Partially dissipative hyperbolic systems in the critical regularity setting : the multi-dimensional case

Timothée Crin-Barat[1,*], Raphaël Danchin[1]

Univ Paris Est Creteil, Univ Gustave Eiffel, CNRS, LAMA UMR8050, F-94010 Creteil, France

Abstract

We are concerned with quasilinear symmetrizable partially dissipative hyperbolic systems in the whole space $\mathbb{R}^d$ with $d \geq 2$. Following our recent work dedicated to the one-dimensional case [11], we establish the existence of global strong solutions and decay estimates in the critical regularity setting whenever the system under consideration satisfies the so-called (SK) (for Shizuta-Kawashima) condition. Our results in particular apply to the compressible Euler system with damping in the velocity equation.

Compared to the recent works devoted to similar issues, our use of *hybrid* Besov norms with *different* regularity exponents in low and high frequency enables us to pinpoint optimal smallness conditions for global well-posedness and to get a more accurate information on the qualitative properties of the constructed solutions.

A great part of our analysis relies on the study of a Lyapunov functional in the spirit of that of Beauchard and Zuazua in [2]. Exhibiting a damped mode with faster time decay than the whole solution also plays a key role.

Résumé

Nous nous intéressons aux systèmes hyperboliques quasi-linéaires symétriques partiellement dissipatifs dans l'espace $\mathbb{R}^d$ avec $d \geq 2$. Dans la continuité de nos travaux récents consacrés au cas unidimensionnel [11], nous établissons l'existence de solutions fortes globales en temps et des estimations de décroissance dans un cadre à régularité critique, lorsque le système considéré satisfait la condition (SK) (pour Shizuta-Kawashima). Nos résultats s'appliquent en particulier au système d'Euler compressible avec terme de friction dans l'équation de vitesse.

Par rapport aux travaux récents consacrés à des questions similaires, l'utilisation de normes de Besov *hybrides* avec des exposants de régularité *différents* en basses et hautes fréquences nous permet d'identifier des conditions optimales de petitesse pour le caractère globalement bien posé et d'obtenir des informations plus précises sur les propriétés qualitatives des solutions construites.

Une grande partie de notre analyse repose sur l'introduction d'une fonctionnelle de Lyapunov dans l'esprit de celle de Beauchard et Zuazua [2]. La mise en évidence d'un mode amorti satisfaisant une meilleure décroissance temporelle que la solution joue également un rôle clé.

Keywords: Hyperbolic systems, partially dissipative, critical regularity, time decay, global solutions.
2020 MSC: 35Q35; 76N10

Introduction

We are concerned with first order n-component systems in $\mathbb{R}^d$ of the type:

$$A^0(V)\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j(V)\frac{\partial V}{\partial x_j} = H(V) \tag{1}$$

*Corresponding author
Email addresses: timothee.crin-barat@deusto.es (Timothée Crin-Barat), danchin@u-pec.fr (Raphaël Danchin)
[1]The authors are partially supported by the ANR project INFAMIE (ANR-15-CE40-0011).

where the smooth[2] matrix-valued functions A^j $(j=0,\cdots,d)$ and vector-valued function H are defined on some open subset $\mathcal{O}_V$ of $\mathbb{R}^n$ and the unknown $V=V(t,x)\in\mathbb{R}^n$ depends on the time variable $t\in\mathbb{R}_+$ and on the space variable $x\in\mathbb{R}^d$ $(d\geq 2)$. We assume that the system is *symmetrizable*. Additional structure assumptions will be specified in the next section.

System (1) is supplemented with initial data $V_0\in\mathcal{O}_V$ at time $t=0$. We are concerned with the existence of global strong solutions in the case where V_0 is close to some constant state $\bar{V}$ such that $H(\bar{V})=0$.

In the nondissipative case, that is if $H\equiv 0$, it is classical that symmetrizable quasilinear hyperbolic systems supplemented with initial data with Sobolev regularity H^s such that $s>1+d/2$ admit local-in-time strong solutions (see e.g. [3]), that may develop singularities in finite time even if the initial data are small (see for instance the works by Majda in [23] or Serre in [26]). By contrast, if in the neighborhood of $\bar{V}$, the term $H(V)$ has the 'good' sign and acts on each component of the solution (that is, all the eigenvalues of $DH(\bar{V})$ have negative real part), then smooth perturbations of $\bar{V}$ give rise to global-in-time solutions that tend exponentially fast to $\bar{V}$ when time goes to ∞.

In most physical situations that may be modelled by systems of the form (1) however, some components of the solution satisfy conservation laws and only *partial* dissipation occurs, that is to say, the term $H(V)$ acts only on a part of the solution. Typically, this happens in gas dynamics where the mass density and entropy are conserved, or in numerical schemes involving conservation laws with relaxation. A well known example is the damped compressible Euler system for isentropic flows that will be addressed at the end of the paper. For this system, it is known from the works of Wang and Tang [30] or Sideris, Thomases and Wang [28] that the dissipative mechanism, albeit only present in the velocity equation, prevents the formation of singularities that would occur for $H\equiv 0$.

Looking for sufficient conditions on the dissipation term H guaranteeing the global existence of strong solutions for perturbations of a constant state $\bar{V}$ goes back to the thesis of Kawashima [18] (further developed in [27]) and to the more recent work by Yong in [34]. Two main conditions arise. The first one is the so-called (SK)(for *Shizuta-Kawashima*) stability condition that ensures that the damping is strong enough to prevent the solutions emanating from small perturbations of $\bar{V}$ from blowing up. The second one is the existence of a (dissipative) entropy which provides a symmetrisation of the system that is compatible with H. Putting together those two conditions, Yong [34] obtained a global existence result for a class of partially dissipative systems that is more general than that considered by Kawashima.

More recently, by taking advantage of the properties of the Green kernel of the linearized system around $\bar{V}$ and of the Duhamel formula, Bianchini, Hanouzet and Natalini in [5] pointed out that global solutions converge to $\bar{V}$ in L^p, with the rate $\mathcal{O}(t^{-\frac{d}{2}(1-\frac{1}{p})})$ when $t\to\infty$, for all $p\in[\min\{d,2\},\infty]$. Shortly after, in [20], Kawashima and Yong obtained decay estimates for a family of Sobolev norms.

A few years ago, Kawashima and Xu in [32] and [33] extended the prior works to the larger setting of *critical* non-homogeneous Besov spaces. To obtain their results, they used the symmetrization from [19] and applied a frequency localization argument relying on the Littlewood-Paley decomposition. In their work, the equivalence between Condition (SK) and the existence of a compensating function allowed to exhibit the global-in-time L^2 integrability properties of all the components of the solution.

The present paper aims at going beyond the results by Kawashima and Xu, by specifying precisely the regularity of the low frequencies of the data, in the spirit of our recent work [11] devoted to one-dimensional two components systems. Being able to consider a larger class of data leading to global existence is not our only motivation. In fact, our new functional setting will provide us with a better control of the solution with, in particular, global-in-time L^1 bounds, more accurate rate of convergence to the reference equilibrium state when the time goes to infinity and, last but not least, a user friendly framework to tackle the relaxation limit (see [12]).

Before giving an overview of our approach, let us explain the meaning of critical regularity in the context of hyperbolic systems. It has been observed by many authors that controlling ∇V in $L^1(0,T;L^\infty)$ prevents blow-up of Sobolev norms of the solution at time T. For that reason, owing to

[2]We just need that $A^j(V)$ and $H(V)$ have the same regularity as V. According to the classical literature on composition estimates, in our functional setting, assuming that A^j and H are of class $C^{2+[\frac{d}{2}]}$ is enough.

the embedding $H^s(\mathbb{R}^d) \hookrightarrow C^{0,1}(\mathbb{R}^d)$ if and only if $s > d/2+1$, it is generally considered that the critical regularity exponent for systems of type (1) is $d/2+1$. As first observed by D. Iftimie in the appendix of [17], this value may be achieved for Besov spaces $\mathbb{B}^s_{2,1}(\mathbb{R}^d)$, in connection with the chain of embedding:

$$H^s(\mathbb{R}^d) \hookrightarrow \mathbb{B}^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d) \hookrightarrow C^{0,1}(\mathbb{R}^d), \qquad s > 1 + d/2.$$

At the same time, examples of ill-posedness results in H^s with $s < 1 + d/2$ have been pointed out recently in [22, 21].

In order to go beyond the recent results by Xu and Kawashima [32, 33], we shall take advantage of the new connection established by Beauchard and Zuazua in [2] between (SK) condition and the Kalman rank condition in control theory. In this paper, the authors developed a new and systematic approach that allows to establish global existence and to describe large time behavior of solutions to partially dissipative systems that need not satisfy Condition (SK)[3]. Looking at the linearization of System (1) around a constant state (denoting from now on $\partial_t \triangleq \frac{\partial}{\partial t}$ and $\partial_j \triangleq \frac{\partial}{\partial x_j}$),

$$\partial_t Z + \sum_{j=1}^{m} A^j \partial_j Z = -LZ, \tag{2}$$

they showed that Condition (SK) is equivalent to the Kalman maximal rank condition on the matrices A^j and L, and introduced a Lyapunov functional that encodes enough information to recover the dissipative properties of (2). Considering such a functional is motivated by the classical (linear) control theory of ODEs, and is also related to Villani's work in [29]. Back to the nonlinear system (1), Beauchard and Zuazua obtained the existence of global smooth solutions for H^s perturbations of a constant equilibrium $\bar{V}$ that satisfies Condition (SK). Furthermore, using arguments borrowed from Coron's return method [10], they were able to achieve certain cases where (SK) does not hold.

Our aim here is to extend the results we obtained recently in [11] to *multi-dimensional* partially dissipative hyperbolic systems. More precisely, under Condition (SK), we shall develop Beauchard and Zuazua's approach as sketched by the second author in [14] and prove the global well-posedness of (1) supplemented with data that are close to $\bar{V}$ in an optimal critical regularity setting. As in the study of the compressible Navier-Stokes system and related models (see e.g. [8, 9, 13, 15]), it will appear naturally that, in order to get optimal results, one has to use functional spaces with *different* regularity exponents in low and high frequencies. Here, Beauchard and Zuazua's approach will give us the information that the low frequencies (resp. high frequencies) of the solution of the linearized system behave like the heat flow (resp. are exponentially damped). Furthermore, in order to improve our low frequency analysis, we will exhibit a damped mode with better decay properties than the whole solution. Thanks to it, we will end up with more accurate estimates and a weaker smallness condition than in [32] and refine the decay estimates that were obtained in [33]. In particular, our functional framework allows to keep track of the parameters of the system (if any).

The paper is arranged as follows. In the first section, we specify the structure of the class of partially dissipative hyperbolic systems we aim at considering, and explain the construction of a Lyapunov functional that will be the key to our global results. In passing, we exhibit a 'damped mode' that is expected to have better time integrability than the overall solution. In Section 2, we state the main results of the paper. Section 3 is devoted to the proof of a first global existence result and time decay estimates for general partially dissipative systems satisfying the Shizuta-Kawashima condition. In Section 4, under additional structure assumptions (that are satisfied by the compressible Euler system with damping), we obtain a more accurate global existence result. Some technical results are proved or recalled in Appendix.

[3]Condition (SK) is not optimal in the sense that there are many systems that do not satisfy it but still admit global strong solutions, see e.g. [25, 4, 7] as well as the recent work [6] by the first author dedicated to the relaxation limit of a non conservative multi-fluid system.

1. Hypotheses and method

In this section, we specify our assumptions on the system under consideration, and describe the main steps of our approach.

1.1. *Partially dissipative symmetrizable systems*

First, to ensure the local well-posedness we assume that System (1) is *Friedrichs-symmetrizable*: there exists a smooth function $S : V \mapsto S(V)$ defined on $\mathcal{O}_V$, valued in the set of symmetric and positive definite matrices such that for all $V \in \mathcal{O}_V$, the matrices $(SA^0)(V), \cdots, (SA^d)(V)$ are symmetric and, in addition, $(SA^0(V))$ is positive definite.

Denoting $\widetilde{H} \triangleq SH$ and $\widetilde{A}^j \triangleq SA^j$ for $j \in \{0, \cdots, d\}$, System (1) rewrites

$$\widetilde{A}^0(V)\partial_t V + \sum_{j=1}^{d} \widetilde{A}^j(V)\partial_j V = \widetilde{H}(V).$$

Next, fix some $\bar{V} \in \mathcal{O}_V$ such that $H(\bar{V}) = 0$. Clearly, this is a constant solution of (1). Since we want to study whether perturbations of $\bar{V}$ lead to global-in-time solutions, it is natural to look at the system satisfied by $Z \triangleq V - \bar{V}$, namely

$$\widetilde{A}^0(V)\partial_t Z + \sum_{j=1}^{d} \widetilde{A}^j(V)\partial_j Z + LZ = Q(Z) \tag{3}$$

with $L \triangleq -D_V\widetilde{H}(\bar{V})$ and $Q(Z) \triangleq \widetilde{H}(\bar{V} + Z) + LZ$.

