big)\int_0^\infty \|\nabla\rho_1(t)\|_{H^m}^2\,dt\\ &+ \sup_{t\in\mathbb{R}^+} \|\rho_1(t)\|_{H^m}^2 \int_0^\infty \|\nabla\rho^*(t)\|_{H^m}^2\,dt \leq C.\end{aligned}$$

Finally, since

$$\partial_t q_1 = -\nabla\big(p''(\rho^*)\partial_t\rho^* + p'(\rho^*)\partial_t\rho_1\big) = \nabla\big(p''(\rho^*)\operatorname{div} q^*\rho_1 + p'(\rho^*)\operatorname{div} q_1\big),$$

it follows that

$$\begin{aligned}\int_0^\infty \|\partial_t q_1(t')\|_{H^{m-2}}^2\,dt' &\leq C\int_0^\infty \big(\|p''(\rho^*)\operatorname{div} q^*\rho_1\|_{H^{m-1}}^2 + \|p'(\rho^*)\operatorname{div} q_1(t')\|_{H^{m-1}}^2\big)\,dt'\\ &\leq C\big(1 + \sup_{t\in\mathbb{R}^+} \|\rho^*(t) - 1\|_{H^{m-1}}^2\big)\sup_{t\in\mathbb{R}^+} \|\rho_1(t')\|_{H^{m-1}}^2 \int_0^\infty \|q^*(t')\|_{H^m}^2\,dt'\\ &\quad + C(1 + \sup_{t\in\mathbb{R}^+} \|\rho^*(t) - 1\|_{H^{m-1}}^2)\int_0^\infty \|q_1(t')\|_{H^m}^2\,dt' \leq C.\end{aligned}$$

This completes the proof of Lemma 5.1. □

5.3. **Error estimates for $q^\varepsilon - q^*$.** Assuming further that (2.15) holds, we can prove an improved estimate of the error $q^\varepsilon - q^*$, which will be useful for deriving the faster convergence rate of the error $(\tilde\rho_a^\varepsilon, \tilde q_a^\varepsilon)$.

Lemma 5.2. *Under the conditions of Theorem 2.2, it holds*

$$\begin{cases}\sup_{t\in\mathbb{R}^+} \|(q^\varepsilon - q^*)(t)\|_{H^{m-1}}^2 \leq C,\\ \int_0^\infty \|(q^\varepsilon - q^*)(t)\|_{H^{m-1}}^2\,dt \leq C\varepsilon^2,\end{cases} \tag{5.8}$$

for a uniform constant $C > 0$.

Proof. Since $\|q_0^\varepsilon - q_0^*\|_{H^{m-1}} \leq \varepsilon$, we know that q_0^ε is uniformly bounded in H^{m-1} with respect to ε. As a consequence,

$$\int_0^\infty \|q_I^\varepsilon(t)\|_{H^{m-1}}^2\,dt \leq \int_0^\infty e^{-\frac{2t}{\varepsilon^2}}\,dt\|q_0^\varepsilon\|_{H^{m-1}}^2 \leq C\varepsilon^2,$$

which, together with (2.7) and $\|\rho_0^\varepsilon-\rho_0^*\|_{H^{m-1}}\le C(1+\|\rho_{1,0}\|_{H^{m-1}})\varepsilon$, gives rise to

$$\int_0^\infty \|(q^\varepsilon-q^*)(t)\|_{H^{m-1}}^2\,dt\le C\int_0^\infty \big(\|(q^\varepsilon-q^*-q_I^\varepsilon)(t)\|_{H^{m-1}}^2+\|q_I^\varepsilon(t)\|_{H^{m-1}}^2\big)\,dt\le C\varepsilon^2.$$

This yields the second estimate in (5.8). Meanwhile, the first estimate can be easily deduced from (4.19), (4.28) and the definition of q_I^ε. □

5.4. **Estimates of $\mathcal{R}_4^\varepsilon$.** We have the following lemma.

Lemma 5.3. *We have*

$$\int_0^\infty \|\mathcal{R}_4(t)\|_{H^{m-2}}^2\,dt\le C\varepsilon^4, \tag{5.9}$$

where $C>0$ is a constant independent of ε.

Proof. We first handle the term $\varepsilon^2\operatorname{div}\Big(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon}\Big)$. In view of the regularities of $\rho^\varepsilon,q^\varepsilon$ and q^* (see (2.2) and (2.5)), $q^\varepsilon=q^\varepsilon-q^*+q^*$ and the improved estimate (5.8), it follows that

$$\begin{aligned}&\int_0^\infty \Big\|\operatorname{div}\Big(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon}\Big)(t)\Big\|_{H^{m-2}}^2\,dt\\ &\le C\big(1+\sup_{t\in\mathbb{R}^+}\|\rho^\varepsilon(t)-1\|_{H^m}^2\big)\sup_{t\in\mathbb{R}^+}\big(\|(q^\varepsilon-q^*)(t)\|_{H^{m-1}}^2+\|q^*(t)\|_{H^{m-1}}^2\big)\int_0^\infty\|q^\varepsilon(t)\|_{H^m}^2\,dt\le C.\end{aligned}$$

To analyze the second term in $\mathcal{R}_4^\varepsilon$, we write

$$p(\rho_a^\varepsilon)-p(\rho^*)-\varepsilon p'(\rho^*)\rho_1=\varepsilon^2\Pi^\varepsilon(\rho^*,\rho_a^\varepsilon)\rho_1^2,$$

where

$$\Pi^\varepsilon(\rho^*,\rho_a^\varepsilon)=\int_0^1 p''(\theta\rho^*+(1-\theta)\rho_a^\varepsilon)\theta d\theta.$$

This implies that

$$\begin{aligned}&\nabla\big(p(\rho_a^\varepsilon)-p(\rho^*)-\varepsilon p'(\rho^*)\rho_1\big)\\ &=\varepsilon^2\big(\partial_{\rho^*}\Pi^\varepsilon(\rho^*,\rho_1)\nabla\rho^*+\partial_{\rho_1}\Pi^\varepsilon(\rho^*,\rho_1)\nabla\rho_1\big)\rho_1^2+2\varepsilon^2\Pi^\varepsilon(\rho^*,\rho_1)\rho_1\nabla\rho_1.\end{aligned}$$

Therefore, (3.6) as well as (2.2) and (2.4) guarantee that

$$\begin{aligned}&\int_0^\infty \big\|\nabla\big(p(\rho_a^\varepsilon)-p(\rho^*)-\varepsilon p'(\rho^*)\rho_1\big)(t)\big\|_{H^m}^2\,dt\\ &\le C\varepsilon^4\int_0^\infty\big(\|\nabla\rho^*(t)\|_{H^m}^2+\|\nabla\rho_1(t)\|_{H^m}^2\big)\,dt\le C\varepsilon^4.\end{aligned}\tag{5.10}$$

The combination of (4.7) and (5.10) leads to (5.9). □

5.5. **Improved error estimates.**

We are going to establish faster convergence rates of $\rho^\varepsilon-\rho^*-\varepsilon\rho_1$ and $q^\varepsilon-q^*-\varepsilon q_1-q_I^\varepsilon$, under the condition (2.15). In what follows, $(\rho^\varepsilon,q^\varepsilon)$ and (ρ^*,q^*) correspond to the solutions of the Euler equations and the porous media equation obtained in Propositions 2.1 and 2.2, respectively.

First, we establish L^2 estimate for $\rho^\varepsilon-\rho^*-\varepsilon\rho_1$ and $q^\varepsilon-q^*-\varepsilon q_1-q_I^\varepsilon$ with a rate ε^2.

Lemma 5.4. *It holds*

$$\sup_{t\in\mathbb{R}^+}\big(\|\tilde\rho_a^\varepsilon(t)\|_{L^2}^2+\varepsilon^2\|\tilde q_a^\varepsilon(t)\|_{L^2}^2\big)+\int_0^\infty\big(\|\nabla\tilde\rho_a^\varepsilon(t)\|_{L^2}^2+\|\tilde q_a^\varepsilon(t)\|_{L^2}^2\big)\,dt\le C\varepsilon^4, \tag{5.11}$$

where the error $(\tilde\rho_a^\varepsilon,\tilde q_a^\varepsilon)$ is defined by (5.2), and $C>0$ is a constant independent of ε.

Proof. The system (5.3) can be reformulated as the inhomogeneous heat flow:

$$\partial_t\tilde{\rho}_a^\varepsilon - \operatorname{div}(p'(\rho^\varepsilon)\nabla\tilde{\rho}_a^\varepsilon) = \operatorname{div}((p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon) - \operatorname{div}\mathcal{R}_4^\varepsilon + \varepsilon^2\partial_t \operatorname{div} q^\varepsilon. \tag{5.12}$$

Taking the L^2 inner product of (5.12) with $\tilde{\rho}^\varepsilon$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + \langle p'(\rho^\varepsilon)\nabla\tilde{\rho}_a^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle\\ &= -\langle(p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle - \langle\mathcal{R}_4^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle - \varepsilon^2\langle\partial_t q^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle.\end{aligned} \tag{5.13}$$

Since ρ^ε is in a neighborhood of 1, the condition (1.3) implies that there exists a constant p_1 such that

$$\langle p'(\rho^\varepsilon)\nabla\tilde{\rho}_a^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle \geq p_1\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2. \tag{5.14}$$

Concerning the first term on the right-hand side of (5.13), we can write

$$p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon) = \pi(\rho^\varepsilon, \rho_a^\varepsilon)\tilde{\rho}_a^\varepsilon, \qquad \pi(\rho^\varepsilon, \rho_a^\varepsilon) := \int_0^1 p''(\theta\rho^\varepsilon + (1-\theta)\rho_a^\varepsilon)d\theta.$$

In the case $d \geq 3$, the Cauchy-Schwarz inequality together with Sobolev's embedding and the uniform upper bounds of ρ^ε, ρ^*, and ρ_1 leads to

$$\begin{aligned}-2\langle(p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon))\nabla\rho^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle &\leq C\|\pi(\rho^\varepsilon, \rho_a^\varepsilon)\|_{L^\infty}\|\tilde{\rho}_a^\varepsilon\|_{L^{\frac{2d}{d-2}}}\|\nabla\rho^\varepsilon\|_{L^d}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\\ &\leq C\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\|\nabla\rho^\varepsilon\|_{H^{m-1}}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\\ &\leq C\sqrt{\delta}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2.\end{aligned} \tag{5.15}$$

In the case $d = 2$, we take advantage of the Gagliardo-Nirenberg-Sobolev inequality to derive

$$\begin{aligned}&-2\langle(p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle\\ &\leq C\|\pi(\rho^\varepsilon, \rho_a^\varepsilon)\|_{L^\infty}\|\tilde{\rho}_a^\varepsilon\|_{L^4}\|\nabla\rho_a^\varepsilon\|_{L^4}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\\ &\leq C\|\tilde{\rho}_a^\varepsilon\|_{L^2}^{\frac12}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^{\frac12}\|\nabla\rho_a^\varepsilon\|_{L^2}^{\frac12}\|\nabla^2\rho_a^\varepsilon\|_{L^2}^{\frac12}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\\ &\leq \frac{1}{8}p_1\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + C\|\nabla\rho_a^\varepsilon\|_{H^1}^4\|\tilde{\rho}_a^\varepsilon\|_{L^2}^2.\end{aligned} \tag{5.16}$$

Finally, the case $d = 1$ can be similarly addressed by

$$\begin{aligned}&-2\langle(p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle\\ &\leq C\|\pi(\rho^\varepsilon, \rho_a^\varepsilon)\|_{L^\infty}\|\tilde{\rho}_a^\varepsilon\|_{L^\infty}\|\nabla\rho_a^\varepsilon\|_{L^2}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\\ &\leq C\|\tilde{\rho}_a^\varepsilon\|_{L^2}^{\frac12}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^{\frac12}\|\nabla\rho_a^\varepsilon\|_{L^2}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}\\ &\leq \frac{1}{8}p_1\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + C\|\nabla\rho_a^\varepsilon\|_{L^2}^4\|\tilde{\rho}_a^\varepsilon\|_{L^2}^2.\end{aligned} \tag{5.17}$$

Also, one has

$$-\langle\mathcal{R}_4^\varepsilon, \nabla\tilde{\rho}_a^\varepsilon\rangle \leq \frac{1}{8}p_1\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + C\|\mathcal{R}_4^\varepsilon\|_{L^2}^2. \tag{5.18}$$

For the last term on the right-hand side of (5.13), we have to overcome the singularity of $\partial_t q^\varepsilon$. To this matter, we use the equation of $\tilde{\rho}_a^\varepsilon$ and obtain

$$\begin{aligned}\varepsilon^2\langle\partial_t \operatorname{div} q^\varepsilon, \tilde{\rho}_a^\varepsilon\rangle &= \varepsilon^2\langle\partial_t \operatorname{div}\tilde{q}_a^\varepsilon, \tilde{\rho}_a^\varepsilon\rangle + \varepsilon^2\langle\partial_t \operatorname{div}(q_* + \varepsilon q_1), \tilde{\rho}_a^\varepsilon\rangle\\ &= \varepsilon^2\frac{d}{dt}\langle\operatorname{div}\tilde{q}_a^\varepsilon, \tilde{\rho}_a^\varepsilon\rangle + \varepsilon^2\langle\tilde{q}_a^\varepsilon, \nabla\partial_t\tilde{\rho}_a^\varepsilon\rangle - \varepsilon^2\langle\partial_t(q_* + \varepsilon q_1), \nabla\tilde{\rho}_a^\varepsilon\rangle\\ &\leq \frac{1}{8}p_1\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 - \frac{d}{dt}\langle\operatorname{div}\tilde{q}_a^\varepsilon, \tilde{\rho}_a^\varepsilon\rangle + C\varepsilon^4\|\partial_t(q_* + \varepsilon q_1)\|_{L^2}^2\\ &\quad + \nu\|\tilde{q}_a^\varepsilon\|_{L^2}^2 + C\nu^{-1}\varepsilon^4\|\nabla\partial_t\tilde{\rho}_a^\varepsilon\|_{L^2}^4.\end{aligned} \tag{5.19}$$

Combining (5.13) with (5.14)-(5.19) and using $\|\rho^\varepsilon\|_{H^m} \le C$ yields

$$
\begin{aligned}
&\frac{d}{dt}\|\tilde\rho_a^\varepsilon\|_{L^2}^2 + (p_- - \sqrt{\delta})\|\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2 \\
&\le -\varepsilon^2\frac{d}{dt}\langle \operatorname{div}\tilde q_a^\varepsilon, \tilde\rho_a^\varepsilon\rangle + C\|\nabla\rho^\varepsilon\|_{H^m}^2\|\tilde\rho_a^\varepsilon\|_{L^2}^2 \\
&\quad + C\|\mathcal{R}_4^\varepsilon\|_{L^2}^2 + C\varepsilon^4\|\partial_t(q_* + \varepsilon q_1)\|_{L^2}^2 + \nu\|\tilde q_a^\varepsilon\|_{L^2}^2 + C\nu^{-1}\varepsilon^4\|\nabla\partial_t\tilde\rho_a^\varepsilon\|_{L^2}^4.
\end{aligned}
$$