We shall restrict our study to the case where:

(i) The kernel of L is the orthogonal complement of the range of L and, furthermore, both $\ker(L)$ and $\mathrm{Range}(L)$ are invariant by $\widetilde{A}^0(V)$ for all V in $\mathcal{O}_V$.
Consequently, one can assume (with no loss of generality) that we have the block decomposition:

$$\widetilde{A}^0 = \begin{pmatrix} \widetilde{A}^0_{1,1} & 0 \\ 0 & \widetilde{A}^0_{2,2} \end{pmatrix}, \quad \widetilde{A}^j = \begin{pmatrix} \widetilde{A}^j_{1,1} & \widetilde{A}^j_{1,2} \\ \widetilde{A}^j_{2,1} & \widetilde{A}^j_{2,2} \end{pmatrix} \text{ for } j = 1, \cdots, d, \qquad L = \begin{pmatrix} 0 \\ L_2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},$$

where the upper left block is of size $n_1 \times n_1$ and the lower right block is of size $n_2 \times n_2$, with $n_1 + n_2 = n$.

(ii) The 0-order term L_2 is dissipative in the sense that there exists $c > 0$ such that

$$\forall \eta_2 \in \mathbb{R}^{n_2}, \; L_2\eta_2 \cdot \eta_2 \geq c|\eta_2|^2. \tag{4}$$

(iii) The remainder term Q (which, by construction, is at least quadratic with respect to Z) satisfies $Q(Z_1, 0) = 0$ for Z_1 close to 0, and is thus at least linear with respect to Z_2.

Note that, thanks to the first structure assumption, System (3) may be rewritten as:

$$\begin{cases} \widetilde{A}^0_{1,1}(V)\partial_t Z_1 + \displaystyle\sum_{j=1}^{d} \Big(\widetilde{A}^j_{1,1}(V)\partial_j Z_1 + \widetilde{A}^j_{1,2}(V)\partial_j Z_2\Big) = Q_1(Z), \\ \widetilde{A}^0_{2,2}(V)\partial_t Z_2 + \displaystyle\sum_{j=1}^{d} \Big(\widetilde{A}^j_{2,1}(V)\partial_j Z_1 + \widetilde{A}^j_{2,2}(V)\partial_j Z_2\Big) + L_2 Z_2 = Q_2(Z). \end{cases} \tag{5}$$

As we shall see in Section 4, the compressible Euler equations with damping, rewritten in suitable variables, enters in this class of systems.

1.2. The Shizuta-Kawashima and Kalman rank conditions

In order to specify the supplementary conditions on the structure of the system ensuring global well-posedness and present the overall strategy, let us consider the linearization of (1) about $\bar{V}$, namely:

$$\bar{A}^0\partial_t Z + \sum_{j=1}^{d} \bar{A}^j \partial_j Z + LZ = 0 \quad \text{with} \quad \bar{A}^j := \widetilde{A}^j(\bar{V}) \quad \text{for} \quad j = 0, \cdots, d. \tag{6}$$

Then, owing to the symmetry of the matrices $\bar{A}^j$, the classical energy method leads to

$$\frac{1}{2}\frac{d}{dt}\|Z\|^2_{L^2_{\bar{A}_0}} + (LZ|Z)_{L^2} = 0 \quad \text{with} \quad \|Z\|^2_{L^2_{\bar{A}_0}} \triangleq (\bar{A}_0 Z|Z)_{L^2}. \tag{7}$$

On the one hand, since the matrix $\bar{A}_0$ is symmetric and positive definite, we have

$$\|Z\|_{L^2_{\bar{A}_0}} \simeq \|Z\|_{L^2}. \tag{8}$$

On the other hand, (4) guarantees that there exists $\kappa_0 > 0$ such that

$$(LZ|Z)_{L^2} \geq \kappa_0 \|Z_2\|^2_{L^2} \quad \text{for all} \quad Z \in L^2(\mathbb{R}^d; \mathbb{R}^n). \tag{9}$$

Hence, (7) yields L^2-in-time integrability on the component of Z experiencing direct dissipation, but not on the whole solution. To compensate this lack of coercivity, following Beauchard and Zuazua in [2], we are going to introduce a lower order corrector $\mathcal{I}$ to track the optimal dissipation of the solution to (6). Since it is more natural to define that corrector on the Fourier side, we rewrite (6) in this setting, that is denoting by $\xi \in \mathbb{R}^d$ the Fourier variable,

$$\bar{A}^0\partial_t \widehat{Z} + i\sum_{j=1}^{d} \bar{A}^j \xi_j \widehat{Z} + L\widehat{Z} = 0. \tag{10}$$

Let us put $\xi = \rho\omega$ with $\omega \in \mathbb{S}^{d-1}$ and $\rho = |\xi|$. Then, the above system rewrites

$$\bar{A}_0\partial_t \widehat{Z} + i\rho A_\omega \widehat{Z} + L\widehat{Z} = 0 \quad \text{with} \quad A_\omega \triangleq \bar{A}_0^{-1}\sum_{j=1}^{d} \omega_j \bar{A}^j. \tag{11}$$

Let $N \triangleq \bar{A}_0^{-1}L$ and $M_\omega \triangleq \bar{A}_0^{-1}A_\omega$. Fix $n-1$ (small) positive parameters $\varepsilon_1, \cdots \varepsilon_{n-1}$, and set

$$\mathcal{I} \triangleq \Re \sum_{k=1}^{n-1} \varepsilon_k \big(NM_\omega^{k-1}\widehat{Z} \cdot NM_\omega^k \widehat{Z}\big) \tag{12}$$

where $\cdot$ designates the Hermitian scalar product in $\mathbb{C}^n$.

Then, differentiating $\mathcal{I}$ with respect to time and using (11) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{I} + \sum_{k=1}^{n-1} \varepsilon_k \rho |NM_\omega^k \widehat{Z}|^2 = &-\Im \sum_{k=1}^{n-1} \varepsilon_k \big(NM_\omega^{k-1}N\widehat{Z} \cdot NM_\omega^k \widehat{Z}\big) \\ &- \Re \sum_{k=1}^{n-1} \varepsilon_k \rho \big(NM_\omega^{k-1}\widehat{Z} \cdot NM_\omega^{k+1}\widehat{Z}\big) - \Im \sum_{k=1}^{n-1} \varepsilon_k \big(NM_\omega^{k-1}\widehat{Z} \cdot NM_\omega^k N\widehat{Z}\big) \cdot\end{aligned} \tag{13}$$

As pointed out in [2] (and recalled in Appendix for the reader's convenience), it is possible to choose positive and arbitrarily small parameters $\varepsilon_1, \cdots, \varepsilon_{n-1}$ so that (13) implies for some $C > 0$,

$$\frac{d}{dt}\mathcal{I} + \frac{1}{2}\sum_{k=1}^{n-1} \varepsilon_k \rho |NM_\omega^k \widehat{Z}|^2 \leq C\varepsilon_0 \max(\rho, \rho^{-1})|N\widehat{Z}|^2. \tag{14}$$

As $\ker N = \ker L$, one can replace $|N\widehat{Z}|$ by $|\widehat{Z}_2|$ in the right-hand side, up to a harmless change of C. At the same time, taking the Hermitian product of (10) with $\widehat{Z}$, using (4) and keeping the real part gives:

$$\frac{1}{2}\frac{d}{dt}(\bar{A}_0\widehat{Z}\cdot\widehat{Z}) + \kappa_0|\widehat{Z}_2|^2 \leq 0.$$

In the end, taking ε_1 small enough, adding up (14) and setting $\varepsilon_0 = \kappa_0/2$, we get

$$\frac{d}{dt}\mathcal{L}_{\rho,\omega} + \min(1,\rho^2)\sum_{k=0}^{n-1}\varepsilon_k|NM_\omega^k\widehat{Z}|^2 \leq 0 \quad \text{with} \quad \mathcal{L}_{\rho,\omega} \triangleq \bar{A}_0\widehat{Z}\cdot\widehat{Z} + 2\min(\rho,\rho^{-1})\mathcal{I}. \tag{15}$$

Clearly, if $\varepsilon_1,\cdots,\varepsilon_{n-1}$ are small enough, then $\mathcal{L}_{\rho,\omega} \simeq |\widehat{Z}|^2$. The question now is whether the rate of dissipation in (15) may be compared to $|\widehat{Z}|^2$. The answer depends on the possible cancellation of the following quantity:

$$\mathcal{N}_{\bar{V}} := \inf\left\{\sum_{k=0}^{n-1}\varepsilon_k|NM_\omega^k x|^2;\ x\in\mathbb{S}^{n-1},\ \omega\in\mathbb{S}^{d-1}\right\}. \tag{16}$$

At this very point, the (SK) (for Shizuta and Kawashima) condition comes into play:

Definition 1.1. *System* (1) *verifies the (SK) condition at $\bar{V}\in\mathbb{R}^n$ if, for all $\omega\in\mathbb{S}^{d-1}$, we have at the same time $N\phi = 0$ and $\lambda\phi + M_\omega\phi = 0$ for some $\lambda\in\mathbb{R}$, if and only if $\phi = 0_{\mathbb{R}^n}$.*

In order to pursue our analysis, we need the following key result (see the proof in e.g. [2]).

Proposition 1.1. *Let M and N be two matrices in $\mathcal{M}_n(\mathbb{R})$. The following assertions are equivalent:*

1. *$N\phi = 0$ and $\lambda\phi + M\phi = 0$ for some $\lambda\in\mathbb{R}$ implies $\phi = 0$;*
2. *For every $\varepsilon_0,\cdots,\varepsilon_{n-1} > 0$, the function*

$$y \longmapsto \sqrt{\sum_{k=0}^{n-1}\varepsilon_k|NM^k y|^2}$$

defines a norm on $\mathbb{R}^n$.

Thanks to the above proposition and observing that the unit sphere $\mathbb{S}^{d-1}$ is compact, one may conclude that Condition (SK) is satisfied by the pair (M_ω, N) for all $\omega\in\mathbb{S}^{d-1}$ if and only if $\mathcal{N}_{\bar{V}} > 0$. If indeed it is the case then, one can deduce from (15) and $\mathcal{L}_{\rho,\omega}\simeq|\widehat{Z}|^2$ that there exists some constant $c_{\bar{V}}$ such that

$$\frac{d}{dt}\mathcal{L}_{\rho,\omega} + c_{\bar{V}}\min(1,\rho^2)\mathcal{L}_{\rho,\omega} \leq 0. \tag{17}$$

Then, integrating on $\mathbb{R}^d$ and using Fourier-Plancherel, we conclude that there exists a Lyapunov functional that is equivalent to $\|Z\|_{L^2}^2$, and an explicit rate of dissipation $\mathcal{H}$. Note however, that the properties of $\mathcal{H}$ strongly depend on the support of $\widehat{Z}$ (or, equivalently, of $\widehat{Z}_0$). More precisely,

- if $\widehat{Z}_0$ is compactly supported then $\mathcal{H} \gtrsim \|\nabla Z\|_{L^2}^2$, which reveals a parabolic behavior of all components of the solution;
- if the support of $\widehat{Z}_0$ is away from the origin, then $\mathcal{H} \gtrsim \|Z\|_{L^2}^2$, which corresponds to exponential decay.

The above analysis reveals that, in order to get optimal dissipative estimates, it is suitable to split the solution into low and high frequencies parts. This will be achieved by means of a Littlewood-Paley decomposition (introduced in the next section). Then, an important part of our work will consist in bounding separately the low and high frequencies of the solution to the nonlinear system (3).

1.3. The damped mode

The second key ingredient for proving global existence results for (3) under the above assumptions is to exhibit a 'damped mode' that corresponds to the part of the solution that experiences maximal dissipation in low frequencies. Although a similar idea has been used before in the context of the compressible Navier-Stokes equations (see [14] and the references therein), it seems to be new for partially dissipative hyperbolic systems. This damped mode is defined as follows :

$$W \triangleq -{L_2}^{-1}\widetilde{A}^0_{2,2}(V)\partial_t Z_2 = Z_2 + \sum_{j=1}^d {L_2}^{-1}\big(\widetilde{A}^j_{2,1}(V)\partial_j Z_1 + \widetilde{A}^j_{2,2}(V)\partial_j Z_2\big) - {L_2}^{-1}Q_2(Z)\cdot \tag{18}$$

The key observation that motivates the definition of W is that it satisfies:

$$\widetilde{A}^0_{2,2}(V)\partial_t W + L_2 W = \widetilde{A}^0_{2,2}(V)L_2^{-1}\sum_{j=1}^d \partial_t\big(\widetilde{A}^j_{2,1}(V)\partial_j Z_1 + \widetilde{A}^j_{2,2}(V)\partial_j Z_2\big) - \widetilde{A}^0_{2,2}(V)L_2^{-1}\partial_t Q_2(Z)\cdot \tag{19}$$

On the left-hand side, Property (4) ensures maximal dissipation on W and the right-hand side of (19) contains only at least quadratic terms, or linear terms with one derivative. Hence, up to negligible terms, we expect the low frequencies of W to behave as the solution to an ODE with exponential decay. Of course, we will not get such fast a decay owing to the right-hand side, but still W will have much better time integrability properties than the overall solution. Since (18) ensures that the low frequencies W are comparable to those of Z_2, we will also get better integrability for Z_2.

2. Main results

Before stating our main results, introducing a few notations is in order. First, we fix a homogeneous Littlewood-Paley decomposition $(\dot\Delta_q)_{q\in\mathbb{Z}}$, setting

$$\dot\Delta_q \triangleq \varphi(2^{-q}D) \quad \text{with} \quad \varphi(\xi) \triangleq \chi(\xi/2) - \chi(\xi)$$

where χ stands for a smooth function with range in $[0,1]$, supported in the open ball $B(0,4/3)$ and such that $\chi \equiv 1$ on the closed ball $\bar{B}(0,3/4)$. We further state

$$\dot S_q \triangleq \chi(2^{-q}D) \quad \text{for all} \quad q\in\mathbb{Z}$$

and define $\mathcal{S}'_h$ to be the set of tempered distributions z such that

$$\lim_{q\to-\infty} \|\dot S_q z\|_{L^\infty} = 0.$$

Following [1], we introduce the homogeneous Besov semi-norms:

$$\|z\|_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \big\|2^{qs}\|\dot\Delta_q z\|_{L^p(\mathbb{R}^d)}\big\|_{\ell^r(\mathbb{Z})},$$

then define the homogeneous Besov spaces $\dot{\mathbb{B}}^s_{p,r}$ (for any $s\in\mathbb{R}$ and $(p,r)\in[1,\infty]^2$) to be the subset of those z in $\mathcal{S}'_h$ such that $\|z\|_{\dot{\mathbb{B}}^s_{p,r}}$ is finite.

Using from now on the shorthand notation

$$z_q \triangleq \dot\Delta_q z, \tag{20}$$

we associate to any element z of $\mathcal{S}'_h$, its low and high frequency parts through

$$z^\ell \triangleq \sum_{q\le 0} z_q = \dot S_1 z \quad \text{and} \quad z^h \triangleq \sum_{q>0} z_q = (\mathrm{Id} - \dot S_1)z.$$

We shall constantly use the following Besov semi-norms for low and high frequencies:

$$\|z\|^{\ell}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{q\leq 0} 2^{qs}\|z_q\|_{L^2} \quad\text{and}\quad \|z\|^{h}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{q>0} 2^{qs}\|z_q\|_{L^2},$$
$$\|z\|^{\ell}_{\dot{\mathbb{B}}^s_{2,\infty}} \triangleq \sup_{q\leq 0} 2^{qs}\|z_q\|_{L^2} \quad\text{and}\quad \|z\|^{h}_{\dot{\mathbb{B}}^s_{2,\infty}} \triangleq \sup_{q>0} 2^{qs}\|z_q\|_{L^2}.$$

Throughout the paper, we shall use repeatedly the following obvious fact:

$$\|z\|^{\ell}_{\dot{\mathbb{B}}^{s'}_{2,r}} \leq \|z\|^{\ell}_{\dot{\mathbb{B}}^{s}_{2,r}} \quad\text{and}\quad \|z\|^{h}_{\dot{\mathbb{B}}^{s'}_{2,r}} \geq \|z\|^{h}_{\dot{\mathbb{B}}^{s}_{2,r}} \quad\text{for}\quad r=1,\infty, \quad\text{whenever}\quad s\leq s'. \tag{21}$$

For any Banach space X, index ρ in $[1,\infty]$ and time $T\in[0,\infty]$, we use the notation $\|z\|_{L^\rho_T(X)} \triangleq \big\|\|z(t)\|_X\big\|_{L^\rho(0,T)}$. If $T=\infty$, then we just write $\|z\|_{L^\rho(X)}$. Finally, in the case where z has n components z_j in X, we keep the notation $\|z\|_X$ for $\sum_{j\in\{1,\cdots,n\}}\|z_j\|_X$.

Before stating our first global existence result, let us motivate our functional framework. Since our general approach is based on energy estimates, one has to use L^2-type spaces. Clearly, one can hardly expect to have a more general framework than the one that is required for the local well-posedness. Hence, as already pointed out, having (at least local) Besov regularity of type $\mathbb{B}^{\frac d2+1}_{2,1}$ is the minimal requirement, and this is the assumption that we shall make throughout on the high frequencies part Z_0^h of the initial data Z_0. There is more room for manoeuvre as regards the regularity index s for Z_0^ℓ. A mandatory upper bound is $s\leq d/2$ in order to stay in the Banach spaces framework (see [1] for more explanations). Then, a rather natural choice is $s=d/2-1$ because, as we shall see later on, combining the parabolic properties of the system in low frequencies with the dissipative properties in high frequencies will ensure a control of the gradient of the solution in $L^1(0,T;\dot{\mathbb{B}}^{\frac d2}_{2,1})$ and thus, by embedding, in $L^1(0,T;L^\infty)$, which will prevent the finite time blow-up (see Proposition 3.4 below).

Our first global existence result for System (3) exactly corresponds to this framework.

Theorem 2.1. *Let $\bar V$ in $\mathcal{O}_V$ satisfy $H(\bar V)=0$. Suppose that the structure conditions of paragraph 1.1 and Condition (SK) of Definition 1.1 are satisfied. Then, there exists a positive constant α such that for all $Z_0\in\dot{\mathbb{B}}^{\frac d2-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac d2+1}_{2,1}$ satisfying*

$$\mathcal{Z}_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2-1}_{2,1}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} \leq \alpha, \tag{22}$$

System (3) supplemented with initial data Z_0 admits a unique global-in-time solution Z in the space E defined by

$$Z\in\mathcal{C}_b(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac d2-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac d2+1}_{2,1}),\quad Z^h\in L^1(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac d2+1}_{2,1}),\quad Z_1^\ell\in L^1(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac d2+1}_{2,1}) \ \text{ and }\ W\in L^1(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac d2-1}_{2,1}),$$

with W defined according to (18).

Besides, Z_2^ℓ belongs to $L^1(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac d2}_{2,1})\cap L^2(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac d2-1}_{2,1})$, there exists a Lyapunov functional [4] $\mathcal{L}$ that is equivalent to $\|Z\|_{\dot{\mathbb{B}}^{\frac d2-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}$, and a constant C depending only on the matrices A^j and on H, such that

$$\mathcal{Z}(t)\leq C\mathcal{Z}_0 \quad \text{for all } t\geq 0 \tag{23}$$

where

$$\mathcal{Z}(t) \triangleq \|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac d2-1}_{2,1})} + \|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac d2-1}_{2,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac d2-1}_{2,1})}. \tag{24}$$

[4] Here we mean that $\mathcal{L}$ is a nonincreasing functional.

Remark 2.1. *As is, the above theorem does not extend to the case $d = 1$. The reason why is that the low frequency regularity index then becomes negative, so that some nonlinear terms cannot be bounded in the proper spaces. For more details, the reader may refer to [11].*

Remark 2.2. *Thanks to* (23)*, one can control the term $\|\nabla Z\|_{L^1(L^\infty)}$ that pops up whenever one wants to bound 'reasonable' norms of the solution. In particular, it allows to propagate any Sobolev norm H^s with $s \geq 0$ and thus to recover all the prior results on partially dissipative systems. We will see in the next statement that one can also propagate some norms with negative index of regularity.*

Our second result concerns the time-decay estimates of the solutions of Theorem 2.1.

Theorem 2.2. *Under the hypotheses of Theorem 2.1 and if, additionally, $Z_0 \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ then, there exists a constant C depending only on σ_1 and such that*

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0. \tag{25}$$

Furthermore, if $\sigma_1 > 1 - d/2$ then, denoting

$$\langle t\rangle \triangleq \sqrt{1+t^2}, \quad \alpha_1 \triangleq \frac{\sigma_1 + \frac{d}{2} - 1}{2} \quad \text{and} \quad C_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}},$$

we have the following decay estimates:

$$\begin{aligned}
\sup_{t\geq 0} \left\| \langle t\rangle^{\frac{\sigma+\sigma_1}{2}} Z(t)\right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_0 \quad \text{if} \quad -\sigma_1 < \sigma \leq d/2-1,\\
\sup_{t\geq 0} \left\| \langle t\rangle^{\frac{\sigma+\sigma_1}{2}+\frac{1}{2}} Z_2(t)\right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_0 \quad \text{if} \quad -\sigma_1 < \sigma \leq d/2-2,\\
\sup_{t\geq 0} \|\langle t\rangle^{\alpha_1} Z_2(t)\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_0 \quad \text{if} \quad \min(d/2-2, -\sigma_1) < \sigma \leq d/2-1\\
\text{and} \quad \sup_{t\geq 0} \left\|\langle t\rangle^{2\alpha_1} Z(t)\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\leq CC_0.
\end{aligned}$$

Remark 2.3. *Since we have the embedding $L^1 \hookrightarrow \dot{\mathbb{B}}^{-\frac{d}{2}}_{2,\infty}$, the above statement encompasses the classical integrability condition $Z_0 \in L^1$ (see e.g. [24] in a slightly different context).*

Remark 2.4. *Owing to the presence of dissipation in the equation of Z_2, the decay rate of the low frequencies of Z_2 is stronger than the decay of the whole solution.*

As already said, we do not know whether the above statements extend to the one-dimensional case. Another drawback of the above functional setting is that it is not appropriate for studying the infinite relaxation limit (that is $\lambda \to \infty$ if the dissipative term is replaced by λH). The reason for that is a bad dependence with respect to λ of the corresponding functional $\mathcal{Z}$ defined in (24).

Making the weaker assumption that Z_0^ℓ is in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ instead of $\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}$ allows to treat the one-dimensional case (see [11]) and to study the relaxation limit (see [12]). Our last global existence result consists in solving (3) in this more general framework. To achieve it, the following additional structure conditions are needed:

(iv) for all $j \in \{1, \cdots, d\}$, $\widetilde{A}^j_{1,1}(\bar V) = 0$ and $D_{V_1}\widetilde{A}^j_{1,1}(\bar V) = 0$;

(v) for all $j \in \{1, \cdots, d\}$, $D_{V_1}\widetilde{A}^j_{2,1}(\bar V) = 0$ (and thus also $D_{V_1}\widetilde{A}^j_{1,2}(\bar V) = 0$);

(vi) the function Q is quadratic with respect to Z_2 (i.e. $D^2_{V_i,V_j}Q_2(0) = 0$ for $(i,j) \neq (2,2)$).

Theorem 2.3. *Let the structure assumptions of Theorem 2.1 be in force and assume in addition that (iv), (v) and (vi) hold true.*

Then, there exists a positive constant α such that for all $Z_0 \in \dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ satisfying

$$\mathcal{Z}'_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \alpha, \tag{26}$$

System (3) *supplemented with initial data Z_0 admits a unique global-in-time solution Z in the space F defined by*

$$Z \in \mathcal{C}_b(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z^h \in L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z_1^{\ell} \in L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}) \ \text{ and } \ W \in L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}).$$

Moreover, Z_2^{ℓ} belongs to $L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}_{2,1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}}_{2,1})$, there exists a Lyapunov functional that is equivalent to $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$, and we have the following a priori estimate:

$$\begin{aligned} \mathcal{Z}'(t) \leq C\mathcal{Z}'_0 \quad \text{where} \quad \mathcal{Z}'(t) &\triangleq \|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} \\ &+ \|Z_1\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z\|^{h}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}. \end{aligned} \tag{27}$$

Finally, if, additionally, $Z_0 \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ then (25) *is satisfied as well as the decay estimates that follow,* up to $\sigma = d/2$ *for the first one, and with $d/2-1$ and $d/2$ instead of $d/2-2$ and $d/2-1$ for the next two ones, with α_1 replaced by $(\sigma_1 + d/2)/2$.*

Remark 2.5. *The above result applies to the compressible Euler with damping (see Theorem 4.1).*

3. Proof of Theorems 2.1 and 2.2

This section is devoted to proving the global existence of strong solutions and decay estimates for System (1) supplemented with initial data that are close to the reference solution $\bar{V}$, in the general case where the structural assumptions listed in Subsection 1.1 and (SK) condition are satisfied.

The bulk of the proof consists in establishing a priori estimates, the other steps (proving existence and uniqueness) being more classical. As explained before, our strategy is to first work out a Lyapunov functional in Beauchard-Zuazua's style, that is equivalent to the norm that we aim at controlling, then to combine with the study of the damped mode W defined in (18) so as to close the estimates.