Recalling that δ is suitably small and $\tilde\rho_a^\varepsilon(0) = \rho_0^\varepsilon - \rho_0^* - \varepsilon\rho_{1,0} = \mathcal{O}(\varepsilon^2)$ in H^{m-2}, we get

$$
\begin{aligned}
&\sup_{t'\in[0,t]}\|\tilde\rho_a^\varepsilon(t')\|_{L^2}^2 + C\int_0^t \|\nabla\tilde\rho_a^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\le C\varepsilon^4 - \langle \operatorname{div}\tilde q_a^\varepsilon, \tilde\rho_a^\varepsilon\rangle\Big|_0^t + C\int_0^t \|\mathcal{R}_4^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\quad + C\sup_{t'\in[0,t]}\|\rho^\varepsilon(t')\|_{H^m}^2\int_0^t \|\nabla\rho^\varepsilon(t')\|_{H^{m-1}}^2\|\tilde\rho_a^\varepsilon(t')\|_{L^2}^2\,dt' + C\nu\int_0^t \|\tilde q_a^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\quad + C\varepsilon^4\int_0^t \left(\|\partial_t(q_* + \varepsilon q_1)(t')\|_{L^2}^2 + \nu^{-1}\|\partial_t\tilde\rho_a^\varepsilon(t')\|_{L^2}^4\right)dt'.
\end{aligned}
\tag{5.20}
$$

With the uniform bounds (2.2), (2.4), (4.8) and (5.5) at hand, it holds that

$$
\varepsilon^4\int_0^t \|\partial_t(q_* + \varepsilon q_1)(t')\|_{H^{m-2}}^2\,dt' \le C\varepsilon^4. \tag{5.21}
$$

As

$$
\partial_t\tilde\rho_a^\varepsilon = \partial_t(\rho^\varepsilon - \rho^* - \varepsilon\rho_1) = \operatorname{div}(-q^\varepsilon + q^* + \varepsilon q_1),
$$

one also has

$$
\begin{aligned}
&\varepsilon^4\int_0^t \|\nabla\partial_t\tilde\rho_a^\varepsilon(t')\|_{H^{m-2}}^2\,dt' \\
&\le C\varepsilon^4\int_0^t \left(\|q^\varepsilon(t')\|_{H^m}^2 + \|q^*(t')\|_{H^m}^2 + \|q_1(t)\|_{H^m}^2\right)dt' \le C\varepsilon^4.
\end{aligned}
$$

Since $\|\tilde\rho_a^\varepsilon\|_{H^{m-2}} \le \varepsilon^2$ and $\|\tilde q_a^\varepsilon|_{t=0}\|_{H^{m-1}} \le \varepsilon + \varepsilon\|q_{1,0}\|_{H^{m-1}}$ one gets from (2.15) and (5.8) that

$$
\begin{aligned}
-\varepsilon^2\langle \operatorname{div}\tilde q_a^\varepsilon, \tilde\rho_a^\varepsilon\rangle\Big|_0^t &\le \frac14\sup_{t'\in[0,t]}\|\tilde\rho_a^\varepsilon(t')\|_{L^2}^2 + \varepsilon^4\sup_{t'\in[0,t]}\left(\|(q^\varepsilon - q^*)(t')\|_{H^1}^2 + \varepsilon^2\|q_1(t')\|_{H^1}^2\right) \\
&\quad + \varepsilon^5\left(1 + \|\operatorname{div} q_{1,0}\|_{L^2}\right) \le C\varepsilon^4.
\end{aligned}
\tag{5.22}
$$

It thus follows from (2.2), (5.9) and (5.20)-(5.22) that

$$
\sup_{t'\in[0,t]}\|\tilde\rho_a^\varepsilon(t')\|_{L^2}^2 + C\int_0^t \|\tilde\rho_a^\varepsilon(t')\|_{L^2}^2\,dt' \le C(1+\nu^{-1})\varepsilon^4 + \nu\int_0^t \|\tilde q_a^\varepsilon(t')\|_{L^2}^2\,dt'. \tag{5.23}
$$

We then turn to analyze the error $\tilde q_a^\varepsilon$ that satisfies a damped equation

$$
\varepsilon^2\partial_t\tilde q_a^\varepsilon + \tilde q_a^\varepsilon = -\nabla(p(\rho^\varepsilon) - p(\rho_a^\varepsilon)) + \mathcal{R}_4^\varepsilon - (\partial_t q^* + \varepsilon\partial_t q_1). \tag{5.24}
$$

Taking the inner product of (5.24) with $\tilde q_a^\varepsilon$ yields

$$
\begin{aligned}
&\varepsilon^2\frac{d}{dt}\|\tilde q_a^\varepsilon\|_{L^2}^2 + \|\tilde q_a^\varepsilon\|_{L^2}^2 \\
&\le \left(\|\nabla(p(\rho^\varepsilon) - p(\rho_a^\varepsilon))\|_{L^2} + \|\mathcal{R}_4^\varepsilon\|_{L^2} + \varepsilon^2\|(\partial_t q^* + \varepsilon\partial_t q_1)\|_{L^2}\right)\|\tilde q_a^\varepsilon\|_{L^2} \\
&\le \frac12\|\tilde q_a^\varepsilon\|_{L^2}^2 + C\|\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2 + C\|\mathcal{R}_4^\varepsilon\|_{L^2}^2 + C\varepsilon^4\|(\partial_t q^* + \varepsilon\partial_t q_1)\|_{L^2}^2.
\end{aligned}
$$

Since $\tilde{q}_a^\varepsilon|_{t=0} = \mathcal{O}(\varepsilon)$ in H^{m-1}, this leads to

$$\begin{aligned}
&\varepsilon^2 \sup_{t'\in[0,t]} \|\tilde{q}_a^\varepsilon(t')\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\tilde{q}_a^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\quad \le C\varepsilon^4 + C\varepsilon \int_0^t \|\nabla\tilde{\rho}_a^\varepsilon(t')\|_{L^2}^2\,dt' + C\int_0^t \|\mathcal{R}_4^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\qquad + C\varepsilon^4 \int_0^t \|(\partial_t q^* + \varepsilon\partial_t q_1)(t')\|_{L^2}^2\,dt' \\
&\quad \le C(1+\nu^{-1})\varepsilon^4 + \nu\int_0^t \|\tilde{q}_a^\varepsilon(t')\|_{L^2}^2\,dt',
\end{aligned}$$

where (5.9), (5.21) and (5.23) have been used. Choosing ν sufficiently small and combining (5.25) with (5.23), we prove (5.11). □

5.6. **Higher-order error estimates.** We have the following lemma.

Lemma 5.5. *Let $d \ge 2$ be such that $m \ge 3$. It holds*

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \left(\|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-3}}^2 + \varepsilon^2\|\nabla\tilde{q}_a^\varepsilon(t)\|_{H^{m-3}}^2\right) \\
&\quad + \int_0^\infty \left(\|\nabla^2\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-3}}^2 + \|\nabla\tilde{q}_a^\varepsilon(t)\|_{H^{m-3}}^2\right)dt \le C\varepsilon^4, \qquad (5.25)
\end{aligned}$$

where $C > 0$ is a constant independent of ε.

Proof. Applying ∂^α with $1 \le |\alpha| \le m-2$ to (5.12), we have

$$\begin{aligned}
&\partial_t\partial^\alpha\tilde{\rho}_a^\varepsilon - p'(1)\Delta\partial^\alpha\tilde{\rho}_a^\varepsilon \\
&\quad = \operatorname{div}\partial^\alpha\Big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde{\rho}_a^\varepsilon + (p'(\rho^\varepsilon)-p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon - \mathcal{R}_4^\varepsilon + \varepsilon^2\partial_t\partial^\alpha q^\varepsilon\Big). \qquad (5.26)
\end{aligned}$$

This implies the following energy inequality

$$\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + 2p'(1)\|\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 \\
&= -2\Big\langle\Big(\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde{\rho}_a^\varepsilon + (p'(\rho^\varepsilon)-p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon - \mathcal{R}_4^\varepsilon + \varepsilon^2\partial_t\partial^\alpha q^\varepsilon\Big),\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\Big\rangle \\
&\le \frac{1}{2}p'(1)\|\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + C\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde{\rho}_a^\varepsilon\big)\big\|_{L^2}^2 \\
&\quad + C\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(\rho_a^\varepsilon))\nabla\rho_a^\varepsilon\big)\big\|_{L^2}^2 + \|\partial^\alpha\mathcal{R}_4^\varepsilon\|_{L^2}^2 + \varepsilon^2\langle\partial_t\partial^\alpha q^\varepsilon,\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle. \qquad (5.27)
\end{aligned}$$

We now handle the terms on the right-hand side of (5.27). The term $\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde{\rho}_a^\varepsilon\big)\big\|_{L^2}^2$ is analyzed in the following three cases.

- Case 1: $m \ge [\frac{d}{2}]+3$ or m is even when $m = [\frac{d}{2}]+2$ for $d \ge 2$.

By (2.2) and the Moser-type inequality (3.3), we have

$$\begin{aligned}
&\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde{\rho}_a^\varepsilon\big)\big\|_{L^2}^2 \\
&\le \|p'(\rho^\varepsilon)-p'(1)\|_{L^\infty}^2\|\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2 + \|\partial^\alpha(p'(\rho^\varepsilon)-p'(1))\|_{L^2}^2\|\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\|_{L^\infty}^2. \qquad (5.28)
\end{aligned}$$

In this case, note that $m-2>\frac{d}{2}$ implies the embedding $H^{m-2}\hookrightarrow L^\infty$. Consequently, by (2.2) and (3.6), for $1\le|\alpha|\le m-1$ we obtain

$$\begin{aligned}&\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde\rho_a^\varepsilon\big)\big\|_{L^2}^2\\ \le&\|p'(\rho^\varepsilon)-p'(1)\|_{H^{m-2}}^2\|\partial^\alpha\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2+\|\partial^\alpha(p'(\rho^\varepsilon)-p'(1))\|_{L^2}^2\|\nabla\tilde\rho_a^\varepsilon\|_{H^{m-2}}^2\\ \le&C\|\rho^\varepsilon-1\|_{H^m}\|\nabla^2\tilde\rho_a^\varepsilon\|_{H^{m-2}}^2\\ \le&C\delta\|\nabla^2\tilde\rho_a^\varepsilon\|_{H^{m-2}}^2.\end{aligned}\tag{5.29}$$

- Case 2: $m=[\frac{d}{2}]+2$ and m is odd for $d\ge3$.

In this case, we have $[\frac{d}{2}]=\frac{d}{2}-\frac12$ and $m-2<\frac{d}{2}$. It follows from (2.2) and the product law (3.5) that

$$\begin{aligned}\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde\rho_a^\varepsilon\big)\big\|_{L^2}^2&\le\|p'(\rho^\varepsilon)-p'(1)\|_{H^{m-1}}^2\|\partial^\alpha\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2\\ &\le C\delta\|\partial^\alpha\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2.\end{aligned}\tag{5.30}$$

- Case 3: $m=3$ for $d=2$.

Since $m-2=1$, we know $|\alpha|=1$. Thus, it is easy to verify that

$$\begin{aligned}&\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde\rho_a^\varepsilon\big)\big\|_{L^2}^2\\ \le&\|p'(\rho^\varepsilon)-p'(1)\|_{L^\infty}^2\|\nabla^2\tilde\rho_a^\varepsilon\|_{L^2}^2+\|\nabla(p'(\rho^\varepsilon)-p'(1))\|_{L^\infty}^2\|\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2\\ \le&C\delta\|\nabla^2\tilde\rho_a^\varepsilon\|_{L^2}^2+C\|\nabla\rho^\varepsilon\|_{H^m}^2\|\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2.\end{aligned}\tag{5.31}$$

Combining the above three cases, we know

$$\big\|\partial^\alpha\big((p'(\rho^\varepsilon)-p'(1))\nabla\tilde\rho_a^\varepsilon\big)\big\|_{L^2}^2\le C\delta\|\nabla^2\tilde\rho_a^\varepsilon\|_{H^{m-2}}^2+C\|\nabla\rho^\varepsilon\|_{H^m}^2\|\nabla\tilde\rho_a^\varepsilon\|_{H^{m-2}}^2.\tag{5.32}$$

With respect to the last term $\varepsilon^2\langle\partial^\alpha\partial_t\partial^\alpha q^\varepsilon,\nabla\partial^\alpha\tilde\rho_a^\varepsilon\rangle$, we use a similar argument as in (5.19) to get

$$\begin{aligned}&\varepsilon^2\langle\partial^\alpha\partial_t q^\varepsilon,\partial^\alpha\operatorname{div}\tilde\rho_a^\varepsilon\rangle\\ \le&\frac18p'(1)\|\nabla\partial^\alpha\tilde\rho_a^\varepsilon\|_{L^2}^2-\frac{d}{dt}\langle\operatorname{div}\partial^\alpha\tilde q_a^\varepsilon,\partial^\alpha\tilde\rho_a^\varepsilon\rangle\\ &+\nu\|\partial^\alpha\tilde q_a^\varepsilon\|_{L^2}^2+C\nu^{-1}\varepsilon^2\|\partial_t\nabla\tilde\rho_a^\varepsilon\|_{L^2}^2+C\varepsilon^4\|\partial_t\partial^\alpha(q_*+\varepsilon q_1)\|_{L^2}^2.\end{aligned}\tag{5.33}$$

Recall that δ and δ_* are suitably small, and $\partial_t(q_*+\varepsilon q_1)$ and $\partial_t\nabla\tilde\rho^\varepsilon$ are uniformly bounded in $L^2(\mathbb{R}^+;H^{m-2})$ due to (5.21) and (5.22). Then, after a computation similar to (5.4), we integrate (5.27) in time, make use of (5.9) and (5.21), and then derive

$$\sup_{t\in\mathbb{R}^+}\|\nabla\tilde\rho_a^\varepsilon(t)\|_{H^{m-2}}^2+\int_0^\infty\|\nabla^2\partial^\alpha\tilde\rho_a^\varepsilon(t)\|_{L^2}^2\,dt\le C(1+\nu^{-1})\varepsilon^4+\nu\int_0^\infty\|\nabla\tilde q_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt.$$

Going back to (5.24) and performing similar calculations for (5.25), we also have the desired estimates of $\tilde q_a^\varepsilon$. Finally, choosing a suitably small $\nu>0$ leads to the expected convergence rate; the details are omitted. □

6. The Euler-Maxwell system: Proof of Theorem 2.3

6.1. The error equations.

Let $(\rho^\varepsilon,u^\varepsilon,E^\varepsilon,B^\varepsilon)$ and (ρ^*,u^*,E^*,B^*) be the global solutions to (1.12) and (1.14) given by Propositions 2.3 and 2.4, respectively. Introducing the error unknowns

$$(\tilde\rho^\varepsilon,\tilde q^\varepsilon,\tilde E^\varepsilon,\tilde B^\varepsilon)=(\rho^\varepsilon-\rho^*,q^\varepsilon-q^*,E^\varepsilon-E^*,B^\varepsilon-\bar B),$$

we have the error system

$$
\begin{cases}
\partial_t\tilde{\rho}^\varepsilon + \operatorname{div}\tilde{q}^\varepsilon = 0, \\
\varepsilon^2\partial_t\tilde{q}^\varepsilon + \tilde{q}^\varepsilon + \tilde{\rho}^\varepsilon E^\varepsilon + \rho^*\tilde{E}^\varepsilon + \nabla\big(p(\rho^\varepsilon) - p(\rho^*)\big) \\
\qquad = -\varepsilon q^\varepsilon \times B^\varepsilon - \varepsilon^2\partial_t q^* - \varepsilon^2 \operatorname{div}\Big(\dfrac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon}\Big), \\
\partial_t\tilde{E}^\varepsilon - \dfrac{1}{\varepsilon}\nabla\times\tilde{B}^\varepsilon = -\partial_t E^* + \tilde{q}^\varepsilon, \\
\partial_t\tilde{B}^\varepsilon + \dfrac{1}{\varepsilon}\nabla\times\tilde{E}^\varepsilon = 0, \\
\operatorname{div}\tilde{E}^\varepsilon = -\tilde{\rho}^\varepsilon, \qquad \operatorname{div}\tilde{B}^\varepsilon = 0.
\end{cases} \tag{6.1}
$$

Due to the low-order term $-\partial_t E^*$ in the third equation, it is difficult to establish the desired convergence rate of $(\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon)$. In order to overcome this difficulty, we construct the asymptotic expansion

$$\big(\rho_a^\varepsilon, q_a^\varepsilon, E_a^\varepsilon, B_a^\varepsilon\big) = \big(\rho^*, q^*, E^*, \bar{B}\big) + \varepsilon\big(\rho_1, q_1, E_1, B_1\big).$$

Looking for the equations satisfied by (ρ_1, q_1, E_1, B_1), we formally see that

$$
\begin{aligned}
p(\rho_a^\varepsilon) = p(\rho^* + \varepsilon\rho_1) = p(\rho^*) + \varepsilon p'(\rho^*)\rho_1 + \frac{1}{2}\varepsilon^2 p''(\rho^*)\rho_1^2 + O(\varepsilon^3), \\
\rho_a^\varepsilon E_a^\varepsilon = \rho^* E^* + \varepsilon\big(\rho^* E_1 + \rho_1 E^*\big) + \varepsilon^2\big(\rho^* E_2 + \rho_1 E_1\big) + O(\varepsilon^3), \\
\varepsilon q_a^\varepsilon \times B_a^\varepsilon = \varepsilon q^* \times \bar{B} + \varepsilon^2(q^*\times B_1 + q_1 \times \bar{B}) + O(\varepsilon^3),
\end{aligned}
$$

and

$$\varepsilon^2 \operatorname{div}\Big(\frac{q_a^\varepsilon\otimes q_a^\varepsilon}{\rho_a^\varepsilon}\Big) = \varepsilon^2 \operatorname{div}\Big(\frac{q^*\otimes q^*}{\rho^*}\Big) + O(\varepsilon^3).$$

Inserting these expressions into (1.12) and identifying the coefficients in terms of the powers of ε, we obtain

$$
\begin{cases}
\partial_t\rho_1 + \operatorname{div} q_1 = 0, \\
q_1 + \nabla\big(p'(\rho^*)\rho_1\big) = -\big(\rho^* E_1 + \rho_1 E^*\big) - q^*\times\bar{B}, \\
\nabla\times B_1 = \partial_t E^* - q^*, \qquad \operatorname{div} E_1 = -\rho_1, \\
\nabla\times E_1 = 0, \qquad \operatorname{div} B_1 = 0.
\end{cases} \tag{6.2}
$$

Note that E_1 is given by $-\Delta E_1 = \nabla\times\nabla E_1 - \nabla\operatorname{div} E_1 = \nabla\rho_1$, or, equivalently,

$$E_1 = \Lambda^{-2}\nabla\rho_1. \tag{6.3}$$

Since $(\rho^*, q^*, E^*, \bar{B})$ is known by (1.14), we have

$$\partial_t E^* = \Lambda^{-2}\nabla\partial_t\rho^* = -\Lambda^{-2}\nabla\operatorname{div} q^*.$$

Similarly, from (6.2) and (6.3), it is clear that

$$-\Delta B_1 = \nabla\times\nabla B_1 - \nabla\operatorname{div} B_1 = \nabla\times(\partial_t E^* - q^*) = -\nabla\times q^*.$$

Consequently, we have

$$B_1 = -\Lambda^{-2}\nabla\times q^*. \tag{6.4}$$

Then, ρ_1 satisfies the drift-diffusion type equation

$$\partial_t\rho_1 - \operatorname{div}\big(\nabla(p'(\rho^*)\rho_1) + \rho_1 E^* + \rho^*\nabla\Lambda^{-2}\rho_1\big) = \operatorname{div}(q^*\times\bar{B}). \tag{6.5}$$

Without loss of generality, (ρ_1, q_1, E_1, B_1) is supplemented by the initial datum $\rho_1|_{t=0} = 0$, which implies

$$q_1|_{t=0} = -q^*|_{t=0} \times \bar{B}, \quad E_1|_{t=0} = 0, \quad B_1|_{t=0} = -\Lambda^{-2}\nabla \times q^*|_{t=0}. \tag{6.6}$$

Then, by (1.12) and (6.2), $(\rho_a^\varepsilon, q_a^\varepsilon, E_a^\varepsilon, B_a^\varepsilon)$ solves

$$\begin{cases} \partial_t \rho_a^\varepsilon + \operatorname{div} q_a^\varepsilon = 0, \\ \varepsilon^2 \partial_t q_a^\varepsilon + q_a^\varepsilon + \nabla\big(p(\rho^*) + \varepsilon p'(\rho^*)\rho_1\big) = -\rho^* E_a^\varepsilon - \varepsilon \rho_1 E^* - \varepsilon q^* \times \bar{B} - \varepsilon^2 \operatorname{div}\left(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}\right), \\ \partial_t E_a^\varepsilon - \dfrac{1}{\varepsilon}\nabla \times B_a^\varepsilon = q_a^\varepsilon + \varepsilon\big(\partial_t E_1 - q_1\big), \\ \partial_t B_a^\varepsilon + \dfrac{1}{\varepsilon}\nabla \times E_a^\varepsilon = \varepsilon \partial_t B_1, \\ \operatorname{div} E_a^\varepsilon = 1 - \rho_a^\varepsilon \qquad \operatorname{div} B_a^\varepsilon = 0. \end{cases} \tag{6.7}$$

Recall the initial layer correction q_I^ε defined in (2.26). In order to derive the convergence rate ε of $(\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon)$, we turn to analyze the error

$$(\tilde{\rho}_a^\varepsilon, \tilde{q}_a^\varepsilon, \tilde{E}_a^\varepsilon, \tilde{B}_a^\varepsilon) = (\rho^\varepsilon - \rho_a^\varepsilon, q^\varepsilon - q_a^\varepsilon - q_I^\varepsilon, E^\varepsilon - E_a^\varepsilon, B^\varepsilon - B_a^\varepsilon), \tag{6.8}$$

such that (6.1) becomes

$$\begin{cases} \partial_t \tilde{\rho}_a^\varepsilon + \operatorname{div} \tilde{q}_a^\varepsilon = J_1^\varepsilon, \\ \varepsilon^2 \partial_t \tilde{q}_a^\varepsilon + \tilde{q}_a^\varepsilon + p'(1)\nabla \tilde{\rho}_a^\varepsilon + \tilde{E}_a^\varepsilon = J_0^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon, \\ \partial_t \tilde{E}_a^\varepsilon - \dfrac{1}{\varepsilon}\nabla \times \tilde{B}_a^\varepsilon = \tilde{q}_a^\varepsilon + J_5^\varepsilon \\ \partial_t \tilde{B}_a^\varepsilon + \dfrac{1}{\varepsilon}\nabla \times \tilde{E}_a^\varepsilon = J_6^\varepsilon, \\ \operatorname{div} \tilde{E}_a^\varepsilon = -\tilde{\rho}_a^\varepsilon, \qquad \operatorname{div} \tilde{B}_a^\varepsilon = 0, \end{cases} \tag{6.9}$$

with

$$J_0^\varepsilon = \big(p'(\rho^\varepsilon) - p'(\rho_a^\varepsilon)\big)\nabla \rho_a^\varepsilon + \big(p'(\rho^\varepsilon) - p'(1)\big)\nabla \tilde{\rho}_a^\varepsilon - \tilde{\rho}_a^\varepsilon E^\varepsilon - (\rho^* - 1)\tilde{E}_a^\varepsilon, \tag{6.10}$$

and

$$\begin{cases} J_1^\varepsilon = \operatorname{div} q_I^\varepsilon, \\ J_2^\varepsilon = -\varepsilon^2 \operatorname{div}\left(\dfrac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}\right) - \nabla\big(p(\rho_a^\varepsilon) - p(\rho^*) - \varepsilon p'(\rho^*)\rho_1\big), \\ J_3^\varepsilon = \varepsilon\big(\rho_1(E^\varepsilon - E^*) - q^\varepsilon \times B^\varepsilon + q^* \times \bar{B}\big), \\ J_4^\varepsilon = -\varepsilon^2 \partial_t q^* - \varepsilon^3 \partial_t q_1, \\ J_5^\varepsilon = \varepsilon(\partial_t E_1 - q_1), \\ J_6^\varepsilon = \varepsilon \partial_t B_1. \end{cases} \tag{6.11}$$

Note that all the sources on the right-hand side of (6.9) have the order ε, so it is possible to derive the $\mathcal{O}(\varepsilon)$-bounds from (6.9).

6.2. **Regularities of the profiles.** Before deriving bounds for (ρ_1, q_1, E_1, B_1), we need the L^2 regularity of ρ^* and E^*. Due to $E^* = \Lambda^{-2}\nabla \rho^*$, assuming that $\rho_0^* - 1 \in \dot{H}^{-1}$ seems necessary. We have the following lemma.

Lemma 6.1. *Let (ρ^*, E^*, q^*) be given by Proposition 2.4 and assume that $\rho_0^* - 1 \in \dot{H}^{-1}$. We have (2.23) and*

$$\sup_{t\in\mathbb{R}^+} \left(\|\rho^*(t)-1\|_{\dot{H}^{-1}}^2 + \|q^*(t)\|_{L^2}^2 + \|E^*(t)\|_{L^2}^2\right) + \int_0^\infty \left(\|\rho^*(t)-1\|_{\dot{H}^{-1}}^2 + \|q^*(t)\|_{L^2}^2 + \|E^*(t)\|_{L^2}^2\right) dt \leq C\|\rho_0^*-1\|_{\dot{H}^{-1}\cap H^m}^2. \tag{6.12}$$

Proof. Recall that $\phi^* = \Lambda^{-2}(\rho^*-1)$ and $E^* = \Lambda^{-2}\nabla(\rho^*-1)$. Applying Λ^{-1} to (1.15) gives

$$\begin{aligned} &\partial_t \Lambda^{-1}(\rho^*-1) - p'(1)\Delta\Lambda^{-1}(\rho^*-1) + \Lambda^{-1}(\rho^*-1) \\ &\quad = -\Lambda^{-1}\operatorname{div}\left((p'(\rho^*)-p'(1))\nabla\rho^* + (\rho^*-1)\Lambda^{-2}\nabla\rho^*\right). \end{aligned} \tag{6.13}$$

Performing L^2 energy estimates on (6.13) and using the fact that $\Lambda^{-1}\operatorname{div}$ is a bounded operator from L^2 to L^2, we have

$$\begin{aligned} &\frac{d}{dt}\|\Lambda^{-1}(\rho^*-1)\|_{L^2}^2 + 2p'(1)\|\rho^*-1\|_{L^2}^2 + \|\Lambda^{-1}(\rho^*-1)\|_{L^2}^2 \\ &\leq C\|(p'(\rho^*)-p'(1))\nabla\rho^*\|_{L^2}^2 + C\|(\rho^*-1)\Lambda^{-2}\nabla\rho^*\|_{L^2}^2 \\ &\leq C\|\rho^*-1\|_{L^\infty}^2\left(\|\nabla\rho^*\|_{L^2}^2 + \|\Lambda^{-2}\nabla\rho^*\|_{L^2}^2\right) \\ &\leq C\delta_1^*\|\nabla\rho^*\|_{L^2}^2 + C\|\rho^*-1\|_{H^{m-1}}^2\|\rho^*-1\|_{\dot{H}^{-1}}^2), \end{aligned} \tag{6.14}$$

from which and (2.23) we obtain

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+}\|\Lambda^{-1}\rho^*(t)-1\|_{L^2}^2 + \int_0^\infty \left(\|\rho^*(t)-1\|_{L^2}^2 + \|\Lambda^{-1}\rho^*(t)-1\|_{L^2}^2\right) dt \\ &\leq Ce^{\int_0^\infty \|\rho^*(t)\|_{H^{m-1}}^2\,dt}\Big(\|\Lambda^{-1}(\rho_0^*-1)\|_{L^2}^2 + \int_0^\infty \|\nabla\rho^*(t)\|_{L^2}^2\,dt\Big) \\ &\leq C\|\rho_0^*-1\|_{\dot{H}^{-1}\cap H^m}^2. \end{aligned} \tag{6.15}$$

Since $E^* = \Lambda^{-2}\nabla(\rho^*-1)$, (6.15) directly implies

$$\sup_{t\in\mathbb{R}^+}\|E^*(t)\|_{L^2}^2 + \int_0^\infty \|E^*(t)\|_{L^2}^2\,dt \leq C\|\rho^*-1\|_{\dot{H}^{-1}\cap H^m}^2. \tag{6.16}$$

Furthermore, employing (2.23), (6.16) and $q^* = -\nabla p(\rho^*) - \rho^* E^*$, we arrive at

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+}\|q^*(t)\|_{L^2}^2 + \int_0^\infty \|q^*(t)\|_{L^2}^2\,dt \\ &\leq C(1+\sup_{t\in\mathbb{R}^+}\|\rho^*(t)-1\|_{H^m}^2)\Big(\sup_{t\in\mathbb{R}^+}\left(\|\rho^*(t)-1\|_{H^1}^2 + \|E^*(t)\|_{L^2}^2\right) \\ &\quad + \int_0^\infty \left(\|\rho^*(t)-1\|_{H^1}^2 + \|E^*(t)\|_{L^2}^2\right) dt\Big) \\ &\leq C\|\rho_0^*-1\|_{\dot{H}^{-1}\cap H^m}^2. \end{aligned} \tag{6.17}$$

Combining the above estimates (6.15), (6.16) and (6.17) yields (6.12). □

6.3. **Uniform estimates for** (ρ_1, q_1, E_1, B_1)**.** We have the following lemma.

Lemma 6.2. *The Cauchy problem of (6.5) with $\rho_1|_{t=0} = 0$ admits a unique global solution $\rho_1 \in C(\mathbb{R}^+; H^m)$. Let (q_1, E_1, B_1) be determined by $(6.2)_2$, (6.3) and (6.4). Then, it holds that*

$$\sup_{t\in\mathbb{R}^+}\left(\|\rho_1(t)\|_{\dot{H}^{-1}\cap H^m}^2 + \|q_1(t)\|_{H^{m-1}}^2 + \|E_1(t)\|_{H^{m+1}}^2 + \|B_1(t)\|_{H^{m-1}}^2\right) + \int_0^\infty \left(\|\rho_1(t)\|_{\dot{H}^{-1}\cap H^{m+1}}^2 + \|q_1(t)\|_{H^m}^2 + \|E_1(t)\|_{H^{m+1}}^2 + \|B_1(t)\|_{H^{m+1}}^2\right) dt \leq C_*, \tag{6.18}$$

where C_* is a positive constant depending on ρ_0^*.