3.1. Establishing the a priori estimates

Throughout this part, we assume that we are given a smooth (and decaying) solution Z of (3) on $[0,T] \times \mathbb{R}^d$ with Z_0 as initial data, satisfying

$$\sup_{t\in[0,T]} \|Z(t)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \ll 1. \tag{28}$$

We shall use repeatedly that, owing to the embedding $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$, we have also

$$\sup_{t\in[0,T]} \|Z(t)\|_{L^\infty} \ll 1. \tag{29}$$

From now on, $C>0$ designates a generic harmless constant, the value of which depends on the context and we denote by $(c_q)_{q\in\mathbb{Z}}$ nonnegative sequences such that $\sum_{q\in\mathbb{Z}} c_q = 1$.

To start with, let us rewrite (3) as:

$$\bar{A}^0 \partial_t Z + \sum_{j=1}^d \bar{A}^j \partial_j Z + LZ = G \tag{30}$$

with $G \triangleq G_1 + G_2 + G_3$ and

$$
\begin{aligned}
G_1 &\triangleq -\sum_{j=1}^{d} \bar{A}^0 \left((\widetilde{A}^0(V))^{-1} \widetilde{A}^j(V) - (\bar{A}^0)^{-1} \bar{A}^j \right) \partial_j Z, \\
G_2 &\triangleq -\bar{A}^0 \left((\widetilde{A}^0(V))^{-1} - (\bar{A}^0)^{-1} \right) LZ, \\
G_3 &\triangleq \bar{A}^0 (\widetilde{A}^0(V))^{-1} Q(Z).
\end{aligned}
$$

For $q \in \mathbb{Z}$, applying $\dot{\Delta}_q$ to (30) yields

$$
\bar{A}^0 \partial_t Z_q + \sum_{j=1}^{d} \bar{A}^j \partial_j Z_q + LZ_q = \dot{\Delta}_q G \quad \text{with} \quad Z_q \triangleq \dot{\Delta}_q Z. \tag{31}
$$

Our analysis will mainly consist in estimating for all $q \in \mathbb{Z}$ a functional $\mathcal{L}_q$ that is equivalent to the $L^2(\mathbb{R}^d; \mathbb{R}^n)$ norm of Z_q and encodes informations on the dissipative properties of the system. That functional will be built from (15) and, since Condition (SK) is satisfied, the number $\mathcal{N}_{\bar{V}}$ defined in (16) will be positive. Furthermore, since the Fourier transform of Z_q is localized near the frequencies of magnitude 2^q, Inequality (17) will become after time integration and application of Fourier-Plancherel theorem,

$$
\frac{d}{dt} \mathcal{L}_q + \mathcal{H}_q \leq 0 \quad \text{with} \quad \mathcal{H}_q \gtrsim \min(1, 2^{2q}) \mathcal{L}_q. \tag{32}
$$

The prefactor $\min(1, 2^{2q})$ may be seen as a gain of two derivatives in low frequencies after time integration (like for the heat equation) whereas it corresponds to exponential decay for high frequencies. In our setting where the low and high frequencies of Z_0 belong to the spaces $\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}$ and $\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$, respectively, we will eventually get from (32) after summing up on $q \leq 0$ or $q > 0$ and integrating on $[0, t]$,

$$
\begin{aligned}
\|Z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \int_0^t \|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\lesssim \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \int_0^t \|G\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}, \\
\|Z(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\lesssim \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|G\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.
\end{aligned}
$$

A rapid examination reveals that the part G_1 of G may entail a loss of one derivative (since it is a combination of components of ∇Z) while G_2 and G_3 contain products of components of Z and Z_2 owing to our structure assumptions. Overcoming the difficulty with G_1 will be achieved by exploiting the symmetrizable character of the system under consideration and changing slightly the weight $\bar{A}_0$ in the definition of $\mathcal{L}_q$ for the high frequencies: we shall take

$$
\mathcal{L}_q \triangleq \|Z_q\|^2_{L^2_{\widetilde{A}_0(V)}} + 2^{-q} \mathcal{I}_q \quad \text{if} \quad q \geq 0, \tag{33}
$$

with

$$
\mathcal{I}_q \triangleq \int_{\mathbb{R}^d} \sum_{k=1}^{n-1} \varepsilon_k \Im \left((N M_\omega^{k-1} \widehat{Z_q}) \cdot (N M_\omega^k \widehat{Z_q}) \right), \tag{34}
$$

where $\varepsilon_1, \cdots, \varepsilon_{n-1} > 0$ will be chosen small enough (according to the Appendix).

For the low frequencies, we shall keep the original definition corresponding to (15) which, after integrating on the whole space and using Fourier-Plancherel theorem yields,

$$
\mathcal{L}_q \triangleq \|Z_q\|^2_{L^2_{\bar{A}_0}} + 2^q \mathcal{I}_q \quad \text{if} \quad q \leq 0. \tag{35}
$$

However, we will discover that the terms G_2 and G_3 cannot be controlled properly in the space $L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})$ because Z_2 is, somehow, too regular ! The way to overcome the difficulty is to look for an estimate of the low frequencies of the damped mode W, then to compare with Z_2.

We shall keep in mind all the time that if choosing the coefficients ε_k small enough, then we have

$$\sum_{k=1}^{n-1}\varepsilon_k\big|((M_\omega)^t)^k N^t N M_\omega^{k-1}\big| \leq \frac{1}{2}\frac{1}{(2\pi)^d},$$

whence, owing to Fourier-Plancherel theorem,

$$|\mathcal{I}_q| \leq \frac{1}{2}\|Z_q\|_{L^2}.$$

Furthermore, as $\bar{A}_0 = A_0(\bar{V})$ is positive definite and $V \mapsto \widetilde{A}_0(V)$, continuous, Condition (29) ensures that $\|Z_q\|_{L^2_{\bar{A}_0}} \simeq \|Z_q\|_{L^2}$ and $\|Z_q\|_{L^2_{\widetilde{A}_0(V)}} \simeq \|Z_q\|_{L^2}$. Therefore, we have

$$\mathcal{L}_q \simeq \|Z_q\|_{L^2}^2 \text{ for all } q \in \mathbb{Z}. \tag{36}$$

3.1.1. Basic energy estimates

The first step is devoted to studying the time evolution of $\|Z_q\|^2_{L^2_{\widetilde{A}_0(V)}}$ and $\|Z_q\|^2_{L^2_{\bar{A}_0}}$. The outcome is given in the following proposition.

Proposition 3.1. *Let Z be a smooth solution of* (3) *on* $[0,T]\times\mathbb{R}^d$ *satisfying* (28). *Then, for all* $s \in \left[\frac{d}{2}, \frac{d}{2}+1\right]$ *and* $q \geq 0$, *we have:*

$$\begin{aligned}\frac{d}{dt}\|Z_q\|^2_{L^2_{\widetilde{A}_0(V)}} + \kappa_0\|Z_{2,q}\|^2_{L^2} &\lesssim \|(\nabla Z, Z_2)\|_{L^\infty}\|Z_q\|^2_{L^2} + c_q 2^{-qs}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^s_{2,1}}\|Z_q\|_{L^2}\\ &+ c_q 2^{-qs}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}\|Z_q\|_{L^2} + c_q 2^{-qs}\big(\|Z\|_{\dot{\mathbb{B}}^s_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^s_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|Z_q\|_{L^2}.\end{aligned} \tag{37}$$

Furthermore, for all $s' \in \left[\frac{d}{2}-1, \frac{d}{2}\right]$ *and* $q \leq 0$, *we have:*

$$\begin{aligned}\frac{d}{dt}\|Z_q\|^2_{L^2_{\bar{A}_0}} + \kappa_0\|Z_{2,q}\|^2_{L^2} &\lesssim \|\nabla Z\|_{L^\infty}\|Z_q\|^2_{L^2} + c_q 2^{-qs'}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z_q\|_{L^2}\\ &+ c_q 2^{-qs'}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z_q\|_{L^2} + c_q 2^{-qs'}\|Z_2\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_q\|_{L^2}.\end{aligned} \tag{38}$$

Proof. It relies on an energy method implemented on (3) after localization in the Fourier space, and on classical commutator estimates.

In order to prove (37), apply operator $\dot{\Delta}_q$ to (3) to get:

$$\widetilde{A}^0(V)\partial_t Z_q + \sum_{j=1}^d \widetilde{A}^j(V)\partial_j Z_q + LZ_q = R^1_q + R^2_q + \dot{\Delta}_q(Q(Z))$$

with $R^1_q \triangleq \sum_{j=1}^d[\widetilde{A}^j(V), \dot{\Delta}_q]\partial_j Z$ and $R^2_q \triangleq [\widetilde{A}^0(V), \dot{\Delta}_q]\partial_t Z$.

Taking the $L^2(\mathbb{R}^d;\mathbb{R}^n)$ scalar product with Z_q, integrating by parts in the second term and using the fact that the matrices $\widetilde{A}^j(V)$ are symmetric yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\widetilde{A}_0(V)Z_q\cdot Z_q + \int_{\mathbb{R}^d} LZ_q\cdot Z_q = \frac{1}{2}\int_{\mathbb{R}^d}\Big(\partial_t\tilde{A}^0(V) + \sum_j \partial_j(\widetilde{A}^j(V))\Big)Z_q\cdot Z_q&\\ + \int_{\mathbb{R}^d}(R^1_q + R^2_q)\cdot Z_q + \int_{\mathbb{R}^d}\dot{\Delta}_q Q(Z)\cdot Z_q.&\end{aligned}$$

Hence, thanks to Property (4), we obtain

$$\frac{1}{2}\frac{d}{dt}\|Z_q\|^2_{L^2_{\widetilde{A}^0(V)}}+\kappa_0\|Z_{2,q}\|^2_{L^2}\le\frac{1}{2}\int_{\mathbb{R}^d}\Big(\partial_t\widetilde{A}^0(V)+\sum_j\partial_j(\widetilde{A}^j(V))\Big)Z_q\cdot Z_q$$
$$+\int_{\mathbb{R}^d}(R^1_q+R^2_q)\cdot Z_q+\int_{\mathbb{R}^d}\dot{\Delta}_qQ(Z)\cdot Z_q. \quad (39)$$

For the first term in the right-hand side, we have

$$\int_{\mathbb{R}^d}\partial_t(\widetilde{A}^0(V))Z_q\cdot Z_q\lesssim\|\partial_tZ\|_{L^\infty}\|Z_q\|^2_{L^2}. \quad (40)$$

Hence, using the fact that

$$\partial_tZ=(\widetilde{A}^0(V))^{-1}\Big(\widetilde{H}(\bar{V}+Z)-\sum_{j=1}^d\widetilde{A}^j(V)\partial_jZ\Big), \quad (41)$$

the smallness condition (29) and the structure of $\widetilde{H}$, we get

$$\int_{\mathbb{R}^d}\partial_t(\widetilde{A}^0(V))Z_q\cdot Z_q\lesssim\|(\nabla Z,Z_2)\|_{L^\infty}\|Z_q\|^2_{L^2}. \quad (42)$$

For the second term in the right-hand side of (39), we may write

$$\int_{\mathbb{R}^d}\sum_{j=1}^d\partial_j(\widetilde{A}^j(V))Z_q\cdot Z_q\lesssim\|\nabla Z\|_{L^\infty}\|Z_q\|^2_{L^2}. \quad (43)$$

Bounding the commutators terms in (39) relies on Cauchy-Schwarz inequality and Inequality (103) that give

$$\int_{\mathbb{R}^d}(R^1_q+R^2_q)\cdot Z_q\lesssim c_q2^{-qs}\Big(\sum_j\big\|\nabla(\widetilde{A}^j(V))\big\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}+\big\|\nabla(\widetilde{A}^0(V))\big\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\partial_tZ\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}\Big)\|Z_q\|_{L^2}$$
$$\lesssim c_q2^{-qs}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big(\|Z\|_{\dot{\mathbb{B}}^{s}_{2,1}}+\|\partial_tZ\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}\big)\|Z_q\|_{L^2}.$$

To bound ∂_tZ, we need the following lemma.

Lemma 3.1. *Under assumption* (28)*, we have for all* $\sigma\in]-d/2,d/2]$,

$$\|\partial_tZ\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\min\big(\|W\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}},\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\big)\cdot$$

Proof. Using (41), Propositions 5.2, 5.3 and 5.4 yields

$$\|\partial_tZ\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\le\Big\|\sum_{j=1}^d\widetilde{A}^j(V)\partial_jZ\Big\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|LZ\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Q(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}$$
$$\lesssim\big(1+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.$$

Since we assumed that $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$ is small, we readily have

$$\|\partial_tZ\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.$$

To get the bound with W, we observe that $\partial_tZ_2=-(\widetilde{A}^0_{2,2}(V))^{-1}L_2W$ by definition of W and that the first line of (5) combined with Propositions 5.2, 5.3 and 5.4 yields

$$\|\partial_tZ_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.$$

Hence, since from (18) we have

$$\|W - Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}},$$

we get the desired inequality. ☐

Finally, Proposition 5.4 ensures that

$$\begin{aligned}\int_{\mathbb{R}^d} \dot{\Delta}_q Q(Z)\cdot Z_q &\lesssim c_q 2^{-qs}\, \|Q(Z)\|_{\dot{\mathbb{B}}^{s}_{2,1}} \|Z_q\|_{L^2}\\ &\lesssim c_q 2^{-qs}\Big(\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{s}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{s}_{2,1}}\Big) \|Z_q\|_{L^2}\cdot\end{aligned}$$

Putting all the above estimates together completes the proof of (37).