Proof. Note that (6.5) can be rewritten as

$$\begin{aligned}&\partial_t\rho_1 - P'(1)\Delta\rho_1 + \rho_1\\ &\quad = -\operatorname{div}\big(p'(\rho^*) - P'(1))\nabla\rho_1\big) + \operatorname{div}\big(q^*\times\bar{B} + \rho_1 E^* - (\rho^*-1)\Lambda^{-2}\nabla\rho_1\big),\end{aligned} \tag{6.19}$$

with the initial datum $\rho_1|_{t=0} = 0$. According to the standard theorem for parabolic systems, there is a unique solution $\rho_1 \in C(\mathbb{R}^+; H^m)$ to (6.19) with $\rho_1|_{t=0} = 0$. Then, similarly to (6.14)-(6.15), we have

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\|\Lambda^{-1}\rho_1(t)\|_{L^2}^2 + \int_0^\infty \big(\|\Lambda^{-1}\rho_1(t)\|_{L^2}^2 + \|\rho_1(t)\|_{L^2}^2\big)\,dt\\
&\le \int_0^\infty \big(\|(\rho^*-1)\nabla\Lambda^{-2}\rho_1(t)\|_{L^2}^2 + \|\rho_1 E^*(t)\|_{L^2}^2 + \|q^*(t)\times\bar{B}\|_{L^2}^2\big)\,dt\\
&\le C\int_0^\infty \Big(\|\rho^*(t)-1\|_{L^\infty}^2\|\nabla\rho_1(t)\|_{L^2}^2 + \|\rho^*(t)-1\|_{L^3}^2\|\nabla\Lambda^{-2}\rho_1(t)\|_{L^6}^2\\
&\quad + \|\rho_1(t)\|_{L^3}^2\|E^*(t)\|_{L^6}^2 + \|q^*(t)\|_{L^2}^2\Big)\,dt\\
&\le \frac{1}{4}\int_0^\infty \|\rho_1(t)\|_{H^1}^2\,dt\\
&\quad + C\int_0^\infty \big((\|\rho^*(t)-1\|_{H^{m-1}}^2 + \|\nabla E^*(t)\|^2)\|\rho_1(t)\|_{H^1}^2 + \|q^*(t)\|_{L^2}^2\big)\,dt,\end{aligned} \tag{6.20}$$

where the embeddings $H^1 \hookrightarrow L^3$, $\dot{H}^1 \hookrightarrow L^6$ and $H^{m-1} \hookrightarrow L^\infty$ have been used. Concerning the higher-order estimates, one deduces from (3.4) and (3.6) that

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\|\nabla\rho_1(t)\|_{H^{m-1}}^2 + \int_0^\infty \|\nabla\rho_1(t)\|_{H^m}^2\,dt\\
&\le C\int_0^\infty \Big(\|(p'(\rho^*) - P'(1))\nabla\rho_1(t)\|_{H^{m-1}} + \|(\rho^*-1)\nabla\Lambda^{-2}\rho_1(t)\|_{H^{m-1}}\\
&\quad + \|\rho_1 E^*(t)\|_{H^{m-1}} + \|q^*(t)\times\bar{B}\|_{H^{m-1}}\Big)\|\nabla\rho_1(t)\|_{H^m}\,dt\\
&\le \frac{1}{4}\int_0^\infty \|\nabla\rho_1(t)\|_{H^m}^2\,dt\\
&\quad + C\int_0^\infty \Big(\big(\|\rho^*(t)-1\|_{H^{m-1}}^2 + \|E^*(t)\|_{H^{m-1}}^2\big)\|\rho_1(t)\|_{H^{m-1}}^2 + \|q^*(t)\|_{H^{m-1}}^2\Big)\,dt.\end{aligned} \tag{6.21}$$

Adding (6.20) and (6.21) together, using Grönwall's lemma, (2.23) and (6.12), we infer that

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\|\rho_1(t)\|_{\dot{H}^{-1}\cap H^m}^2 + \int_0^\infty \|\rho_1(t)\|_{\dot{H}^{-1}\cap H^{m+1}}^2\,dt\\
&\le Ce^{C\int_0^\infty(\|\rho^*(t)-1\|_{H^{m-1}}^2 + \|\nabla E^*(t)\|_{H^{m-2}}^2)\,dt}\int_0^\infty \|q^*(t)\|_{H^{m-1}}^2\,dt \le C_*.\end{aligned} \tag{6.22}$$

This, combined with $E_1 = \Lambda^{-2}\nabla\rho_1$, gives rise to

$$\sup_{t\in\mathbb{R}^+}\|E_1(t)\|_{H^{m+1}}^2 + \int_0^\infty \|E_1(t)\|_{H^{m+2}}^2\,dt \le C_*. \tag{6.23}$$

For B_1, note that

$$B_1 = -\Lambda^{-2}\nabla\times q^* = -\Lambda^{-2}\nabla\times\big((\rho^*-1)\Lambda^{-2}\nabla(\rho^*-1)\big).$$

As $\|\Lambda^{-2}\nabla\times\cdot\|_{L^2}\leq C\|\cdot\|_{\dot{H}^{-1}}\leq C\|\cdot\|_{L^{\frac{6}{5}}}$ due to $L^{\frac{6}{5}}\hookrightarrow\dot{H}^{-1}$, one deduces from (2.23) that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\|B_1(t)\|_{L^2}^2+\int_0^\infty\|B_1(t)\|_{L^2}^2\,dt\\
&\leq C\sup_{t\in\mathbb{R}^+}\|(\rho^*-1)\Lambda^{-2}\nabla(\rho^*-1)(t)\|_{L^{\frac{6}{5}}}^2+C\int_0^\infty\|(\rho^*-1)\Lambda^{-2}\nabla(\rho^*-1)(t)\|_{L^{\frac{6}{5}}}^2\,dt\\
&\leq C\sup_{t\in\mathbb{R}^+}\big(\|\rho^*(t)-1\|_{L^3}^2\|\Lambda^{-2}\nabla\times(\rho^*(t)-1)\|_{L^2}^2\big)\\
&\quad+\sup_{t\in\mathbb{R}^+}\|\rho^*(t)-1\|_{L^3}^2\int_0^\infty\|\Lambda^{-2}\nabla\times(\rho^*(t)-1)\|_{L^2}^2\,dt\\
&\leq C\sup_{t\in\mathbb{R}^+}\big(\|\rho^*(t)-1\|_{H^m}^2\|\rho^*(t)-1\|_{\dot{H}^{-1}}^2\big)+\sup_{t\in\mathbb{R}^+}\|\rho^*(t)-1\|_{H^m}^2\int_0^\infty\|\rho^*(t)-1\|_{\dot{H}^{-1}}^2\,dt\leq C_*.
\end{aligned}
$$

Similarly, it follows that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\|\nabla B_1(t)\|_{\dot{H}^m}^2+\int_0^\infty\|\nabla B_1(t)\|_{H^{m-1}}^2\,dt\\
&\leq C\sup_{t\in\mathbb{R}^+}\Big(\|\rho^*(t)-1\|_{H^m}^2\|\rho^*(t)-1\|_{H^m}^2\big)+\sup_{t\in\mathbb{R}^+}\|\rho^*(t)-1\|_{H^m}^2\int_0^\infty\|\rho^*(t)-1\|_{H^m}^2\,dt\leq C_*.
\end{aligned}
$$

Noting that $q_1=-\nabla\big(p'(\rho^*)\rho_1\big)-\big(\rho^*E_1+\rho_1E^*\big)-q^*\times\bar{B}$, we obtain

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\|q_1(t)\|_{H^{m-1}}\\
&\leq C\sup_{t\in\mathbb{R}^+}\big(\|p'(\rho^*)\rho_1(t)\|_{H^m}+\|\rho^*E_1(t)\|_{H^{m-1}}+\|\rho_1E^*(t)\|_{H^{m-1}}+\|q^*(t)\|_{H^{m-1}}\big)\\
&\leq C\sup_{t\in\mathbb{R}^+}\Big(\big(1+\|\rho^*(t)-1\|_{H^m}+\|\rho_1(t)\|_{H^m}\big)\big(\|\rho_1(t)\|_{H^{m-1}}+\|E_1(t)\|_{H^{m-1}}+\|E^*(t)\|_{H^{m-1}}\big)\\
&\quad+\|q^*(t)\|_{H^{m-1}}\Big)\leq C_*,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty\|q_1(t)\|_{H^m}^2\,dt\\
&\leq C\int_0^\infty\Big(\big(1+\|\rho^*(t)-1\|_{H^m}^2+\|\rho_1(t)\|_{H^m}^2\big)\big(\|\rho_1(t)\|_{H^m}^2+\|E_1(t)\|_{H^m}^2+\|E^*(t)\|_{H^m}^2\big)\\
&\quad+\|q^*(t)\|_{H^m}^2\Big)\,dt\leq C_*.
\end{aligned}
$$

Combining these estimates, we end up with (6.18) and finish the proof of Lemma 6.2. □

The convergence estimates of the remainder terms J_i^ε $(i=1,2,3,4,5)$ are as follows.

Lemma 6.3. *Let J_i $(i=1,2,3,4,5)$ be given by (6.11). Then, we have*

$$\int_0^\infty\|J_1^\varepsilon(t)\|_{H^{m-1}}^2\,dt\leq C,\qquad\int_0^\infty\|J_1^\varepsilon(t)\|_{H^{m-1}}\,dt\leq C\varepsilon,\tag{6.24}$$

$$\int_0^\infty\big(\|J_2^\varepsilon(t)\|_{H^{m-1}}^2+\|J_3^\varepsilon(t)\|_{H^m}^2+\|J_5^\varepsilon(t)\|_{H^m}^2\big)\leq C\varepsilon^2,\tag{6.25}$$

$$\int_0^\infty\|J_4^\varepsilon(t)\|_{H^{m-2}}^2\leq C\varepsilon^4,\tag{6.26}$$

$$\int_0^\infty\|J_6^\varepsilon(t)\|_{H^m}^2\,dt\leq C\varepsilon^2,\qquad\int_0^\infty\|J_6^\varepsilon(t)\|_{H^m}\,dt\leq C\varepsilon,\tag{6.27}$$

where $C > 0$ is a constant independent of ε.

Proof. The estimates concerning J_1^ε, J_2^ε and J_3^ε are similar to that of Lemma 4.1, so we omit the details. Recall that

$$q^* = -\nabla p(\rho^*) - \rho^* E^*, \qquad E^* = \Lambda^{-2}\nabla(\rho^* - 1),$$

$$q_1 = -\nabla\big(p'(\rho^*)\rho_1\big) - \rho^* E_1 - \rho_1 E^* - q^* \times \bar{B}, \qquad E_1 = \Lambda^{-2}\nabla\rho_1.$$

Thus, in accordance with (1.14) and (6.2), we have

$$\partial_t q^* = \partial_t\big(-\nabla p(\rho^*) - \rho^* E^*\big) = \nabla(p'(\rho^*)\operatorname{div} q^*) + \operatorname{div} q_1 E^* + \rho^*\Lambda^{-2}\nabla \operatorname{div} q_1,$$

and

$$\begin{aligned}\partial_t q_1 &= \partial_t\big(-\nabla(p'(\rho^*)\rho_1) - \rho^* E_1 - \rho_1 E^* - q^* \times \bar{B}\big)\\ &= -\nabla\big(p''(\rho^*)\rho_1 \operatorname{div} q_1 + p'(\rho^*)\operatorname{div} q_1\big) + \operatorname{div} q^* E_1 + \rho^*\Lambda^{-2}\nabla \operatorname{div} q_1\\ &\quad + \operatorname{div} q_1 E^* + \rho_1\Lambda^{-2}\nabla\operatorname{div} q^* - \partial_t q^* \times \bar{B}.\end{aligned}$$

Consequently, using (2.23) and (6.18), we have

$$\int_0^\infty \|J_4^\varepsilon(t)\|_{H^{m-2}}^2\,dt \le \varepsilon^4 \sup_{t\in\mathbb{R}^+}(\|\rho_1(t)\|_{H^m}^2)\int_0^\infty \big(\|q^*(t)\|_{H^m}^2 + \|q_1(t)\|_{H^m}^2\big)\,dt \le C\varepsilon^4, \tag{6.28}$$

and

$$\int_0^\infty \|J_4^\varepsilon(t)\|_{H^{m-2}}^2\,dt \le \varepsilon^4 \int_0^\infty \big(\|q^*(t)\|_{H^m}^2 + \|q_1(t)\|_{H^m}^2\big)\,dt \le C\varepsilon^4. \tag{6.29}$$

As

$$J_5^\varepsilon = \varepsilon(\partial_t E_1 - q_1) = \varepsilon(\Lambda^{-2}\nabla\operatorname{div} q_1 - q_1),$$

one verifies that

$$\int_0^\infty \|J_5^\varepsilon(t)\|_{H^m}^2\,dt \le C\varepsilon^2\int_0^\infty \|q_1(t)\|_{H^m}^2\,dt \le C\varepsilon^2. \tag{6.30}$$

Finally,

$$J_6^\varepsilon = \varepsilon\partial_t B_1 = -\varepsilon\Lambda^{-2}\nabla\times\partial_t q^* = \varepsilon\Lambda^{-2}\nabla\times\big(\operatorname{div} q_1 E^* + (\rho^*-1)\Lambda^{-2}\nabla\operatorname{div} q_1\big).$$

It holds by (6.12), (6.18) and $L^{\frac{6}{5}} \hookrightarrow \dot{H}^{-1}$ that

$$\begin{aligned}\int_0^\infty \|J_6^\varepsilon(t)\|_{L^2}^2\,dt &\le C\varepsilon^2\int_0^\infty \big(\|\operatorname{div} q_1 E^*(t)\|_{\dot{H}^{-1}}^2 + \|(\rho^*-1)\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{\dot{H}^{-1}}^2\big)\,dt\\ &\le C\varepsilon^2\int_0^\infty \big(\|\operatorname{div} q_1 E^*(t)\|_{L^{\frac{6}{5}}}^2 + \|(\rho^*-1)\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{L^{\frac{6}{5}}}^2\big)\,dt\\ &\le C\varepsilon^2\int_0^\infty \big(\|\operatorname{div} q_1(t)\|_{L^3}^2\|E^*(t)\|_{L^2}^2 + \|\rho^*(t)-1\|_{L^3}^2\|\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{L^2}^2\big)\,dt\\ &\le C\varepsilon^2\int_0^\infty \|q_1(t)\|_{H^1}^2\,dt \le C\varepsilon^2.\end{aligned}$$

For the higher-order estimates, one also deduces from (3.3) and $H^{m-1}\hookrightarrow L^\infty$ that

$$\begin{aligned}\int_0^\infty \|J_6^\varepsilon(t)\|_{\dot H^{m+1}}^2\,dt &\le C\varepsilon^2\int_0^\infty \big(\|\operatorname{div} q_1 E^*(t)\|_{\dot H^m}^2+\|(\rho^*-1)\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{\dot H^m}^2\big)\,dt\\ &\le C\varepsilon^2\int_0^\infty \big(\|\operatorname{div} q_1(t)\|_{L^\infty}^2\|E^*(t)\|_{\dot H^m}^2+\|\operatorname{div} q_1(t)\|_{\dot H^m}^2\|E^*(t)\|_{L^\infty}^2\\ &\quad+\|\rho^*(t)-1\|_{L^\infty}^2\|\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{\dot H^m}^2+\|\rho^*(t)-1\|_{\dot H^m}^2\|\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{L^\infty}^2\big)\,dt\\ &\le C\varepsilon^2\int_0^\infty \big(\|E^*(t)\|_{H^m}^2+\|\rho^*(t)-1\|_{H^m}^2\big)\|q_1(t)\|_{H^m}^2\,dt\\ &\le C\varepsilon^2\int_0^\infty \|q_1(t)\|_{H^m}^2\,dt\le C\varepsilon^2.\end{aligned}$$

Consequently, we have the first estimate in (6.27). Similarly, $\|J_6^\varepsilon(t)\|_{H^m}$ has the L^1 time integrability:

$$\begin{aligned}\int_0^\infty \|J_6^\varepsilon(t)\|_{H^m}\,dt &\le C\varepsilon\int_0^\infty \big(\|\operatorname{div} q_1 E^*(t)\|_{\dot H^{-1}\cap\dot H^{m-1}}+\|(\rho^*-1)\Lambda^{-2}\nabla\operatorname{div} q_1(t)\|_{\dot H^{-1}\cap\dot H^{m-1}}\big)\,dt\\ &\le C\varepsilon\int_0^\infty \big(\|E^*(t)\|_{H^m}+\|\rho^*(t)-1\|_{H^m}\big)\|q_1(t)\|_{H^m}\big)\,dt\\ &\le C\varepsilon\Big(\int_0^\infty \big(\|E^*(t)\|_{H^m}^2+\|\rho^*(t)-1\|_{H^m}^2\big)\,dt\Big)^{\frac12}\Big(\int_0^\infty \|q_1(t)\|_{H^m}^2\,dt\Big)^{\frac12}\le C\varepsilon.\end{aligned}$$

This finishes the proof of Lemma 6.3. □

6.4. **Low-order error estimates.** We prove the error estimates in L^2 of the solutions $(\rho^\varepsilon,q^\varepsilon,E^\varepsilon,B^\varepsilon)$ and (ρ^*,q^*,E^*,B^*) associated with the systems (1.12) and (1.14), respectively.