For proving (38), since we do not know how to control $\partial_t Z$ in $L^1_T(\dot{\mathbb{B}}^{s'-1}_{2,1})$ for $s' = d/2-1$ (which is the value that we will take eventually), we proceed slightly differently, writing the equation satisfied by Z_q as follows:

$$\bar{A}^0\partial_t Z_q + \sum_{j=1}^{d} \widetilde{A}^j(V)\partial_j Z_q + LZ_q = R^1_q + R^3_q + \dot{\Delta}_q(Q(Z)) \quad \text{with} \quad R^3_q \triangleq \dot{\Delta}_q\Big(\big(\bar{A}^0 - \widetilde{A}^0(V)\big)\partial_t Z\Big)\cdot$$

Arguing as for proving (39), we now get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_q\|^2_{L^2_{\bar{A}^0}} + \kappa_0\|Z_{2,q}\|^2_{L^2} \le \frac{1}{2}\int_{\mathbb{R}^d}\Big(\sum_j \partial_j(\widetilde{A}^j(V))\Big) Z_q\cdot Z_q&\\ + \int_{\mathbb{R}^d}(R^1_q + R^3_q)\cdot Z_q + \int_{\mathbb{R}^d}\dot{\Delta}_q Q(Z)\cdot Z_q.&\end{aligned} \tag{44}$$

The term R^1_q may be estimated as above (with s' instead of s), and $Q(Z)$ may be bounded by means of (109). As regards R^3_q, we write that

$$\big\|R^3_q\big\|_{L^2} \lesssim c_q 2^{-qs'}\, \Big\|\widetilde{A}^0(V) - \widetilde{A}^0(\bar{V})\Big\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\partial_t Z\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\cdot$$

Thus, using composition, product estimates and Lemma 3.1, we obtain

$$\left|\int_{\mathbb{R}^d} R^3_q\cdot Z_q\right| \;\lesssim\; c_q 2^{-qs'}\, \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{s'}_{2,1}} \|Z_q\|_{L^2},$$

which leads to the desired estimate. ☐

3.1.2. Cross estimates

Proposition 3.1 only allows to exhibit the integrability properties of the components of Z experiencing direct dissipation. To recover the dissipation for all the components, we have to look at the time derivative of the quantity $\mathcal{I}_q$ defined in (34). To achieve it, we apply to (31) the method that has been explained in Section 1 and leads to (14) and (15). The only change lies in the (harmless) additional source term $\dot{\Delta}_q G$. In the end, integrating on $\mathbb{R}^d$ the obtained identity, then using the fact that Supp $\widehat{Z}_q \subset \big\{3\cdot 2^q/4 \le |\xi| \le 8\cdot 2^q/3\big\}$ yields

$$\frac{d}{dt}\mathcal{I}_q + \frac{2^q}{2}\sum_{k=1}^{n-1}\varepsilon_k\int_{\mathbb{R}^d}|NM^k_\omega \widehat{Z}_q|^2\,d\xi \le \frac{\kappa_0}{2}\max(1,2^{-q})\|Z_{2,q}\|^2_{L^2} + C\|\dot{\Delta}_q G\|_{L^2}\|Z_q\|_{L^2}.$$

The last term will be bounded by means of Propositions 5.2, 5.3 and 5.4 (keeping all the time in mind that (28) is satisfied). More precisely, for G_1, we have for all $\sigma \in]-d/2, d/2]$,

$$\begin{aligned}\|G_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\lesssim \sum_{j=1}^{d} \big\|\big(\widetilde{A}^0(V)^{-1}\widetilde{A}^j(V) - (\bar{A}^0)^{-1}\bar{A}^j\big)\partial_j Z\big\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \\ &\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}. \end{aligned} \tag{45}$$

Similarly,

$$\|G_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \left\|\left((\widetilde{A}^0(V))^{-1} - (\bar{A}_0)^{-1}\right) LZ\right\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \tag{46}$$

and, using Proposition 5.4,

$$\|G_3\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} = \left\|\bar{A}^0\widetilde{A}^0(V)^{-1}Q(Z)\right\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}. \tag{47}$$

Hence, one can conclude that for all $\sigma \in]-d/2, d/2]$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_q + \frac{2^q}{2}\sum_{k=1}^{n-1}\varepsilon_k \int_{\mathbb{R}^d} |NM_\omega^k \widehat{Z}_q|^2\,d\xi \\ \leq \frac{\kappa_0}{2}\max(1,2^{-q})\|Z_{2,q}\|_{L^2}^2 + Cc_q 2^{-q\sigma}\|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_q\|_{L^2}.\end{aligned} \tag{48}$$

3.1.3. Closure of the estimates : a first attempt

Remember that since Condition (SK) is satisfied, the quantity $\mathcal{N}_{\bar{V}}$ defined in (16) is positive for any choice of positive parameters $\varepsilon_0, \cdots, \varepsilon_{n-1}$. Consequently, if we set

$$\mathcal{H}_q := \frac{\kappa_0}{2}\|NZ_q\|^2 + \min(1, 2^{2q})\sum_{k=1}^{n-1}\varepsilon_k\int_{\mathbb{R}^d}|NM_\omega^k\widehat{Z}_q|^2\,d\xi$$

and use Fourier-Plancherel theorem and the equivalence (36), we see that (up to a change of κ_0), we have for all $q \in \mathbb{Z}$,

$$\mathcal{H}_q \geq \kappa_0 \min(1, 2^{2q})\mathcal{L}_q \tag{49}$$

where $\mathcal{L}_q$ has been defined in (33) and (35).

We shall use this inequality to bound the quantity $\mathcal{Z}$ defined in (24) in terms of $\mathcal{Z}_0$ only.

Let us start with the bounds for the low frequencies. Putting together Inequality (38) with $s' = d/2 - 1$ and the cross estimate (48) with $\sigma = d/2$ then, using (49), we get for all $q \leq 0$,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_q + \kappa_0 2^{2q}\mathcal{L}_q \lesssim \|\nabla Z\|_{L^\infty}\|Z_q\|_{L^2}^2 + c_q 2^{-q(\frac{d}{2}-1)}\bigg(\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \\ + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\bigg)\|Z_q\|_{L^2}.\end{aligned} \tag{50}$$

Hence, using (36), applying Lemma 5.1, multiplying by $2^{q(\frac{d}{2}-1)}$, using the embedding $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$ and summing up on $q \leq 0$ gives

$$\begin{aligned}\|Z(t)\|^{\ell}_{\dot{\mathcal{B}}^{\frac{d}{2}-1}_{2,1}} + \kappa_0\int_0^t \|Z\|^{\ell}_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}}\,d\tau \leq \|Z_0\|^{\ell}_{\dot{\mathcal{B}}^{\frac{d}{2}-1}_{2,1}} \\ + \int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big),\end{aligned}$$

where we used the notation

$$\|Z\|^{\ell}_{\dot{\mathcal{B}}^{\sigma}_{2,1}} \triangleq \sum_{q\leq 0} 2^{q\sigma}\sqrt{\mathcal{L}_q}. \tag{51}$$

To handle the high frequencies, we combine Inequality (37) with $s = d/2 + 1$, the cross estimate (48) with $\sigma = d/2$ and (49), to get for all $q > 0$,

$$\frac{d}{dt}\mathcal{L}_q + \kappa_0 \mathcal{L}_q \lesssim \|(\nabla Z, Z_2)\|_{L^\infty} \|Z_q\|_{L^2}^2 + c_q 2^{-q(\frac{d}{2}+1)} \Big(\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \Big) \|Z_q\|_{L^2} \cdot \tag{52}$$

Hence, using the equivalence (36), Lemma 5.1, multiplying by $2^{q(\frac{d}{2}+1)}$ and summing up on $q > 0$ gives

$$\|Z(t)\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} + \kappa_0 \int_0^t \|Z\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} \leq \|Z_0\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \Big(\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \Big) \tag{53}$$

where we used the notation

$$\|Z\|^h_{\dot{\mathcal{B}}^{\sigma}_{2,1}} \triangleq \sum_{q>0} 2^{q\sigma} \sqrt{\mathcal{L}_q}.$$

Let us introduce the functional

$$\mathcal{L} \triangleq \|Z\|^\ell_{\dot{\mathcal{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} \tag{54}$$

which, in light of (36), is equivalent to $\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$, and thus to $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$.

Adding up (50) and (53) we get up to a change of κ_0 and for all $t \in [0, T]$,

$$\mathcal{L}(t) + \kappa_0 \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{L}(0) + C \int_0^t \Big(\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \Big) \cdot$$

Hence, using the interpolation inequality

$$\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \sqrt{\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}} \lesssim \mathcal{L}, \tag{55}$$

and eliminating some redundant terms, we end up with

$$\mathcal{L}(t) + \kappa_0 \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{L}(0) + C \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \mathcal{L} + C \int_0^t \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} . \tag{56}$$

As we start from small data, we expect $\mathcal{L}$ to be small as well so that the first term in the first integral in the right-hand side may be absorbed by the second term on the left. However, at this stage, we have no proper control on $\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}$. Studying the evolution of the damped mode W, which is the aim of the next section, will enable us to overcome the difficulty.

3.1.4. The damped mode

As underlined in the introduction, the function

$$W \triangleq -{L_2}^{-1} A^0_{2,2}(V) \partial_t Z_2 = Z_2 + \sum_{j=1}^d {L_2}^{-1} \big(\widetilde{A}^j_{2,1}(V) \partial_j Z_1 + \widetilde{A}^j_{2,2}(V) \partial_j Z_2 \big) - {L_2}^{-1} Q_2(Z) \cdot$$

is expected to have *better* integrability properties in low frequencies than the whole solution. This will be a consequence of the following proposition.

Proposition 3.2. *Let Z be a smooth solution of* (3) *on* $[0,T]\times\mathbb{R}^d$ *satisfying* (28)*, and denote* $\bar{A}^0_{2,2}\triangleq A^0_{2,2}(\bar{V})$*. Assume that* $\sigma\in]-d/2,d/2]$*. Then we have for all* $q\leq 0$*,*

$$\frac{1}{2}\frac{d}{dt}\|W_q\|^2_{L^2_{\bar{A}^0_{2,2}}}+\kappa_0\|W_q\|^2_{L^2}\lesssim\Big(\|\nabla^2 Z_q\|_{L^2}+\|\nabla W_q\|_{L^2}\Big)\|W_q\|_{L^2}$$
$$+c_q2^{-q\sigma}\|W_q\|_{L^2}\Big(\|(\nabla Z,W)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|(\nabla Z,W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\min(\sigma+1,\frac{d}{2})}_{2,1}}\Big)\cdot$$

Proof. From (19), we gather that

$$\bar{A}^0_{2,2}\partial_t W+L_2W=h \tag{57}$$

with $h\triangleq h_1+\bar{A}^0_{2,2}{L_2}^{-1}(h_2+h_3)$ and

$$h_1\triangleq\big(\mathrm{Id}-\bar{A}^0_{2,2}(\widetilde{A}^0_{2,2}(V))^{-1}\big)L_2W,$$
$$h_2\triangleq\sum_{j=1}^d\partial_t\big(\widetilde{A}^j_{2,1}(V)\partial_jZ_1+\widetilde{A}^j_{2,2}(V)\partial_jZ_2\big),$$
$$h_3=-\partial_tQ_2(Z)\cdot$$

Applying $\dot{\Delta}_q$ to (57) and taking the scalar product with $W_q\triangleq\dot{\Delta}_qW$ yields, thanks to (9),

$$\frac{1}{2}\frac{d}{dt}\|W_q\|^2_{L^2_{\bar{A}^0_{2,2}}}+\kappa_0\|W_q\|^2_{L^2}\leq\big(\|\dot{\Delta}_qh_1\|_{L^2}+C\|\dot{\Delta}_qh_2\|_{L^2}+C\|\dot{\Delta}_qh_3\|_{L^2}\big)\|W_q\|_{L^2}. \tag{58}$$

As (28) is satisfied, Propositions 5.2 and 5.3 readily give that for all $\sigma\in]-d/2,d/2]$,

$$\|h_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|W\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}. \tag{59}$$

For bounding h_2, we use that for all $j\in\{1,\cdots,d\}$,

$$\partial_t(\widetilde{A}^j_{2,1}(V)\partial_jZ_1+\widetilde{A}^j_{2,2}(V)\partial_jZ_2)=D_V\widetilde{A}^j_{2,1}(V)\partial_tZ\partial_jZ_1+\bar{A}^j_{2,1}\partial_t\partial_jZ_1$$
$$+\big(\widetilde{A}^j_{2,1}(V)-\bar{A}^j_{2,1}\big)\partial_t\partial_jZ_1+D_V\widetilde{A}^j_{2,2}(V)\partial_tZ\partial_jZ_2+\bar{A}^j_{2,2}\partial_t\partial_jZ_2+\big(\widetilde{A}^j_{2,2}(V)-\bar{A}^j_{2,2}\big)\partial_t\partial_jZ_2.$$

For $k=1,2$, we have, according to product and composition laws, and Lemma 3.1,

$$\|D_V\widetilde{A}^j_{2,k}(V)\partial_tZ\partial_jZ_k\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|\partial_tZ\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}$$
$$\lesssim\|(\nabla Z,W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}$$

as well as (provided we also have $\sigma\leq d/2-1$):

$$\|\big(\widetilde{A}^j_{2,k}(V)-\bar{A}^j_{2,k}\big)\partial_t\partial_jZ_k\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|\partial_t\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}$$
$$\lesssim\|(\nabla Z,W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}.$$

Multiplying the first equation of (5) (on the left) by the matrix $(\widetilde{A}^0_{1,1}(V))^{-1}$ then differentiating with respect to x_j, we discover that $\partial_t\partial_jZ_1$ is a combination of terms of type $A(Z)\,Z\otimes DZ$, $B(Z)\,D^2Z$ and $C(Z)\,DZ\otimes DZ$ where A, B and Z are smooth functions.