Lemma 6.4. *Let $(\tilde\rho^\varepsilon,\tilde q^\varepsilon,\tilde E^\varepsilon,\tilde B^\varepsilon)=(\rho^\varepsilon-\rho^*,q^\varepsilon-q^*,E^\varepsilon-E^*,B^\varepsilon-\bar B)$. For all $t\ge0$, it holds*

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\big(\|\tilde\rho^\varepsilon(t)\|_{L^2}^2+\varepsilon^2\|\tilde q^\varepsilon(t)\|_{L^2}^2+\|\tilde E^\varepsilon(t)\|_{L^2}^2+\|\tilde B^\varepsilon(t)\|_{L^2}^2\big)\\ &\quad+\int_0^\infty\big(\|\tilde\rho^\varepsilon(t)\|_{H^1}^2+\|\tilde q^\varepsilon(t)\|_{L^2}^2+\|\tilde E^\varepsilon(t)\|_{L^2}^2\big)\,dt\\ &\le C(\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+\|E_0^\varepsilon-E_0^*\|_{L^2}^2+\|B_0^\varepsilon-B^e\|_{L^2}^2)+C\varepsilon^2,\end{aligned}\tag{6.31}$$

where $C>0$ is a uniform constant.

Proof. As emphasized in Subsection 6.1, our idea is to derive the estimates of $(\tilde\rho_a^\varepsilon,\tilde q_a^\varepsilon,\tilde E_a^\varepsilon,\tilde B_a^\varepsilon)$ defined in (6.8) and then recover the desired estimates for $(\tilde\rho^\varepsilon,\tilde q^\varepsilon,\tilde E^\varepsilon,\tilde B^\varepsilon)$ due to the bounds for profiles ρ_1, q_1, E_1 and B_1 obtained in Lemma 6.2.

Recall that equations $(6.9)_1$-$(6.9)_2$ are

$$\begin{cases}\partial_t\tilde\rho_a^\varepsilon+\operatorname{div}\tilde q_a^\varepsilon=J_1^\varepsilon,\\ \varepsilon^2\partial_t\tilde q_a^\varepsilon+\tilde q_a^\varepsilon+p'(1)\nabla\tilde\rho_a^\varepsilon+\tilde E_a^\varepsilon=J_0^\varepsilon+J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon.\end{cases}\tag{6.32}$$

Taking the inner products of $(6.32)_1$ and $(6.32)_2$ with $p'(1)\tilde\rho_a^\varepsilon$ and $\tilde\rho_a^\varepsilon$, respectively, we have

$$\begin{aligned}&\frac{d}{dt}\big(p'(1)\|\tilde\rho_a^\varepsilon\|_{L^2}^2+\varepsilon^2\|\tilde q_a^\varepsilon\|_{L^2}^2\big)+2\|\tilde q_a^\varepsilon\|_{L^2}^2+2\langle\tilde E_a^\varepsilon,\tilde q_a^\varepsilon\rangle\\ &=2p'(1)\langle J_1^\varepsilon,\tilde\rho_a^\varepsilon\rangle+2\langle J_0^\varepsilon+J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon,\tilde q_a^\varepsilon\rangle.\end{aligned}\tag{6.33}$$

To cancel the last term on the left-hand side of (6.33), we shall make use of the structure of (6.9). In fact, from $(6.9)_3$-$(6.9)_4$ one has

$$\frac{d}{dt}\Big(\|\tilde E_a^\varepsilon\|_{L^2}^2+\|\tilde B_a^\varepsilon\|_{L^2}^2\Big)-2\langle\tilde q_a^\varepsilon,\tilde E_a^\varepsilon\rangle=2\langle J_5^\varepsilon,\tilde E_a^\varepsilon\rangle+2\langle J_6^\varepsilon,\tilde B_a^\varepsilon\rangle.\tag{6.34}$$

Adding (6.33) with (6.34), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(p'(1)\|\tilde\rho_a^\varepsilon\|_{L^2}^2+\varepsilon^2\|\tilde q_a^\varepsilon\|_{L^2}^2+\|\tilde E_a^\varepsilon\|_{L^2}^2+\|\tilde B_a^\varepsilon\|_{L^2}^2\Big)+2\|\tilde q_a^\varepsilon\|_{L^2}^2\\
&= 2p'(1)\langle J_1^\varepsilon,\tilde\rho_a^\varepsilon\rangle+2\langle J_0^\varepsilon+J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon,\tilde q^\varepsilon\rangle+2\langle J_5^\varepsilon,\tilde E_a^\varepsilon\rangle+2\langle J_6^\varepsilon,\tilde B_a^\varepsilon\rangle\\
&\le \frac12\|\tilde q_a^\varepsilon\|_{L^2}^2+2p'(1)\|J_1^\varepsilon\|_{L^2}\|\tilde\rho_a^\varepsilon\|_{L^2}+C\nu^{-1}\|J_5^\varepsilon\|_{L^2}^2+\nu\|\tilde E_a^\varepsilon\|_{L^2}^2+2\|J_6^\varepsilon\|_{L^2}\|\tilde B_a^\varepsilon\|_{L^2}\\
&\quad +C(\|J_0^\varepsilon\|_{L^2}^2+\|J_2^\varepsilon\|_{L^2}^2+\|J_3^\varepsilon\|_{L^2}^2+\|J_4^\varepsilon\|_{L^2}^2).
\end{aligned}\tag{6.35}$$

Here, $\nu>0$ is a small constant to be chosen later. Similarly to that in (4.21), using the regularity estimate of $(\rho^\varepsilon,E^\varepsilon,B^\varepsilon)$, (ρ^*,E^*,B^*) and (ρ_1,E_1,B_1), we have

$$\begin{aligned}
\int_0^t\|J_0^\varepsilon(t')\|_{L^2}^2\,dt&\le C\sup_{t'\in[0,t]}\big(\|\nabla\rho_a^\varepsilon(t')\|_{L^\infty}^2+\|\rho^*(t')-1\|_{L^\infty}^2+\|\rho^\varepsilon(t')-1\|_{L^\infty}^2+\|E^\varepsilon(t')\|_{L^\infty}^2\big)\\
&\qquad\times\int_0^t\big(\|\tilde\rho_a(t')\|_{H^1}^2+\|\tilde E_a(t')\|_{L^2}^2\big)\,dt'\\
&\le C(\delta_1+\delta_1^*+\varepsilon)\int_0^t\big(\|\tilde\rho_a(t')\|_{H^1}^2+\|\tilde E_a(t')\|_{L^2}^2\big)\,dt'.
\end{aligned}\tag{6.36}$$

Thus, integrating (6.35) over $[0,t]$ and using (6.24)-(6.27) and (6.36), we arrive at

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\big(\|\tilde\rho_a^\varepsilon(t)\|_{L^2}^2+\varepsilon^2\|\tilde q_a^\varepsilon(t)\|_{L^2}^2+\|\tilde E_a^\varepsilon(t)\|_{L^2}^2+\|\tilde B_a^\varepsilon(t)\|_{L^2}^2\big)+\int_0^\infty\|\tilde q_a^\varepsilon(t)\|_{L^2}^2\,dt\\
&\le C\big(\|\tilde\rho_a^\varepsilon(0)\|_{L^2}^2+\varepsilon^2\|\tilde q_a^\varepsilon(0)\|_{L^2}^2+\|\tilde E_a^\varepsilon(0)\|_{L^2}^2+\|\tilde B_a^\varepsilon(0)\|_{L^2}^2\big)\\
&\quad+C(\delta_1+\delta_1^*+\varepsilon+\nu)\int_0^t\big(\|\tilde\rho_a(t')\|_{H^1}^2+\|\tilde E_a(t')\|_{L^2}^2\big)\,dt'\\
&\quad+C\int_0^\infty\big(\|J_2^\varepsilon(t)\|_{L^2}^2+\|J_3^\varepsilon(t)\|_{L^2}^2+\|J_4^\varepsilon(t)\|_{L^2}^2+\nu^{-1}\|J_5^\varepsilon(t)\|_{L^2}^2\big)\,dt\\
&\quad\quad+C\int_0^\infty\big(\|J_1^\varepsilon(t)\|_{L^2}+\|J_6^\varepsilon(t)\|_{L^2}\big)\,dt\,\sup_{t\in\mathbb{R}^+}\big(\|\tilde\rho_a^\varepsilon(t)\|_{L^2}+\|\tilde B_a^\varepsilon(t)\|_{L^2}\big)\\
&\le C\big(\|\tilde\rho_a^\varepsilon(0)\|^2+\varepsilon^2\|\tilde q^\varepsilon(0)\|_{L^2}^2+\|\tilde E_a^\varepsilon(0)\|^2+\|\tilde B_a^\varepsilon(0)\|^2\big)\\
&\quad+C(\delta_1+\delta_1^*+\varepsilon)\int_0^\infty\big(\|\tilde\rho_a(t)\|_{H^1}^2+\|\tilde E_a(t)\|_{L^2}^2\big)\,dt\\
&\quad+\Big(\delta_1+\delta_1^*+\varepsilon+\frac12\Big)\sup_{t\in\mathbb{R}^+}\big(\|\tilde\rho_a^\varepsilon(t)\|_{L^2}^2+\|\tilde B_a^\varepsilon(t)\|_{L^2}^2\big)+C(1+\nu^{-1})\varepsilon^2.
\end{aligned}$$

Note that

$$\big(\tilde\rho_a^\varepsilon(0),\tilde q_a^\varepsilon(0),\tilde E_a^\varepsilon(0),\tilde B_a^\varepsilon(0)\big)=\big(\rho_0^\varepsilon-\rho_0^*,\nabla p(\rho_0^*),E_0^\varepsilon-E_0^*,B_0^\varepsilon-B^e\big)+\varepsilon\big(0,-q^*|_{t=0}\times\bar B,-\Lambda^{-2}\nabla\times q^*|_{t=0}\big).$$

Using

$$q^*=-\nabla p(\rho^*)-\rho^*E^*,\qquad E^*=\Lambda^{-2}\nabla(\rho^*-1)=\nabla\Lambda^{-2}(\rho^*-1),$$

and the regularity condition $\rho_0^*-1\in\dot H^{-1}\cap H^m$, we know

$$q^*|_{t=0}=\nabla p(\rho_0^*)+\rho_0^*\nabla\Lambda^{-2}(\rho_0^*-1)\in L^2.$$

In addition, we have

$$\nabla\times q^*|_{t=0}=\nabla\times\big((\rho_0^*-1)\nabla\Lambda^{-2}(\rho_0^*-1)\big)$$

and

$$
\begin{aligned}
\|\Lambda^{-2}\nabla\times q^*|_{t=0})\|_{L^2} &\le C\|(\rho_0^*-1)\nabla\Lambda^{-2}(\rho_0^*-1)\|_{\dot{H}^{-1}}\\
&\le C\|(\rho_0^*-1)\nabla\Lambda^{-2}(\rho_0^*-1)\|_{L^{\frac{6}{5}}}\\
&\le C\|\rho_0^*-1\|_{L^3}\|\nabla\Lambda^{-2}(\rho_0^*-1)\|_{L^2}\\
&\le C\|\rho_0^*-1\|_{H^m}\|\rho_0^*-1\|_{\dot{H}^{-1}}, \qquad (6.37)
\end{aligned}
$$

due to $L^{\frac{6}{5}}\hookrightarrow\dot{H}^{-1}$. Consequently, it holds

$$
\begin{aligned}
&\|\tilde{\rho}_a^\varepsilon(0)\|_{L^2}+\|\tilde{q}_a^\varepsilon(0)\|_{L^2}^2+\|\tilde{E}_a^\varepsilon(0)\|_{L^2}^2+\|\tilde{B}_a^\varepsilon(0)\|_{L^2}\\
&\quad\le C\big(\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+\|E_0^\varepsilon-E_0^*\|_{L^2}^2+\|B_0^\varepsilon-B^e\|_{L^2}^2\big)+C\varepsilon^2.
\end{aligned}
$$

As δ_1, δ_1^* and ε are sufficiently small, the following estimate holds:

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\big(\|\tilde{\rho}_a^\varepsilon(t)\|_{L^2}^2+\varepsilon^2\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2+\|\tilde{E}_a^\varepsilon(t)\|_{L^2}^2+\|\tilde{B}_a^\varepsilon(t)\|_{L^2}^2\big)+\int_0^\infty\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2\,dt\\
&\le C\big(\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+\|E_0^\varepsilon-E_0^*\|_{L^2}^2+\|B_0^\varepsilon-B^e\|_{L^2}^2\big)+C(1+\nu^{-1})\varepsilon^2\\
&\quad+(\delta_1+\delta_1^*+\varepsilon+\nu)\int_0^\infty\big(\|\tilde{\rho}_a^\varepsilon(t)\|_{H^1}^2+\|\tilde{E}_a^\varepsilon(t)\|_{L^2}^2\big)\,dt. \qquad (6.38)
\end{aligned}
$$

Next, the dissipation estimates of $\tilde{\rho}_a^\varepsilon$ are established as follows. Taking the inner product of $(6.9)_2$ by $\nabla\tilde{\rho}_a^\varepsilon$ and making use of $(6.9)_1$ and $\operatorname{div}\tilde{E}_a^\varepsilon=-\tilde{\rho}_a^\varepsilon$, we have

$$
\begin{aligned}
&p'(1)\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2+\|\tilde{\rho}_a^\varepsilon\|_{L^2}^2\\
&=-\varepsilon^2\langle\partial_t\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle-\langle\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle+\langle J_0^\varepsilon+J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle\\
&=-\varepsilon^2\frac{d}{dt}\langle\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle+\varepsilon^2\|\operatorname{div}\tilde{q}_a^\varepsilon\|_{L^2}^2+\langle J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle-\varepsilon^2\langle\operatorname{div}\tilde{q}_a^\varepsilon,J_1^\varepsilon\rangle\\
&\le-\varepsilon^2\frac{d}{dt}\langle\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle+C\|\tilde{q}^\varepsilon\|_{L^2}^2+C\varepsilon^2\|\operatorname{div}\tilde{q}_a^\varepsilon\|_{L^2}^2+C\|J_0^\varepsilon\|_{L^2}^2+C\|J_2^\varepsilon\|_{L^2}^2+\|J_3^\varepsilon\|_{L^2}^2\\
&\quad+\|J_4^\varepsilon\|_{L^2}^2+C\varepsilon^2\|J_1^\varepsilon\|_{L^2}^2+\frac{p'(1)}{2}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2. \qquad (6.39)
\end{aligned}
$$

Thus, integrating (6.39) in time and recalling (6.36), we discover that

$$
\begin{aligned}
&\int_0^t\Big(\frac{p'(1)}{2}\|\nabla\tilde{\rho}_a^\varepsilon(t')\|_{L^2}^2+\|\tilde{\rho}_a^\varepsilon(t')\|_{L^2}^2\Big)\,dt'\\
&\le\varepsilon^2\langle\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle\Big|_0^t+C(\delta_1+\delta_1^*+\varepsilon)\int_0^t\big(\|\tilde{\rho}_a(t')\|_{H^1}^2+\|\tilde{E}_a(t')\|_{L^2}^2\big)\,dt'+C\int_0^t\|\tilde{q}_a^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\quad+C\int_0^t\big(\varepsilon^2\|\operatorname{div}\tilde{q}_a^\varepsilon(t')\|_{L^2}^2+\varepsilon^2\|J_1^\varepsilon(t')\|_{L^2}^2+\|J_2^\varepsilon(t')\|_{L^2}^2+\|J_3^\varepsilon(t')\|_{L^2}^2+\|J_4^\varepsilon(t')\|_{L^2}^2\big)\,dt'.
\end{aligned}
$$

As the bounds of $\rho^\varepsilon,\rho^*,\rho_1,q^\varepsilon,q^*$ and q_1 have been obtained, it holds

$$
\sup_{t\in\mathbb{R}^+}\|\tilde{\rho}_a^\varepsilon(t)\|_{H^m}^2+\int_0^\infty\big(\|\tilde{q}_a^\varepsilon(t)\|_{H^m}^2+\|\nabla\tilde{B}_a^\varepsilon(t)\|_{H^{m-1}}^2\big)\,dt\le C. \qquad (6.40)
$$

By (6.6), (6.40) and $\rho_1|_{t=0}=0$ we have

$$
\varepsilon^2\langle\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle\Big|_0^t\le\varepsilon^2\|\tilde{q}_a^\varepsilon\|_{L^2}\|\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}+\varepsilon^3\|q^*|_{t=0}\|_{L^2}\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}\le C\varepsilon^2\|\tilde{q}_a^\varepsilon\|_{L^2}^2+C\varepsilon^2. \qquad (6.41)
$$

Here we used

$$
\begin{aligned}
\|q^*|_{t=0}\|_{L^2}&\le\|\nabla p(\rho^*)\|_{L^2}+\|\rho_0^*E_0^*\|_{L^2}\\
&\le C\|\rho^*-1\|_{H^1}+C(1+\|\rho_0^*-1\|_{L^\infty})\|E_0^*\|_{L^2}\le C.
\end{aligned}
$$

Putting (6.24)-(6.25) and (6.52)-(6.41) into (6.40), we derive

$$\begin{aligned}
&\int_0^\infty \left(\|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{L^2}^2+\|\tilde{\rho}_a^\varepsilon(t)\|_{L^2}^2\right)dt\\
&\le C\sup_{t\in\mathbb{R}^+}\varepsilon^2\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2+C\int_0^\infty\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2\,dt'\\
&\quad+C(\delta_1+\delta_1^*+\varepsilon)\int_0^\infty\left(\|\tilde{\rho}_a(t)\|_{H^1}^2+\|\tilde{E}_a(t)\|_{L^2}^2\right)dt+C\varepsilon^2.
\end{aligned}\tag{6.42}$$

Another key point is to establish the dissipation estimate of the error $\tilde{E}_a^\varepsilon$ required in (6.38). Taking the inner product of $(6.9)_2$ by $\tilde{E}_a^\varepsilon$, making use of $(6.9)_3$ and keeping in mind that $\operatorname{div}\tilde{E}_a^\varepsilon=-\tilde{\rho}_a^\varepsilon$, we obtain

$$\begin{aligned}
&p'(1)\|\operatorname{div}\tilde{E}_a^\varepsilon\|_{L^2}^2+\|\tilde{E}_a^\varepsilon\|_{L^2}^2\\
&=-\varepsilon^2\langle\partial_t\tilde{q}_a^\varepsilon,\tilde{E}_a^\varepsilon\rangle-\langle\tilde{q}_a^\varepsilon,\tilde{E}_a^\varepsilon\rangle+\langle J_0^\varepsilon+J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon,\tilde{E}_a^\varepsilon\rangle\\
&=-\varepsilon^2\frac{d}{dt}\langle\tilde{q}_a^\varepsilon,\tilde{E}_a^\varepsilon\rangle-\langle\tilde{q}_a^\varepsilon,\tilde{E}_a^\varepsilon\rangle+\langle J_0^\varepsilon+J_2^\varepsilon+J_3^\varepsilon+J_4^\varepsilon,\tilde{E}_a^\varepsilon\rangle\\
&\quad+\varepsilon\langle\tilde{q}_a^\varepsilon,\nabla\times\tilde{B}_a^\varepsilon\rangle+\varepsilon^2\langle\tilde{q}_a^\varepsilon,J_5^\varepsilon\rangle\\
&\le-\varepsilon^2\frac{d}{dt}\langle\tilde{q}_a^\varepsilon,\tilde{E}_a^\varepsilon\rangle+C\|\tilde{q}_a^\varepsilon\|_{L^2}^2+C\varepsilon^2\|\tilde{q}_a^\varepsilon\|_{L^2}^2+C\varepsilon^2\|\nabla\times\tilde{B}_a^\varepsilon\|_{L^2}^2\\
&\quad+C\|J_0^\varepsilon\|_{L^2}^2+C\|J_2^\varepsilon\|_{L^2}^2+\|J_3^\varepsilon\|_{L^2}^2+\|J_4^\varepsilon\|_{L^2}^2+\varepsilon^2\|J_5^\varepsilon\|_{L^2}^2.
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^t\left(p'(1)\|\tilde{\rho}_a^\varepsilon(t')\|_{L^2}^2+\|\tilde{E}_a^\varepsilon(t')\|_{L^2}^2\right)dt'\\
&\le-\varepsilon^2\langle\tilde{q}_a^\varepsilon,\nabla\tilde{\rho}_a^\varepsilon\rangle\Big|_0^t+C\int_0^t\|\tilde{q}_a^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\quad+C\int_0^t\|J_0^\varepsilon(t')\|_{L^2}^2\,dt'\\
&\quad+\int_0^t\left(\varepsilon^2\|\nabla\tilde{B}_a^\varepsilon(t')\|_{L^2}^2+\|J_2^\varepsilon(t')\|_{L^2}^2+\|J_3^\varepsilon(t')\|_{L^2}^2+\|J_4^\varepsilon(t')\|_{L^2}^2+\varepsilon^2\|J_5(t')\|_{L^2}^2\right)dt',
\end{aligned}$$

which, together with (6.25)-(6.27), (6.36) and (6.40), leads to

$$\begin{aligned}
\int_0^\infty\|\tilde{E}_a^\varepsilon(t')\|_{L^2}^2\,dt'&\le C\sup_{t\in\mathbb{R}^+}\varepsilon^2\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2+C\int_0^\infty\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2\,dt'\\
&\quad+C(\delta_1+\delta_1^*)\int_0^\infty\left(\|\tilde{\rho}_a^\varepsilon(t)\|_{H^1}^2+\|\tilde{E}_a^\varepsilon(t)\|_{L^2}^2\right)dt+C\varepsilon^2.
\end{aligned}\tag{6.43}$$

Finally, we let $(6.38)+\eta(6.42)+\eta(6.43)$ with a uniform small constant $0<\eta<<1$ and then obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\left(\|\tilde{\rho}_a^\varepsilon(t)\|_{L^2}^2+\varepsilon^2\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2+\|\tilde{E}_a^\varepsilon(t)\|_{L^2}^2+\|\tilde{B}_a^\varepsilon(t)\|_{L^2}^2\right)\\
&\quad+\int_0^\infty\left(\|\tilde{\rho}_a^\varepsilon(t)\|_{H^1}^2+\|\tilde{q}_a^\varepsilon(t)\|_{L^2}^2+\|\tilde{E}_a^\varepsilon(t)\|_{L^2}^2\right)dt\\
&\le C(\|\rho_0^\varepsilon-\rho_0^*\|_{L^2}^2+\|E_0^\varepsilon-E_0^*\|_{L^2}^2+\|B_0^\varepsilon-B^e\|_{L^2}^2)+C(1+\nu^{-1})\varepsilon^2\\
&\quad+C(\delta_1+\delta_1^*+\varepsilon+\nu)\int_0^\infty\left(\|\tilde{\rho}_a^\varepsilon(t)\|_{H^1}^2+\|\tilde{E}_a^\varepsilon(t)\|_{L^2}^2\right)dt.
\end{aligned}$$

Let $\varepsilon_0 \in (0,1)$ be a suitably small constant. In the case $\varepsilon \geq \varepsilon_0$, one easily gets (6.31) using the uniform estimates of $(\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon)$. When $\varepsilon \leq \varepsilon_0$, choosing a suitably small constant $\nu > 0$ and making use of the smallness of δ^1, δ^1_* and ε, we conclude that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \big(\|\tilde{\rho}^\varepsilon_a(t)\|^2_{L^2} + \varepsilon^2\|\tilde{q}^\varepsilon_a(t)\|^2_{L^2} + \|\tilde{E}^\varepsilon_a(t)\|^2_{L^2} + \|\tilde{B}^\varepsilon_a(t)\|^2_{L^2}\big) \\
&\quad + \int_0^\infty \big(\|\tilde{\rho}^\varepsilon_a(t)\|^2_{H^1} + \|\tilde{q}^\varepsilon_a(t)\|^2_{L^2} + \|\tilde{E}^\varepsilon_a(t)\|^2_{L^2}\big)\, dt \\
&\leq C\big(\|\rho^\varepsilon_0 - \rho^*_0\|^2_{L^2} + \|E^\varepsilon_0 - E^*_0\|^2_{L^2} + \|B^\varepsilon_0 - B^e\|^2_{L^2}\big) + C\varepsilon^2.
\end{aligned} \tag{6.44}$$

In light of (6.44), (6.18) and the fact that $(\tilde{\rho}^\varepsilon_a, \tilde{q}^\varepsilon_a, \tilde{E}^\varepsilon_a, \tilde{B}^\varepsilon_a) = (\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon) + \varepsilon(\rho_1, q_1, E_1, B_1)$, (6.31) holds. □

6.5. **High-order error estimates.**

Lemma 6.5. *It holds*

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \big(\|\nabla\tilde{\rho}^\varepsilon(t)\|^2_{H^{m-2}} + \varepsilon^2\|\nabla\tilde{q}^\varepsilon(t)\|^2_{H^{m-2}} + \|\nabla\tilde{E}^\varepsilon(t)\|^2_{H^{m-2}} + \|\nabla\tilde{B}^\varepsilon(t)\|^2_{H^{m-2}}\big) \\
&\quad + \int_0^\infty \big(\|\nabla\tilde{\rho}^\varepsilon(t)\|^2_{H^{m-1}} + \|\nabla\tilde{q}^\varepsilon(t)\|^2_{H^{m-2}} + \|\nabla\tilde{E}^\varepsilon(t)\|^2_{H^{m-2}}\big)\, dt \\
&\leq C\big(\|\nabla(\rho^\varepsilon_0 - \rho^*_0)\|^2_{H^{m-2}} + \|\nabla(E^\varepsilon_0 - E^*_0)\|^2_{H^{m-2}} + \|\nabla(B^\varepsilon_0 - B^e)\|^2_{H^{m-2}}\big) + C\varepsilon^2,
\end{aligned} \tag{6.45}$$

where $C > 0$ is a constant independent of ε.

Proof. Before deriving (6.45), we need to establish estimates for $(\tilde{\rho}^\varepsilon_a, \tilde{q}^\varepsilon_a, \tilde{E}^\varepsilon_a, \tilde{B}^\varepsilon_a)$. Let $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq m-1$. Applying ∂^α to (6.9) leads to

$$\begin{cases}
\partial_t\partial^\alpha\tilde{\rho}^\varepsilon_a + \operatorname{div}\partial^\alpha\tilde{q}^\varepsilon_a = \partial^\alpha J^\varepsilon_1, \\
\varepsilon^2\partial_t\partial^\alpha\tilde{q}^\varepsilon_a + \partial^\alpha\tilde{q}^\varepsilon_a + p'(1)\nabla\partial^\alpha\tilde{\rho}^\varepsilon_a + \partial^\alpha\tilde{E}^\varepsilon_a = \partial^\alpha J^\varepsilon_0 + \partial^\alpha J^\varepsilon_2 + \partial^\alpha J^\varepsilon_3 + \partial^\alpha J^\varepsilon_4, \\
\partial_t\partial^\alpha\tilde{E}^\varepsilon_a - \dfrac{1}{\varepsilon}\nabla\times\partial^\alpha\tilde{B}^\varepsilon_a = \partial^\alpha\tilde{q}^\varepsilon_a + \partial^\alpha J^\varepsilon_5, \\
\partial_t\partial^\alpha\tilde{B}^\varepsilon_a + \dfrac{1}{\varepsilon}\nabla\times\partial^\alpha\tilde{E}^\varepsilon_a = \partial^\alpha J^\varepsilon_6, \\
\operatorname{div}\partial^\alpha\tilde{E}^\varepsilon_a = -\partial^\alpha\tilde{\rho}^\varepsilon_a, \qquad \operatorname{div}\partial^\alpha\tilde{B}^\varepsilon_a = 0.
\end{cases} \tag{6.46}$$

Let $\nu > 0$ be chosen later. Carrying out the L^2 energy estimate for (6.46) as in (6.33)-(6.35), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(p'(1)\|\partial^\alpha\tilde{\rho}^\varepsilon_a\|^2_{L^2} + \varepsilon^2\|\partial^\alpha\tilde{q}^\varepsilon_a\|^2_{L^2} + \|\partial^\alpha\tilde{E}^\varepsilon_a\|^2_{L^2} + \|\partial^\alpha\tilde{B}^\varepsilon_a\|^2_{L^2}\Big) + 2\|\partial^\alpha\tilde{q}^\varepsilon_a\|^2_{L^2} \\
&\leq \frac{1}{2}\|\partial^\alpha\tilde{q}^\varepsilon_a\|^2_{L^2} + C\big(\|\partial^\alpha J^\varepsilon_0\|^2_{L^2} + \|\partial^\alpha J^\varepsilon_2\|^2_{L^2} + \|\partial^\alpha J^\varepsilon_3\|^2_{L^2}\big) \\
&\quad + 2p'(1)\|\partial^\alpha J^\varepsilon_1\|_{L^2}\|\partial^\alpha\tilde{\rho}^\varepsilon_a\|_{L^2} + \nu\|\partial^\alpha\tilde{E}^\varepsilon_a\|^2_{L^2} + C\nu^{-1}\|\partial^\alpha J^\varepsilon_5\|^2_{L^2} \\
&\quad + 2\|\partial^\alpha J^\varepsilon_6\|_{L^2}\|\partial^\alpha\tilde{B}^\varepsilon_a\|_{L^2} + 2\langle\partial^\alpha J^\varepsilon_4, \partial^\alpha\tilde{q}^\varepsilon_a\rangle.
\end{aligned} \tag{6.47}$$

Here, for any $1 \leq |\alpha| \leq m-1$, the combination of (3.3) and Sobolev's inequality

$$\|f\|_{L^\infty} \leq C\|\nabla f\|^{1/2}_{L^2}\|f\|^{1/2}_{\dot{H}^2} \leq C\|\nabla f\|_{H^{m-2}}$$

yields

$$\begin{aligned}
\|\partial^\alpha J^\varepsilon_0\|^2_{L^2} &\leq C(\|\nabla^2\rho^\varepsilon_a\|^2_{L^2} + \|\nabla\rho^*\|^2_{H^{m-2}} + \|\nabla E^\varepsilon\|^2_{H^{m-2}})(\|\nabla\tilde{\rho}^\varepsilon\|^2_{H^{m-2}} + \|\nabla\tilde{E}^\varepsilon_a\|^2_{H^{m-2}}) \\
&\quad + \|\nabla\rho^\varepsilon\|^2_{H^{m-2}}\|\nabla^2\tilde{\rho}^\varepsilon_a\|^2_{H^{m-2}}.
\end{aligned} \tag{6.48}$$