We have, owing to product and composition laws,

$$\|A(Z)Z\otimes DZ\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}$$

while for all $q\leq 0$ if $\sigma\leq d/2-1$:

$$\|\dot{\Delta}_q(B(Z)\,D^2Z)\|_{L^2}+\|\dot{\Delta}_q(C(Z)\,DZ\otimes DZ)\|_{L^2}\lesssim\|\nabla^2Z_q\|_{L^2}+c_q2^{-q\sigma}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Note that if $\sigma\in]d/2-1,d/2]$, then the above inequalities are valid (owing to (21)) if we change $\sigma+1$ to $d/2$.

Finally, we have

$$\partial_t \partial_j Z_2 = -\partial_j\big((\bar A^0_{2,2})^{-1} L_2 W\big) + \partial_j\big((\bar A^0_{2,2})^{-1} - \big(\widetilde A^0_{2,2}(V)\big)^{-1}\big) L_2 W.$$

Hence, for all $q \le 0$, and thanks to (21),

$$\begin{aligned}\|\dot\Delta_q(\partial_t\partial_j Z_2)\|_{L^2} &\lesssim \|\nabla W_q\|_{L^2} + c_q 2^{-q\sigma}\|\big((\bar A^0_{2,2})^{-1} - \big(\widetilde A^0_{2,2}(V)\big)^{-1}\big)L_2 W\|_{\dot{\mathbb B}^\sigma_{2,1}}\\ &\lesssim \|\nabla W_q\|_{L^2} + c_q 2^{-q\sigma}\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|W\|_{\dot{\mathbb B}^\sigma_{2,1}}.\end{aligned}$$

Putting the above inequalities together, we get for all $q \le 0$ and $\sigma \in]-d/2, d/2]$

$$\begin{aligned}\|\dot\Delta_q h_2\|_{L^2} \lesssim \|\nabla W_q\|_{L^2} + \|\nabla^2 Z_q\|_{L^2}&\\ + c_q 2^{-q\sigma}\Big(\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|(\nabla Z, W)\|_{\dot{\mathbb B}^\sigma_{2,1}} + \|(\nabla Z, W)\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\min(\sigma+1,\frac d2)}_{2,1}}\Big)&\end{aligned} \tag{60}$$

To bound h_3, we use the fact that $\partial_t Q_2(Z) = D_Z Q_2(Z)\partial_t Z$. As $D_Z Q_2$ vanishes at 0, we easily obtain from Propositions 5.2 and 5.3, and Lemma 3.1 that

$$\|h_3\|_{\dot{\mathbb B}^\sigma_{2,1}} \lesssim \|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|(\nabla Z, W)\|_{\dot{\mathbb B}^\sigma_{2,1}} \tag{61}$$

which concludes the proof. □

It is now easy to obtain dissipative estimates for the low frequencies of W. Indeed, starting from the inequality of Proposition 3.2, taking advantage of Lemma 5.1, multiplying the resulting inequality with $2^{q\sigma}$ and summing up on $q \le 0$, we get whenever $\sigma \in]-d/2, d/2]$,

$$\begin{aligned}\mathcal W^\sigma(t) + \kappa_0\int_0^t \|W\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}} \le \mathcal W^\sigma(0) + C\int_0^t \|(\nabla^2 Z, \nabla W)\|^\ell_{\dot{\mathbb B}^\sigma_{2,1}}&\\ + C\int_0^t \|(W,\nabla Z)\|_{\dot{\mathbb B}^\sigma_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}} + C\int_0^t \|(\nabla Z, W)\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\min(\sigma+1,\frac d2)}_{2,1}}&\end{aligned} \tag{62}$$

with $\mathcal W^\sigma \triangleq \sum_{q<0} 2^{q\sigma}\|\dot\Delta_q W\|_{L^2_{\bar A^0_{2,2}}}$.

Let us first apply (62) with $\sigma = d/2$. Then we get (discarding the redundant terms):

$$\mathcal W^{\frac d2}(t) + \kappa_0\int_0^t \|W\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}} \le \mathcal W^{\frac d2}(0) + C\int_0^t \|(\nabla^2 Z, \nabla W)\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}} + C\int_0^t \|(\nabla Z, W)\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}. \tag{63}$$

In order to close the estimates, we also need the inequality corresponding to $\sigma = d/2 - 1$, namely

$$\begin{aligned}\mathcal W^{\frac d2-1}(t) + \kappa_0\int_0^t \|W\|^\ell_{\dot{\mathbb B}^{\frac d2-1}_{2,1}} \le \mathcal W^{\frac d2-1}(0) + C\int_0^t \|(\nabla Z, W)\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}}&\\ + C\int_0^t \|(W,\nabla Z)\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}} + C\int_0^t \|(\nabla Z, W)\|_{\dot{\mathbb B}^{\frac d2-1}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}.&\end{aligned} \tag{64}$$

Since

$$Z_2 = W - \sum_{j=1}^d L_2^{-1}\big(A^j_{2,1}(V)\partial_j Z_1 + A^j_{2,2}(V)\partial_j Z_2\big) + L_2^{-1}Q_2(Z) \tag{65}$$

and $\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}$ is small, we have for all $\sigma \in]-d/2, d/2]$,

$$\|W - Z_2\|_{\dot{\mathbb B}^\sigma_{2,1}} \lesssim \|\nabla Z\|_{\dot{\mathbb B}^\sigma_{2,1}} + \|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z_2\|_{\dot{\mathbb B}^\sigma_{2,1}}. \tag{66}$$

Hence, W may be replaced by Z_2 in the last term of Inequality (63), and (64) becomes

$$\mathcal{W}^{\frac{d}{2}-1}(t)+\kappa_0\int_0^t\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\leq\mathcal{W}^{\frac{d}{2}-1}(0)+C\int_0^t\|(\nabla Z,W)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$
$$+C\int_0^t\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \quad (67)$$

3.1.5. Global a priori estimates

We are now ready to establish the following proposition which will be the key to the proof of the existence part of Theorem 2.1.

Proposition 3.3. *Let Z be a smooth solution of* (3) *on $[0,T]$ satisfying the smallness condition* (28)*. Then, there exist three (small) positive parameters κ_0, ε and ε' such that*

$$\widetilde{\mathcal{L}}\triangleq\mathcal{L}+\varepsilon\mathcal{W}^{\frac{d}{2}}+\varepsilon'\mathcal{W}^{\frac{d}{2}-1}$$

with $\mathcal{L}$ and $\mathcal{W}^{\sigma}$ defined in (56) *and* (62)*, respectively, satisfies for all $0\leq t_0\leq t\leq T$,*

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_{t_0}^t\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big)\leq\widetilde{\mathcal{L}}(t_0). \quad (68)$$

Furthermore, there exists a positive constant C such that

$$\mathcal{Z}(t)\leq C\mathcal{Z}_0\quad\text{for all}\quad t\in[0,T], \quad (69)$$

where $\mathcal{Z}_0$ and $\mathcal{Z}$ have been defined in (22) *and* (24)*, respectively.*

Proof. From (56), (63), (67), we get after a few simplifications,

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\leq\widetilde{\mathcal{L}}(0)+C(\varepsilon+\varepsilon')\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$$
$$+C\varepsilon'\int_0^t\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\mathcal{L}+C\int_0^t\|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Hence, choosing (positive) ε and ε' so that

$$2C\varepsilon'\leq\kappa_0\varepsilon\quad\text{and}\quad 2C(\varepsilon+\varepsilon')\leq\kappa_0,$$

using (66) and the interpolation inequality (55) eventually yields:

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\leq\widetilde{\mathcal{L}}(0)+C\int_0^t\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big)\mathcal{L}. \quad (70)$$

Let us denote

$$T_0\triangleq\sup\big\{t\in[0,T],\ \sup_{\tau\in[0,t]}\widetilde{\mathcal{L}}(\tau)\leq2\widetilde{\mathcal{L}}(0)\big\}\cdot$$

Discarding the trivial case $\widetilde{\mathcal{L}}(0)=0$ (corresponding to the stationary solution $\bar{V}$), the continuity of $\widetilde{\mathcal{L}}$ ensures that $T_0>0$. Now, for all $t\in[0,T_0]$, Inequality (70) ensures that

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\leq\widetilde{\mathcal{L}}(0)+2C\widetilde{\mathcal{L}}(0)\int_0^t\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big)\cdot$$

Consequently, if the initial data are so small that $4C\widetilde{\mathcal{L}}(0)\leq\varepsilon'\kappa_0$, then we deduce that

$$\widetilde{\mathcal{L}}(t)+\frac{\kappa_0}{2}\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\leq\widetilde{\mathcal{L}}(0),$$

and thus $T_0=T$. Hence (68) holds (with $\kappa_0/2$) on $[0,T]$. Clearly, the argument may be started from any time $t_0\in[0,T]$, which gives (68) in full generality.

Let us finally establish (69). First, since $\mathcal{L}$ is equivalent to $\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}+\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}$, it is easy to see that, under Assumption (28), we also have $\widetilde{\mathcal{L}}\simeq\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}+\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$. Putting together with (68), we thus already get

$$\|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}+\|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\leq C\mathcal{Z}_0\quad\text{for all}\quad t\in[0,T].$$

Combining with (66), we discover that

$$\int_0^t\|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\leq\int_0^t\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|\nabla Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\mathcal{Z}_0$$

and

$$\begin{aligned}\|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\leq\|W\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}&+C\|\nabla Z\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\\&+C\|Z\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|Z_2\|^{h}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}+C\|Z\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}.\end{aligned}$$

Owing to (28), the last term may be absorbed by the left-hand side. Furthermore, one can bound the last but one thanks to (21) and, by Hölder inequality, interpolation and (68), we get

$$\|\nabla Z\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\lesssim\sqrt{\|Z\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}}\lesssim\mathcal{Z}_0$$
$$\|W\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\leq\sqrt{\|W\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}}\lesssim\mathcal{Z}_0,$$

which completes the proof of the proposition. □

3.2. Proof of Theorem 2.1

The starting point of the proof of existence is the following local well-posedness result that may be found in [32].

Proposition 3.4. *For any data Z_0 in the nonhomogeneous Besov space $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$, there exists a positive time T_1, depending only the coefficients of the matrices A^j, on H and on $\|Z_0\|_{\mathbb{B}^{\frac{d}{2}+1}_{2,1}}$ such that System (3) has a unique classical solution Z with*

$$Z\in\mathcal{C}^1([0,T_1]\times\mathbb{R}^d)\quad\text{and}\quad Z\in\mathcal{C}([0,T_1];\mathbb{B}^{\frac{d}{2}+1}_{2,1})\cap\mathcal{C}^1([0,T_1];\mathbb{B}^{\frac{d}{2}}_{2,1}).$$

Furthermore, if the maximal time of existence T^* is finite, then

$$\int_0^{T^*}\|\nabla Z\|_{L^\infty}\,dt=\infty.$$

The proof of the existence part of Theorem 2.1 is structured as follows. First, we truncate the low frequencies of the data and use the above theorem to construct a sequence $(Z^n)_{n\in\mathbb{N}}$ of (a priori local) approximate solutions. Then, we take advantage of (68) to establish that those solutions are actually global and uniformly bounded in E. In order to pass to the limit, we show that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})$ for all $T>0$. Then, we eventually check that the limit is indeed a solution of (3) and has the required regularity.