Together with the regularity estimates of $(\rho^\varepsilon, E^\varepsilon, B^\varepsilon)$, (ρ^*, E^*, B^*) and (ρ_1, E_1, B_1), this implies

$$
\begin{aligned}
&\int_0^t \|\partial^\alpha J_0^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\le C \sup_{t'\in[0,t]} \big(\|\nabla\tilde\rho^\varepsilon(t')\|_{H^{m-2}}^2 + \|\nabla\tilde E_a^\varepsilon(t')\|_{H^{m-2}}^2\big) \\
&\quad \times \int_0^t \big(\|\nabla^2\rho^\varepsilon(t')\|_{H^m}^2 + \|\nabla\rho^*(t')\|_{H^{m-2}}^2 + \|\nabla E^\varepsilon(t')\|_{H^{m-2}}^2\big)\,dt \\
&\quad + C \sup_{t'\in[0,t]} \|\nabla\rho^\varepsilon(t')\|_{H^{m-2}}^2 \int_0^\infty \|\nabla^2\tilde\rho_a^\varepsilon(t')\|_{H^{m-2}}^2\,dt \\
&\le C(\delta_1+\delta_1^*+\varepsilon) \sup_{t'\in[0,t]} (\|\nabla\tilde\rho_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde E_a^\varepsilon(t)\|_{H^{m-2}}^2) + \delta_1 \int_0^t \|\nabla^2\tilde\rho_a^\varepsilon(t')\|_{H^{m-2}}^2\,dt'.
\end{aligned} \tag{6.49}
$$

We shall use the faster rate of J_4^ε and overcome its low regularity. The term $\langle\partial^\alpha J_4^\varepsilon, \partial^\alpha\tilde q_a^\varepsilon\rangle$ can be addressed as

$$
\langle\partial^\alpha J_4^\varepsilon, \partial^\alpha\tilde q_a^\varepsilon\rangle = \langle\Lambda^{-1}\partial^\alpha J_4^\varepsilon, \Lambda\partial^\alpha\tilde q_a^\varepsilon\rangle \le C\|J_4^\varepsilon\|_{H^{m-2}}\|\tilde q_a^\varepsilon\|_{H^m}. \tag{6.50}
$$

Moreover, for $1\le|\alpha|\le m-1$ one has

$$
\begin{aligned}
&\|\partial^\alpha\tilde\rho_a^\varepsilon(0)\|_{L^2}^2 + \|\partial^\alpha\tilde q_a^\varepsilon(0)\|_{L^2}^2 + \|\partial^\alpha\tilde E_a^\varepsilon(0)\|_{L^2}^2 + \|\tilde B_a^\varepsilon(0)\|_{L^2}^2 \\
&\le \|\rho_0^\varepsilon-\rho^*\|_{H^{m-1}}^2 + \|E_0^\varepsilon-E_0^*\|_{H^{m-1}}^2 + \|B_0^\varepsilon-B^e\|_{H^{m-1}}^2 + C\varepsilon^2.
\end{aligned} \tag{6.51}
$$

By virtue of the uniform regularity estimates of $(\rho^\varepsilon, E^\varepsilon, B^\varepsilon)$, (ρ^*, E^*, B^*) and (ρ_1, E_1, B_1), it holds that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\|\tilde\rho_a^\varepsilon(t)\|_{H^m}^2 + \int_0^\infty \big(\|\tilde q_a^\varepsilon(t)\|_{H^m}^2 + \|\nabla\tilde B_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde\rho_a^\varepsilon(t)\|_{H^m}^2\big)\,dt \\
&\le C(\delta_1+\delta_1+\varepsilon).
\end{aligned} \tag{6.52}
$$

Integrating (6.47) in time and using the above estimates (6.48)-(6.51), we arrive at

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde\rho_a^\varepsilon(t)\|_{H^{m-2}}^2 + \varepsilon^2\|\nabla\tilde q_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde E_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde B_a^\varepsilon(t)\|_{H^{m-2}}^2\big) \\
&\qquad + \int_0^\infty \|\nabla\tilde q_a(t)\|_{H^{m-2}}^2\,dt \\
&\le C(\|\rho_0^\varepsilon-\rho_0^*\|_{H^{m-1}}^2 + \|E_0^\varepsilon-E_0^*\|_{H^{m-1}}^2 + \|B_0^\varepsilon-B^e\|_{H^{m-1}}^2) + C\varepsilon^2 \\
&\quad + C(\delta_1+\delta_1^*+\varepsilon)\sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde\rho_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde E_a^\varepsilon(t)\|_{H^{m-2}}^2\big) \\
&\quad + C(\delta_1+\nu)\int_0^\infty \big(\|\nabla^2\tilde\rho_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde E_a^\varepsilon(t)\|_{H^{m-1}}^2\big)\,dt \\
&\quad + C\int_0^\infty \big(\|J_2^\varepsilon(t)\|_{H^{m-1}}^2 + \|J_3^\varepsilon(t)\|_{H^{m-1}}^2 + \nu^{-1}\|J_5^\varepsilon(t)\|_{H^{m-1}}^2\big)\,dt \\
&\quad + \Big(\int_0^\infty \|J_4^\varepsilon(t)\|_{H^{m-2}}^2\,dt\Big)^{1/2}\Big(\int_0^\infty \|\tilde q_a^\varepsilon(t)\|_{H^m}^2 dt\Big)^{1/2} \\
&\quad + C\Big(\int_0^\infty \big(\|J_1^\varepsilon(t)\|_{H^{m-1}} + \|J_5^\varepsilon(t)\|_{L^2} + \|J_6^\varepsilon(t)\|_{L^2}\big)\,dt\Big) \\
&\quad \times \sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde\rho_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde E_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde B_a^\varepsilon(t)\|_{H^{m-2}}^2\big),
\end{aligned}
$$

from which as well as (6.24)-(6.27) and (6.52) we infer

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2+\varepsilon^2\|\nabla\tilde{q}_a^\varepsilon(t)\|_{H^{m-2}}^2+\|\nabla\tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2+\|\nabla\tilde{B}_a^\varepsilon(t)\|_{H^{m-2}}^2\big)\\&\qquad+\int_0^\infty\|\nabla\tilde{q}_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt\\&\leq C\big(\|\rho_0^\varepsilon-\rho_0^*\|_{H^{m-1}}^2+\|E_0^\varepsilon-E_0^*\|_{H^{m-1}}^2+\|B_0^\varepsilon-B^e\|_{H^{m-1}}^2\big)+C(1+\nu^{-1})\varepsilon^2\\&\quad+C(\delta_1+\nu)\int_0^\infty\big(\|\nabla^2\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2+\|\nabla\tilde{E}_a^\varepsilon(t)\|_{H^{m-1}}^2\big)\,dt.\end{aligned}\tag{6.53}$$

This proves the result. □

We now establish the dissipation estimates of $\tilde{\rho}_a^\varepsilon$. To this end, we rewrite $(6.46)_2$ as

$$\partial^\alpha\tilde{q}_a^\varepsilon+p'(1)\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon+\partial^\alpha\tilde{E}_a^\varepsilon=\partial^\alpha J_0^\varepsilon+\partial^\alpha J_2^\varepsilon+\partial^\alpha J_3^\varepsilon-\varepsilon^2\partial_t\partial^\alpha q^\varepsilon.\tag{6.54}$$

After taking the inner product of (6.54) by $\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon$ and using

$$\partial_t\tilde{\rho}_a^\varepsilon=-\operatorname{div}(q^\varepsilon-q^*-\varepsilon q_1)\quad\text{and}\quad\partial^\alpha\tilde{\rho}_a^\varepsilon=-\operatorname{div}\partial^\alpha\tilde{E}_a^\varepsilon,$$

we obtain

$$\begin{aligned}&p'(1)\|\nabla\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2+\|\partial^\alpha\nabla\tilde{\rho}_a^\varepsilon\|_{L^2}^2\\&=-\varepsilon^2\frac{d}{dt}\langle\partial^\alpha q^\varepsilon,\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle+\varepsilon^2\langle\partial^\alpha q^\varepsilon,\partial_t\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle\\&\quad+\langle\partial^\alpha J_2^\varepsilon+\partial^\alpha J_3^\varepsilon+\partial^\alpha J_4^\varepsilon,\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle\\&\leq-\varepsilon^2\frac{d}{dt}\langle\partial^\alpha q^\varepsilon,\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle+C\varepsilon^2\big(\|\operatorname{div}\partial^\alpha q^\varepsilon\|_{L^2}^2+\|\operatorname{div}\partial^\alpha q^*\|_{L^2}^2+\varepsilon^2\|\operatorname{div}\partial^\alpha q_1\|_{L^2}^2\big)\\&\quad+C\|\partial^\alpha\tilde{q}^\varepsilon\|_{L^2}^2+C\|\partial^\alpha J_0^\varepsilon\|_{L^2}^2+C\|\partial^\alpha J_2^\varepsilon\|_{L^2}^2+\|\partial^\alpha J_3^\varepsilon\|_{L^2}^2\\&\quad+\frac{p'(1)}{2}\|\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\|_{L^2}^2.\end{aligned}$$

After integrating this in time, we discover that

$$\begin{aligned}&\int_0^t\Big(\frac{1}{2}p'(1)\|\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon(t')\|_{L^2}^2+\|\partial^\alpha\tilde{\rho}_a^\varepsilon(t')\|_{L^2}^2\Big)\,dt'\\&\quad\leq\varepsilon^2\langle\partial^\alpha q^\varepsilon,\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle\Big|_0^t+C\int_0^t\|\partial^\alpha\tilde{q}^\varepsilon(t')\|_{L^2}^2\,dt'\\&\qquad+C\varepsilon^2\int_0^t\big(\|q^\varepsilon(t')\|_{H^m}^2+\|q^*(t')\|_{H^m}^2+\varepsilon^2\|q_1(t')\|_{H^m}^2\big)\,dt'\\&\qquad+C\int_0^t\big(\|J_0^\varepsilon(t')\|_{H^{m-1}}^2+\|J_2^\varepsilon(t')\|_{H^{m-1}}^2+\|J_3^\varepsilon(t')\|_{H^{m-1}}^2\big)\,dt'.\end{aligned}\tag{6.55}$$

By $\rho_1|_{t=0}=0$, (6.6), (6.52) and $\varepsilon\|q_0^\varepsilon\|_{H^m}+\varepsilon\|q^\varepsilon\|_{H^m}\leq C$ we have

$$\begin{aligned}\varepsilon^2\langle\partial^\alpha q^\varepsilon,\nabla\partial^\alpha\tilde{\rho}_a^\varepsilon\rangle\Big|_0^t&\leq\varepsilon^2\|q^\varepsilon\|_{H^m}\|\nabla\tilde{\rho}_a^\varepsilon\|_{H^{m-2}}+\varepsilon\big(\varepsilon\|q_0^\varepsilon\|_{L^2}\big)\|\rho_0^\varepsilon-\rho_0^*\|_{H^{m-1}}\\&\leq C\|\nabla\tilde{\rho}_a^\varepsilon\|_{H^{m-2}}^2+C\|\nabla(\rho_0^\varepsilon-\rho_0^*)\|_{H^{m-2}}^2+C\varepsilon^2.\end{aligned}\tag{6.56}$$

Putting (6.24)-(6.25), (6.49), (6.52) and (6.56) into (6.55) and using Young's inequality, we derive

$$\begin{aligned}
&\int_0^\infty \|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-1}}^2\,dt \\
&\le C\|\nabla(\rho_0^\varepsilon-\rho_0^*)\|_{H^{m-1}}^2 + C\varepsilon^2 + C\sup_{t\in\mathbb{R}^+}\|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2 + C\int_0^\infty \|\tilde{q}_a^\varepsilon(t)\|_{H^m}^2\,dt \\
&\quad + C(\delta_1+\delta_1^*+\varepsilon)\sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2\big) + \delta_1\int_0^\infty \|\nabla^2\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt. \qquad (6.57)
\end{aligned}$$

The next step is to derive the dissipation estimate of the error $\tilde{E}_a^\varepsilon$. Taking the inner product of (6.54) by $\partial^\alpha\tilde{E}_a^\varepsilon$ with $1\le|\alpha|\le m-1$ and using $\tilde{\rho}_a^\varepsilon = -\partial^\alpha \operatorname{div}\tilde{E}_a^\varepsilon$ yields

$$\begin{aligned}
&\|\partial^\alpha\tilde{E}_a^\varepsilon\|_{L^2}^2 + p'(1)\|\partial^\alpha \operatorname{div}\tilde{E}_a^\varepsilon\|_{L^2}^2 \\
&= -\varepsilon^2\frac{d}{dt}\langle\partial^\alpha q^\varepsilon,\partial^\alpha\tilde{E}_a^\varepsilon\rangle + \varepsilon^2\langle\partial^\alpha q^\varepsilon,\partial_t\partial^\alpha\tilde{E}_a^\varepsilon\rangle - \langle\partial^\alpha\tilde{q}_a^\varepsilon,\partial^\alpha\tilde{E}_a^\varepsilon\rangle + \langle\partial^\alpha J_0^\varepsilon+\partial^\alpha J_2^\varepsilon+\partial^\alpha J_3^\varepsilon,\partial^\alpha\tilde{E}_a^\varepsilon\rangle.
\end{aligned}$$

Note that

$$\partial_t\partial^\alpha\tilde{E}_a^\varepsilon = \frac{1}{\varepsilon}\nabla\times\partial^\alpha\tilde{B}_a^\varepsilon + \partial^\alpha\tilde{q}_a^\varepsilon + J_5^\varepsilon,$$

so that

$$\varepsilon^2\langle\partial^\alpha q^\varepsilon,\partial_t\partial^\alpha\tilde{E}_a^\varepsilon\rangle = \varepsilon\langle\nabla\times\partial^\alpha\tilde{q}^\varepsilon,\partial^\alpha\tilde{B}_a^\varepsilon\rangle + \varepsilon^2\langle\partial^\alpha q^\varepsilon,\partial^\alpha\tilde{q}_a^\varepsilon + J_5^\varepsilon\rangle.$$