First step. Construction of approximate solutions

Fix some initial data $Z_0\in\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ satisfying (22) and approximate it by

$$Z_0^n=(\mathrm{Id}-\dot{S}_n)Z_0,\qquad n\geq1.$$

By construction, Z_0^n belongs to $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$. Consequently, Theorem 3.4 provides us with a unique maximal solution $Z^n\in\mathcal{C}([0,T_n[;\mathbb{B}^{\frac{d}{2}+1}_{2,1})\cap\mathcal{C}^1([0,T_n[;\mathbb{B}^{\frac{d}{2}}_{2,1})$.

Second step. Uniform estimates

Taking advantage of Proposition 3.3 and denoting by $\mathscr{Z}^n$ the function $\mathscr{Z}$ pertaining to Z^n, we get $\mathscr{Z}^n \leq C\mathscr{Z}_0^n$ as long as Z^n satisfies the smallness condition (28). Owing to the definition of Z_0^n, we have $\mathscr{Z}_0^n \leq \mathscr{Z}_0$ and we clearly have $\|Z^n(t)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \mathscr{Z}^n(t)$ for all time $t \geq 0$. Hence, using a classical bootstrap argument, one can conclude that, if $\mathscr{Z}_0$ is small enough, then

$$\mathscr{Z}^n(t) \leq C\mathscr{Z}_0, \quad \text{for all} \quad t \in [0, T_n[. \tag{71}$$

In order to show that the solution Z^n is global (that is $T_n = \infty$), we shall use the blow-up criterion of Theorem 3.4. However, we first have to justify that the nonhomogeneous Besov norm $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$ of the solution is under control *up to time* T_n. Indeed, using the classical energy method for (3), then the Gronwall lemma, we discover that for all $t < T_n$,

$$\|Z^n(t)\|_{L^2} \leq C\,\|Z_0^n\|_{L^2} \exp\biggl(C\int_0^t \|\nabla Z^n\|_{L^\infty}\biggr)\cdot$$

Now, (71) and the embedding of $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ in L^∞ ensure that $\nabla\mathscr{Z}^n$ is in $L^1(0, T_n; L^\infty)$, from which we deduce that Z^n is in $L^\infty(0, T_n; L^2)$, and thus in $L^\infty(0, T_n; \mathbb{B}^{\frac{d}{2}+1}_{2,1})$ owing, again, to (71). It is now easy to conclude : we have ∇Z^n in $L^1(0, T_n; L^\infty)$, and Z^n is in $\mathcal{C}([0,T]; \mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T]; \mathbb{B}^{\frac{d}{2}}_{2,1})$ for all $T < T_n$. Hence $T_n = \infty$ and (71) is satisfied for all time.

Third step. Convergence

The following stability result will ensure both the convergence of $(Z^n)_{n\in\mathbb{N}}$ and the uniqueness of our solution.

Proposition 3.5. *Let $\widetilde{Z} = Z^1 - Z^2$ where Z^1 and Z^2 are two solutions of* (3)*, having respectively Z_0^1 and Z_0^2 as initial data, and belonging to the space E. There exists a constant c such that if both $\|Z^1\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}$ and $\|Z^2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}$ are smaller than c, then we have for all $t \in [0,T]$,*

$$\|\widetilde{Z}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \lesssim \|\widetilde{Z}_0\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \Bigl(\|(Z^1,Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(Z^1,Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Bigr)\Bigl\|\widetilde{Z}\Bigr\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{72}$$

Proof. Let $V^1 \triangleq \bar{V} + Z^1$ and $V^2 \triangleq \bar{V} + Z^2$. Observe that $\widetilde{Z}$ is a solution of

$$\widetilde{A}^0(V^1)\partial_t\widetilde{Z} + \sum_{j=1}^d \widetilde{A}^j(V^1)\partial_j\widetilde{Z}$$
$$= -\widetilde{A}^0(V^1)\sum_{j=1}^d\bigl(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^j(V^1) - \widetilde{A}^0(V^2)^{-1}\widetilde{A}^j(V^2)\bigr)\partial_j Z^2 - L\widetilde{Z} + Q(Z^1) - Q(Z^2).$$

Applying $\dot{\Delta}_q$, taking the scalar product with $\widetilde{Z}_q$, integrating on $\mathbb{R}^d$ and using Lemma 5.1, we get for all $q \in \mathbb{Z}$ and $t \in [0,T]$,

$$\Bigl\|\widetilde{Z}_q(t)\Bigr\|_{L^2_{\widetilde{A}^0(V^1)}} + \kappa_0\int_0^t \Bigl\|\widetilde{Z}_{2,q}\Bigr\|_{L^2} \leq \Bigl\|\widetilde{Z}_{0,q}\Bigr\|_{L^2} + \int_0^t \Bigl\|\nabla\widetilde{A}^j(V^1)\Bigr\|_{L^\infty}\Bigl\|\widetilde{Z}_q\Bigr\|_{L^2}$$
$$+ \int_0^t \biggl\|\dot{\Delta}_q\sum_{j=1}^d\bigl(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^j(V^1) - \widetilde{A}^0(V^2)^{-1}\widetilde{A}^j(V^2)\bigr)\partial_j Z^2\biggr\|_{L^2}$$
$$+ \int_0^t \Bigl\|\dot{\Delta}_q(Q(Z^1) - Q(Z^2))\Bigr\|_{L^2} + \int_0^t \sum_j \|[\dot{\Delta}_q, \widetilde{A}_j(V^1)]\partial_j\widetilde{Z}\|_{L^2}.$$

Multiplying this inequality by $2^{q\frac{d}{2}}$ and using the commutator estimate (103) yields

$$2^{q\frac{d}{2}}\left\|\widetilde{Z}_q(t)\right\|_{L^2}\lesssim 2^{q\frac{d}{2}}\left\|\widetilde{Z}_{0,q}\right\|_{L^2}+\int_0^t c_q\|\nabla Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}2^{q\frac{d}{2}}\|\widetilde{Z}_q\|_{L^2}$$
$$+\int_0^t 2^{q\frac{d}{2}}\left\|\dot{\Delta}_q\sum_{j=1}^d\left(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^j(V^1)-\widetilde{A}^0(V^2)^{-1}\widetilde{A}^j(V^2)\right)\partial_jZ^2\right\|_{L^2}$$
$$+\int_0^t 2^{q\frac{d}{2}}\left\|\dot{\Delta}_q(Q(Z^1)-Q(Z^2))\right\|_{L^2}. \quad (73)$$

Thanks to Proposition 5.2 and Inequality (108),

$$\left\|\left(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^j(V^1)-\widetilde{A}^0(V^2)^{-1}\widetilde{A}^j(V^2)\right)\partial_jZ^2\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z^2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$

and, according to Inequality (111), we have

$$\|Q(Z^1)-Q(Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(Z^1,Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Hence, summing (73) on $q\in\mathbb{Z}$, we end up for all $t\in[0,T]$ with

$$\left\|\widetilde{Z}(t)\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\left\|\widetilde{Z}_0\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\int_0^t\left(\|(Z^1,Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|(Z^1,Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\right)\left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Splitting in low and high frequencies yields the desired estimate. □

The above lemma combined with the fact that $(Z_0^n)_{n\in\mathbb{N}}$ converges to Z_0 in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ ensures that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})$ and thus has a limit Z in that space, and passing to the limit in (3) is straightforward. Furthermore, using the Fatou property of Besov spaces, we obtain that $Z^\ell\in L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})\cap L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})$ and $Z^h\in L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})\cap L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})$ for all $T>0$, together with the desired bounds. Time continuity of the solution may be obtained by adapting the arguments of [1, Chap. 4]. This completes the proof of the existence part of Theorem 2.1.

Fourth step. Uniqueness

Knowing that Z^1 and Z^2 are in E, we have for all $T>0$,

$$\int_0^T\left(\|(Z^1,Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|(Z^1,Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\right)<\infty.$$

Furthermore, one can assume with no loss of generality that Z^1 is the solution that we constructed before and thus satisfies the smallness assumption (28). Owing to time continuity and since $Z^2(0)=Z^1(0)$, the solution Z^2 also satisfies (28) on some nontrivial time interval $[0,T]$, and combining Inequality (72) with Gronwall lemma allows to conclude that Z^1 and Z^2 coincide on $[0,T]$. A bootstrap argument then yields uniqueness on the whole half-line $\mathbb{R}_+$. □

3.3. Proof of Theorem 2.2

The overall strategy is inspired by the work of Y. Guo and Y. Wang in [16] (see also [31] for an adaptation of the method to the compressible Navier-Stokes system).

First step : uniform bound in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$

In order to establish (25), let us rewrite System (3) under the form

$$\bar{A}^0\partial_t Z + \sum_{j=1}^{d} \bar{A}^j\partial_j Z + LZ = f + g + h$$

$$\text{with}\quad f \triangleq \sum_{j=1}^{d}\big(\bar{A}^j - \widetilde{A}^j(V)\big)\partial_j Z, \quad g \triangleq Q(Z) \quad\text{and}\quad h \triangleq \big(\bar{A}^0 - \widetilde{A}^0(V)\big)\partial_t Z.$$

Then, apply $\dot{\Delta}_q$ and perform L^2 estimates for each Z_q. After using Lemma 5.1, multiplying by $2^{-q\sigma_1}$ then taking the supremum on $\mathbb{Z}$, we end up for all $t \geq 0$ (omitting the term coming from L that has the 'good' sign) with

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \int_0^t \|(f,g,h)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}. \tag{74}$$

Setting $f_j = \left(\widetilde{A}^j(V) - \bar{A}^j\right)\partial_j Z$ and using Inequality (107) yields

$$\|f_j\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \left\|\widetilde{A}^j(V) - \bar{A}^j\right\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|\partial_j Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

In order to bound $\widetilde{A}^j(V) - \bar{A}^j$ in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$, one cannot use directly Proposition 5.3 as $-\sigma_1$ may be negative. However, applying Taylor formula, product laws and a composition estimate (see the details in the proof of [11, Th. 4.1]), we can still obtain if (28) is satisfied,

$$\left\|\widetilde{A}^j(V) - \bar{A}^j\right\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \tag{75}$$

whence

$$\|f\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

Next, writing g under the form (114), then combining Inequality (107) and the same composition estimate as in (75) yields

$$\|g\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Concerning h, we have, keeping Lemma 3.1 in mind, that

$$\begin{aligned}\|h\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &\lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\\ &\lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.\end{aligned}$$

Thus, regrouping all those estimates, we obtain

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \int_0^t \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \qquad t \geq 0. \tag{76}$$

Since, as pointed out before, we have

$$\int_0^t \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \mathcal{Z}(t) \lesssim \mathcal{Z}_0$$

and because the smallness condition (22) is satisfied, applying Gronwall inequality completes the proof of (25).

For the sake of completeness, one has to justify that if Z_0 is in $\dot{B}^{-\sigma_1}_{2,\infty}$ (in addition to (22)), then the solution constructed in Theorem 2.1 is in $\dot{B}^{-\sigma_1}_{2,\infty}$ for all time. This may be checked by following the construction scheme of the previous subsection. Indeed, recall that the approximated solutions Z^n are in $\mathcal{C}^1(\mathbb{R}_+;\mathbb{B}^{\frac{d}{2}}_{2,1})$. Then, discarding the linear term LZ^n (that may be handled by suitable conjugation), we get $\partial_t Z^n\in\mathcal{C}(\mathbb{R}_+;L^1)$. As $L^1\hookrightarrow \dot{B}^{-\frac{d}{2}}_{2,\infty}$ and $\sigma_1\geq d/2$, the low frequencies of $\partial_t Z^n$ (and thus the whole $\partial_t Z^n$) are in $\mathcal{C}(\mathbb{R}_+;\dot{B}^{-\sigma_1}_{2,\infty})$ As Z_0^n itself is in $\dot{B}^{-\sigma_1}_{2,\infty}$ (since $\mathcal{F}(Z_0^n)$ is supported away from 0), we have $Z^n\in\mathcal{C}^1(\mathbb{R}_+;\dot{B}^{-\sigma_1}_{2,\infty})$. Consequently, (25) holds for Z^n and, passing to the limit, ensures that it holds for Z, too.

Second step : proof of generic decay estimates.