Therefore, for some constant $\nu>0$ to be determined, we obtain

$$\begin{aligned}
&\|\partial^\alpha\tilde{E}_a^\varepsilon\|_{L^2}^2 + p'(1)\|\partial^\alpha \operatorname{div}\tilde{E}_a^\varepsilon\|_{L^2}^2 \\
&= -\varepsilon^2\frac{d}{dt}\langle\partial^\alpha\tilde{q}^\varepsilon,\partial^\alpha\tilde{E}_a^\varepsilon\rangle + \frac{1}{2}\|\partial^\alpha\tilde{E}_a^\varepsilon\|_{L^2}^2 + \nu\|\partial^\alpha\tilde{B}_a^\varepsilon\|_{L^2}^2 + C\|\partial^\alpha\tilde{q}_a^\varepsilon\|_{L^2}^2 \\
&\quad + C\varepsilon^2(\|\partial^\alpha q^\varepsilon\|_{L^2}^2 + \nu^{-1}\|\nabla\times\partial^\alpha q^\varepsilon\|_{L^2}^2) \\
&\quad + C\|\partial^\alpha J_0^\varepsilon\|_{L^2}^2 + C\|\partial^\alpha J_2^\varepsilon\|_{L^2}^2 + C\|\partial^\alpha J_3^\varepsilon\|_{L^2}^2 + C\|\partial^\alpha J_5^\varepsilon\|_{L^2}^2.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|\partial^\alpha\tilde{E}_a^\varepsilon(t')\|_{L^2}^2\,dt' \\
&\le -\varepsilon^2\langle\partial^\alpha\tilde{q}^\varepsilon,\partial^\alpha\tilde{E}_a^\varepsilon\rangle\Big|_0^t + C\int_0^t \|\nabla\tilde{q}_a^\varepsilon(t')\|_{H^{m-2}}^2\,dt' + \nu\int_0^t \|\nabla\tilde{B}_a^\varepsilon(t')\|_{H^{m-2}}^2\,dt' \\
&\quad + C(1+\nu^{-1})\varepsilon^2\int_0^t \|q^\varepsilon(t')\|_{H^m}^2\,dt' + \int_0^t \|J_0^\varepsilon(t')\|_{H^{m-1}}^2\,dt' \\
&\quad + \int_0^t \big(\|J_2^\varepsilon(t')\|_{H^{m-1}}^2 + \|J_3^\varepsilon(t')\|_{H^{m-1}}^2 + \|J_4^\varepsilon(t')\|_{L^2}^2 + \|J_5(t')\|_{H^{m-1}}^2\big)\,dt',
\end{aligned}$$

which, together with (6.25)-(6.27), (6.49) and $\varepsilon\sup_{t\in\mathbb{R}^+}\|q^\varepsilon(t)\|_{H^{m-1}}\le C$, leads to

$$\begin{aligned}
&\int_0^\infty \|\nabla\tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt \\
&\le C\|\nabla(E_0^\varepsilon - E_0)\|_{H^{m-2}}^2 + C\varepsilon^2 \\
&\quad + C(\delta_1+\delta_1^*+\varepsilon)\sup_{t\in\mathbb{R}^+}\big(\|\nabla\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla\tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2\big) + \delta_1\int_0^\infty \|\nabla^2\tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt \\
&\quad + C\int_0^\infty \|\nabla\tilde{q}_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt + \nu\int_0^\infty \|\nabla\tilde{B}_a^\varepsilon(t)\|_{H^{m-2}}^2\,dt. \qquad (6.58)
\end{aligned}$$

We are in a position to obtain the dissipation estimate of $\tilde{B}_a^\varepsilon$ required in (6.58). For $1 \leq |\alpha'| \leq m-2$, we deduce from $(6.46)_3$-$(6.46)_3$ that

$$\begin{aligned}\|\nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\|_{L^2}^2 &= \varepsilon\langle \partial_t \partial^{\alpha'} \tilde{E}_a^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle - \langle \partial^{\alpha'} \tilde{q}_a^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle + \langle \partial^{\alpha'} J_5^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle \\ &= \varepsilon \frac{d}{dt}\langle \partial^{\alpha'} \tilde{E}_a^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle - \varepsilon\langle \partial^{\alpha'} \tilde{q}_a^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle \\ &\quad + \langle \partial^{\alpha'} J_5^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle + \langle \nabla \times \partial^{\alpha'} \tilde{E}_a^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle \\ &\quad - \varepsilon\langle \partial^{\alpha'} \nabla \times \tilde{E}_a^\varepsilon, \partial^{\alpha'} J_6^\varepsilon\rangle.\end{aligned}$$

Consequently, one discovers that

$$\begin{aligned}&\frac{1}{2}\int_0^t \|\nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon(t')\|_{L^2}^2 \, dt' \\ &\leq \varepsilon\langle \partial^{\alpha'} \tilde{E}_a^\varepsilon, \nabla \times \partial^{\alpha'} \tilde{B}_a^\varepsilon\rangle\Big|_0^t + C\int_0^t \left(\varepsilon^2 \|\tilde{q}_a^\varepsilon(t')\|_{H^{m-2}}^2 + \|J_5^\varepsilon(t')\|_{H^{m-2}}^2 + \|J_6^\varepsilon(t')\|_{H^{m-2}}^2\right) dt' \\ &\quad + C\int_0^t \|\nabla \tilde{E}_a^\varepsilon(t')\|_{H^{m-1}}^2 \, dt'. \qquad (6.59)\end{aligned}$$

By (6.27), (6.52), (6.59) and $\operatorname{div} \tilde{B}_a^\varepsilon = 0$, it yields

$$\begin{aligned}\int_0^\infty \|\nabla \tilde{B}_a^\varepsilon(t)\|_{H^{m-2}}^2 \, dt &\leq C\varepsilon \sup_{t\in\mathbb{R}^+} \left(\|\nabla \tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla \tilde{B}_a^\varepsilon(t)\|_{H^{m-2}}^2\right) \\ &\quad + C\varepsilon^2 + C\int_0^\infty \|\nabla \tilde{E}_a^\varepsilon(t')\|_{H^{m-2}}^2 \, dt'. \qquad (6.60)\end{aligned}$$

Collecting (6.53), $\eta \times$ (6.57), $\eta \times$ (6.58) and $\eta \times$ (6.60) with some small constant η, taking ν sufficiently small and using the smallness of δ_1, δ_1^* and $\varepsilon \leq \varepsilon_1$ with some suitably small ε_1, we obtain

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+} \left(\|\nabla \tilde{\rho}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \varepsilon^2 \|\nabla \tilde{q}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla \tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla \tilde{B}_a^\varepsilon(t)\|_{H^{m-2}}^2\right) \\ &\quad + \int_0^\infty \left(\|\nabla \tilde{\rho}_a^\varepsilon(t)\|_{H^{m-1}}^2 + \|\nabla \tilde{q}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla \tilde{E}_a^\varepsilon(t)\|_{H^{m-2}}^2 + \|\nabla \tilde{H}_a^\varepsilon(t)\|_{H^{m-2}}^2\right) dt \\ &\leq C\left(\|\nabla(\rho_0^\varepsilon - \rho_0^*)\|_{H^{m-2}}^2 + \|\nabla(E_0^\varepsilon - E_0^*)\|_{H^{m-2}}^2 + \|\nabla(B_0^\varepsilon - B^e)\|_{H^{m-2}}^2\right) + C\varepsilon^2. \qquad (6.61)\end{aligned}$$

In the case $\varepsilon \geq \varepsilon_1$, it is clear that (6.45) holds thanks to the uniform estimates satisfied by $(\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon)$. Using that $(\tilde{\rho}_a^\varepsilon, \tilde{q}_a^\varepsilon, \tilde{E}_a^\varepsilon, \tilde{B}_a^\varepsilon) = (\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon) + \varepsilon(\rho_1, q_1, E_1, B_1)$, the desired bounds for $(\tilde{\rho}^\varepsilon, \tilde{q}^\varepsilon, \tilde{E}^\varepsilon, \tilde{B}^\varepsilon)$ follow. □

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References

[1] K. Beauchard and E. Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Ration. Mech. Anal.*, 199:177–227, 2011.

[2] S. Bianchini, B. Hanouzet, and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure Appl. Math.*, 60, 1559-1622, 2007.

[3] C. Cercignani. *The Boltzmann Equation and its Applications.* Springer-Verlag, New York, 1988.

[4] G.-Q. Chen, J. W. Jerome, and D. Wang. Compressible Euler–Maxwell equations. *Transp. Theory Stat. Phys.*, 29:311–331, 2000.

[5] G.-Q. Chen, C. Levermore, and T.-P. Liu. Hyperbolic conservation laws with stiff relaxation terms and entropy. *Comm. Pure Appl. Math.*, 47(6):787–830, 1994.

[6] J.-F. Coulombel and T. Goudon. The strong relaxation limit of the multidimensional isothermal Euler equations. *Trans. Amer. Math. Soc.*, 359(2):637–648, 2007.

[7] T. Crin-Barat and R. Danchin. Partially dissipative hyperbolic systems in the critical regularity setting: The multi-dimensional case. *J. Math. Pures Appl. (9)*, 165:1–41, 2022.

[8] T. Crin-Barat and R. Danchin. Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. *Pure Appl. Anal.*, 4(1):85–125, 2022.

[9] T. Crin-Barat and R. Danchin. Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit. *Math. Ann.*, 386:2159–2206, 2023.

[10] T. Crin-Barat, Y.-J. Peng, L.-Y. Shou, and J. Xu. A new characterization of the dissipation structure and the relaxation limit for the compressible Euler-Maxwell system. *J. Funct. Anal.*, 289:110918, 2025, 51 pp.

[11] C. M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*, 3rd edition. Grundlehren Math. Wiss., vol. 325, Springer, Heidelberg, Dordrecht, London and New York, 2010.

[12] Y. Deng, A. D. Ionescu, and B. Pausader. The Euler–Maxwell system for electrons: Global solutions in 2D. *Arch. Ration. Mech. Anal.*, 225:771–871, 2017.

[13] R. Duan. Global smooth flows for the compressible Euler-Maxwell system: The relaxation case. *J. Hyperbolic Differ. Equ.*, 8(2), 375-413, 2011. ´

[14] P. Germain and N. Masmoudi. Global existence for the Euler-Maxwell system. *Ann. Scient. Ec. Norm. Sup.*, 47:469–503, 2014.

[15] Y. Guo, A. D. Ionescu, and B. Pausader. Global solutions of the Euler–Maxwell two-fluid system in 3D. *Ann. Math.* (2), 183:377–498, 2016.

[16] M.-L. Hajjej and Y.-J. Peng. Initial layers and zero-relaxation limits of Euler-Maxwell equations. *J. Differential Equations*, 252(2):1441–1465, 2012.

[17] B. Hanouzet and R. Natalini. Global existence of smooth solutions for partially dissipative hyperbolic systems with convex entropy. *Arch. Ration. Mech. Anal.*, 169, 89-117, 2003.

[18] A. D. Ionescu and V. Lie. Long term regularity of the one-fluid Euler-Maxwell system in 3D with vorticity. *Adv. Math.*, 325:719–769, 2018.

[19] J. W. Jerome. The Cauchy problem for compressible hydrodynamic-Maxwell systems: A local theory for smooth solutions. *Differ. Integral Equ.*, 16:1345–1368, 2003.

[20] S. Junca and M. Rascle. Strong relaxation of the isothermal Euler system to the heat equation. *Z. Angew. Math. Phys.*, 53, 239–264, 2002.

[21] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Ration. Mech. Anal.*, 58:181–205, 1975.

[22] S. Kawashima and W.-A. Yong. Dissipative structure and entropy for hyperbolic systems of balance laws. *Arch. Ration. Mech. Anal.*, 174, 345–364, 2004.

[23] P. Lax. Development of singularities of solutions of nonlinear hyperbolic partial differential equations. *J. Math. Phys.*, 5:611–613, 1964.

[24] Y. Li, Y.-J. Peng, and L. Zhao. Convergence rate from hyperbolic systems of balance laws to parabolic systems. *Appl. Anal.*, 100(5):1079–1095, 2021.

[25] Y. Li, Y.-J. Peng, and L. Zhao. Convergence rates in zero-relaxation limits for Euler-Maxwell and Euler-Poisson systems. *J. Math. Pures Appl. (9)*, 154, 185-211, 2021.

[26] C. Lin and J.-F. Coulombel. The strong relaxation limit of the multidimensional Euler equations. *Nonlin. Diff. Eq. Appl.*, 20:447–461, 2013.

[27] C. Liu, Z. Guo, and Y.-J. Peng. Global stability of large steady-states for an isentropic Euler–Maxwell system in $\mathbb{R}^3$. *Commun. Math. Sci.*, 17(7):1841–1860, 2019.

[28] T. Liu. Hyperbolic conservation laws with relaxation. *Comm. Math. Phys.*, 60:153–175, 1987.

[29] A. Majda. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables.* Appl. Math. Sci., vol. 53, Springer-Verlag, Berlin, New York, 1984.

[30] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, 84, 129-147, 1990.

[31] P. Marcati, A. Milani, and P. Secchi. Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system. *Manuscripta Math.*, 60:49–69, 1988.

[32] P. Marcati and B. Rubino. Hyperbolic to parabolic relaxation theory for quasilinear first order systems. *J. Differential Equations*, 162:359–399, 2000.
[33] H. Peng and C. Zhu. Asymptotic stability of stationary solutions to the compressible Euler-Maxwell equations. *Indiana Univ. Math. J.*, 62(4):1203–1235, 2013.
[34] Y.-J. Peng. Stability of non-constant equilibrium solutions for Euler–Maxwell equations. *J. Math. Pures Appl. (9)*, 103(1), 39–67, 2015.
[35] Y. Peng. Global large smooth solutions for isothermal Euler equations with damping and small parameter. *J. Funct. Anal.*, 287: 110571, 2024.
[36] Y.-J. Peng, S. Wang, and Q. Gu. Relaxation limit and global existence of smooth solutions of compressible Euler–Maxwell equations. *SIAM J. Math. Anal.*, 43(2), 944-970, 2011.
[37] Y.-J. Peng and V. Wasiolek. Parabolic limit with differential constraints of first-order quasilinear hyperbolic systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33:1103–1130, 2016.
[38] Y.-J. Peng and V. Wasiolek. Uniform global existence and parabolic limit for partially dissipative hyperbolic systems. *J. Differential Equations*, 260 (9): 7059–7092, 2016.
[39] S. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14, 249-275, 1985.
[40] T. Sideris, B. Thomases, and D. Wang. Long time behavior of solutions to the 3d compressible Euler equations with damping. *Comm. Partial Differential Equations*, 28, 795-816, 2003.
[41] Y. Ueda, S. Wang, and S. Kawashima. Dissipative structure of the regularity-loss type and time asymptotic decay of solutions for the Euler–Maxwell system. *SIAM J. Math. Anal.*, 44:2002–2017, 2012.
[42] Y. Ueda, R. Duan, and S. Kawashima. Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application. *Arch. Ration. Mech. Anal.*, 205:239–266, 2012.
[43] Y. Ueda and S. Kawashima. Decay property of regularity-loss type for the Euler-Maxwell system. *Methods Appl. Anal.*, 18:245–268, 2011.
[44] W. Vicenti and C. Kruger. *Introduction to Physical Gas Dynamics*. Melbourne: Robert E. Krieger, 1982.
[45] W. Wang and T. Yang. The pointwise estimates of solutions for Euler equations with damping in multi-dimensions.*J Differential Equations* 173(2): 410-450, 2001.
[46] V. Wasiolek. Uniform global existence and convergence of Euler-Maxwell systems with small parameters. *Commun. Pure Appl. Anal.*, 15(6): 2007–2021, 2016.
[47] G. B. Whitham. *Linear and Nonlinear Waves*. Wiley, New York, 1974.
[48] J. Xu. Global classical solutions to the compressible Euler–Maxwell equations. *SIAM J. Math. Anal.*, 43(6), 2688–2718, 2011.
[49] J. Xu and S. Kawashima. Global classical solutions for partially dissipative hyperbolic system of balance laws. *Arch. Ration. Mech. Anal.*, 211, 513–553, 2014.
[50] J. Xu and S. Kawashima. The optimal decay estimates on the framework of Besov spaces for generally dissipative systems. *Arch. Ration. Mech. Anal.*, 218, 275–315, 2015.
[51] J. Xu, N. Mori, and S. Kawashima. L^p–L^q–L^r estimates and minimal decay regularity for compressible Euler–Maxwell equations. J. Math. Pures Appl. (9), 104(2): 965-981, 2015.
[52] J. Xu and Z. Wang. Relaxation limit in Besov spaces for compressible Euler equations. *J. Math. Pures Appl. (9)*, 99:43–61, 2013.
[53] W.-A. Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Ration. Mech. Anal.*, 172: 47–266, 2004.

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