The functional $\widetilde{\mathcal{L}}$ defined in Proposition 3.3 is nonincreasing and equivalent to $\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}+\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\cdot$ Furthermore, there exist positive κ_0, ε and ε' such that $\widetilde{\mathcal{H}}\triangleq\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}$ satisfies

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_{t_0}^t\widetilde{\mathcal{H}}\leq\widetilde{\mathcal{L}}(t_0)\qquad\text{for all}\ \ 0\leq t_0\leq t.$$

Hence, one can conclude as in [11] that $\widetilde{\mathcal{L}}$ is differentiable almost everywhere and satisfies

$$\frac{d}{dt}\widetilde{\mathcal{L}}+c'\widetilde{\mathcal{H}}\leq 0\quad\text{a. e. on }\ \mathbb{R}_+. \tag{77}$$

Granted with this information and (25), one can prove the first decay estimate of Theorem 2.2 by following the general argument of [16]. The starting point is that, provided $-\sigma_1<d/2-1$,

$$\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\lesssim\Bigl(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\Bigr)^{\theta_0}\Bigl(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\Bigr)^{(1-\theta_0)}\quad\text{with}\quad\theta_0=\frac{2}{d/2+1+\sigma_1}\cdot$$

Inequality (25) thus implies that

$$\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\gtrsim\bigl(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\bigr)^{\frac{1}{1-\theta_0}}\|Z_0\|^{-\frac{\theta_0}{1-\theta_0}}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

For the high frequencies term, using the estimate of Theorem 2.1, one can just write:

$$\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\gtrsim\Bigl(\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\Bigr)^{\frac{1}{1-\theta_0}}\|Z_0\|^{-\frac{\theta_0}{1-\theta_0}}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\cdot$$

Hence, there exists a (small) constant c such that

$$\frac{d}{dt}\widetilde{\mathcal{L}}+cC_0^{-\frac{\theta_0}{1-\theta_0}}\widetilde{\mathcal{L}}^{\frac{1}{1-\theta_0}}\leq 0\quad\text{with}\quad C_0\triangleq\|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\cdot$$

Integrating, this gives us

$$\widetilde{\mathcal{L}}(t)\leq\left(1+c\,\frac{\theta_0}{1-\theta_0}\left(\frac{\widetilde{\mathcal{L}}(0)}{C_0}\right)^{\frac{\theta_0}{1-\theta_0}}t\right)^{1-\frac{1}{\theta_0}}\widetilde{\mathcal{L}}(0)$$

whence, since $\widetilde{\mathcal{L}}\leq\widetilde{\mathcal{L}}(0)\lesssim\mathcal{Z}_0\lesssim C_0$,

$$\|Z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}+\|Z(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\lesssim(1+t)^{-\alpha_1}\mathcal{Z}_0\quad\text{with}\quad\alpha_1=\frac{d/2-1+\sigma_1}{2}\cdot \tag{78}$$

The decay rates in $\dot{\mathbb{B}}^{\sigma}_{2,1}$ for all $\sigma\in]-\sigma_1,d/2-1]$ follow from Inequalities (25) and (78), and interpolation inequalities.

Third step: decay enhancement for the damped mode

Since W satisfies (57), one can get from (58) and Lemma 5.1 that for all $\sigma\in]-\sigma_1,d/2-1]$,

$$\mathcal{W}^\sigma(t)\leq e^{-ct}\mathcal{W}^\sigma(0)+C\int_0^t e^{-c(t-\tau)}\,\|h(\tau)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\,d\tau\quad\text{with}\quad \mathcal{W}^\sigma\triangleq\sum_{q\leq0}2^{q\sigma}\|\dot\Delta_qW\|_{L^2_{\bar A^0_{2,2}}}.$$

As $\mathcal{W}^\sigma\approx\|W\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}$, taking advantage of (59), (60) and (61) leads to

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim e^{-ct}\|W_0\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\\ +\int_0^t e^{-c(t-\tau)}\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|(\nabla Z,W)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}+\|(\nabla Z,W)\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}+\|(\nabla^2Z,\nabla W)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\Big),$$

whence, remembering (66),

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim e^{-ct}\|W_0\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}+\int_0^t e^{-c(t-\tau)}\Big(\|(\nabla Z,Z_2)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|(\nabla Z,Z_2)\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}+\|\nabla Z\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\Big)\cdot$$

In light of the previous step, the worst decay comes from the last term. In order to be allowed to use the corresponding estimate however, we need $\sigma+1\leq d/2-1$. If that condition is satisfied then, setting $\beta=(\sigma+\sigma_1+1)/2$, the above inequality implies that

$$\langle t\rangle^\beta\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim\langle t\rangle^\beta e^{-ct}\|W_0\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}+C_0\int_0^t\frac{\langle t\rangle^\beta}{\langle\tau\rangle^\beta}\,e^{-c(t-\tau)}\,d\tau\lesssim C_0.$$

In the case $\sigma+1>d/2-1$, one can use the fact that $\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim\|W(t)\|^\ell_{\dot{\mathbb{B}}^{\frac d2-2}_{2,1}}$, and the above argument thus just implies that

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim(1+t)^{-\alpha_1}.$$

Keeping in mind (66), one can conclude that $\|Z_2\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}$ satisfies the same decay estimates as W.

Last step : high frequencies decay

Let us start from (52). The usual method based on Lemma 5.1 leads after multiplying by $\langle t\rangle^{2\alpha_1}$ (where α_1 comes from (78)) yields

$$\big\|\langle t\rangle^{2\alpha_1}Z(t)\big\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\leq e^{-ct}\,\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\int_0^t\langle t\rangle^{2\alpha_1}e^{-c(t-\tau)}\,\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\Big(\|Z\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\Big)d\tau\\ +\int_0^t\langle t\rangle^{2\alpha_1}e^{-c(t-\tau)}\,\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\,\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\,d\tau.\quad(79)$$

Thanks to (78), the first quadratic term may be bounded as follows:

$$\int_0^t\langle t\rangle^{2\alpha_1}e^{-c(t-\tau)}\,\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\|Z\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\leq\int_0^t\bigg(\frac{\langle t\rangle}{\langle\tau\rangle}\bigg)^{2\alpha_1}e^{-c(t-\tau)}\big(\langle\tau\rangle^{\alpha_1}\,\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\big)^2d\tau\lesssim C_0,$$

and the other terms of the right-hand side of (79) may be bounded similarly. This completes the proof of Theorem 2.2.

4. Proof of Theorem 2.3 and application to the compressible Euler system

This section is devoted to the proof of Theorem 2.3, that is to say to a refinement of Theorem 2.1 corresponding to the case where System (1) satisfies the extra conditions listed in Theorem 2.3. As an application, we shall obtain a global existence statement for the compressible Euler with damping, in a new functional framework, and will specify the dependency of the estimates with respect to the relaxation (or damping) parameter.

4.1. Proof of Theorem 2.3

Proving existence and uniqueness being very similar to what we did before, we focus on establishing a priori estimates for a smooth solution Z of (3) on $[0,T]\times\mathbb{R}^d$, satisfying the smallness condition (28). The general strategy is the same as in the previous section, and we shall mainly underline the places where having the additional structure assumptions comes into play.

The first difference lies in the following refinement of Lemma 3.1

Lemma 4.1. *Under the structure assumptions of Theorem 2.3 and* (28)*, we have for all* $\sigma\in]-d/2,d/2]$*,*

$$\|\partial_t Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big(\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\big),$$
$$\|\partial_t Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|W\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.$$

Proof. The second inequality comes from the definition of W in (18). The first one relies on the decomposition

$$\partial_t Z_1=-\sum_{j=1}^{d}(\widetilde{A}^0_{1,1}(V))^{-1}\left(\widetilde{A}^j_{1,1}(V)\partial_j Z_1+\widetilde{A}^j_{1,2}(V)\partial_j Z_2\right)+Q_1(Z).\tag{80}$$

As the function $V\mapsto\widetilde{A}^0_{1,1}(V))^{-1}\widetilde{A}^j_{1,1}(V)$ vanishes at $\bar{V}$ and is linear with respect to Z_2, Propositions 5.2, 5.3 and Condition (28) guarantee the desired inequality.

□

4.1.1. Basic energy estimates

As for Theorem 2.1, the first step consists in proving estimates for $\|Z_q\|^2_{L^2_{\widetilde{A}_0(V)}}$ and $\|Z_q\|^2_{L^2_{\bar{A}_0}}$.

Proposition 4.1. *Under the structure assumptions of Theorem 2.3, let Z be a smooth solution of* (3) *on* $[0,T]$ *satisfying* (28)*. Then, we have for all* $q\geq 0$*,*

$$\frac{1}{2}\frac{d}{dt}\|Z_q\|^2_{L^2_{\widetilde{A}_0(V)}}+\kappa_0\|Z_{2,q}\|^2_{L^2}\lesssim c_q 2^{-q(\frac{d}{2}+1)}\big(\|(W,\nabla Z)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z_q\|_{L^2},\tag{81}$$

and for all $q\leq 0$*,*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Z_q\|^2_{L^2_{\bar{A}_0}}+\kappa_0\|Z_{2,q}\|^2_{L^2}\\&\qquad\lesssim c_q 2^{-q\frac{d}{2}}\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(W,\nabla Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|Z_q\|_{L^2}.\end{aligned}\tag{82}$$

Proof. The starting point is still (39) but we now take advantage of Lemma 4.1 and refine the estimates for R^1_q and (43). More precisely, we have

$$R^1_q=\sum_{j=1}^{d}\begin{pmatrix}[\widetilde{A}^j_{1,1}(V),\dot{\Delta}_q]\partial_j Z_1+[\widetilde{A}^j_{1,2}(V),\dot{\Delta}_q]\partial_j Z_2\\ [\widetilde{A}^j_{2,1}(V),\dot{\Delta}_q]\partial_j Z_1+[\widetilde{A}^j_{2,2}(V),\dot{\Delta}_q]\partial_j Z_2\end{pmatrix}.$$

Hence, using Inequality (103), we get for all $\sigma\in]-d/2,d/2+1]$,

$$\|R^1_q\|_{L^2}\lesssim c_q 2^{-q\sigma}\Big(\|\nabla(\widetilde{A}^j_{1,1}(V)),\nabla(\widetilde{A}^j_{2,1}(V))\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|\nabla(\widetilde{A}^j_{1,2}(V)),\nabla(\widetilde{A}^j_{2,2}(V))\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\Big).$$

At this point, one can use that for all $j\in\{1,\cdots,d\}$ and $k\in\{1,2\}$, there exist a linear map h and a smooth map F such that $\widetilde{A}^j_{k,1}(V)-\bar{A}^j_{k,1}=h(Z_2)F(Z)$. Consequently, product laws and composition estimates give us

$$\begin{aligned}\|\nabla(\widetilde{A}^j_{k,1}(V))\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}&\lesssim\|\nabla(h(Z_2))\otimes F(Z)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|h(Z_2)\otimes\nabla(F(Z))\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\\&\lesssim\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}(1+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}})+\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\\&\lesssim\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}},\end{aligned}$$

whence

$$\|R_q^1\|_{L^2} \lesssim c_q 2^{-q\sigma}\Big(\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\Big)\|Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}. \tag{83}$$

Let us also observe that Inequality (113) of Proposition 5.4 gives us

$$\|Q(Z)\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} \lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.$$

Remembering that

$$\|R_q^2\|_{L^2} \lesssim c_q 2^{-q(\frac d2+1)}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}},$$

and using Lemma 4.1 as well as (40) and (43), we eventually get (81).

For proving (82), the starting point is (44). The term corresponding to R_q^1 (resp. $Q(Z)$) can be bounded according to (83) (resp. (113)) with $\sigma = d/2$. In order to bound the term corresponding to R_q^3, we observe that, in light of Lemma 4.1,

$$\begin{aligned}
\|(\widetilde{A}^0_{1,1}(V) - \widetilde{A}^0_{1,1}(\bar V))\partial_t Z_1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} &\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\partial_t Z_1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\\
&\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\big(\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\big),\\
\|(\widetilde{A}^0_{2,2}(V) - \widetilde{A}^0_{2,2}(\bar V))\partial_t Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} &\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|W\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.
\end{aligned}$$

Finally, we have to refine Inequality (43). To this end, we use the decomposition

$$\begin{aligned}
\int_{\mathbb{R}^d}\sum_{j=1}^d \partial_j(\widetilde{A}^j(V))Z_q\cdot Z_q = \sum_{j=1}^d\int_{\mathbb{R}^d}\Big(&\partial_j\big(\widetilde{A}^j_{1,1}(V)\big)Z_{1,q}\cdot Z_{1,q} + \partial_j\big(\widetilde{A}^j_{2,1}(V)\big)Z_{2,q}\cdot Z_{1,q}\\
&+\partial_j\big(\widetilde{A}^j_{1,2}(V)\big)Z_{1,q}\cdot Z_{2,q} + \partial_j\big(\widetilde{A}^j_{2,2}(V)\big)Z_{2,q}\cdot Z_{2,q}\Big).
\end{aligned}$$

The structure of the system ensures that

$$\|\partial_j\big(\widetilde{A}^j_{1,1}(V)\big)\|_{L^\infty} + \|\partial_j\big(\widetilde{A}^j_{1,2}(V)\big)\|_{L^\infty} + \|\partial_j\big(\widetilde{A}^j_{2,1}(V)\big)\|_{L^\infty} \lesssim \|Z_2\|_{L^\infty}\|\nabla Z\|_{L^\infty} + \|\nabla Z_2\|_{L^\infty}.$$

Hence, remembering (29),

$$\int_{\mathbb{R}^d}\sum_{j=1}^d \partial_j(\widetilde{A}^j(V))Z_q\cdot Z_q \lesssim \big(\|Z_2\|_{L^\infty}\|\nabla Z\|_{L^\infty} + \|\nabla Z_2\|_{L^\infty}\big)\|Z_{1,q}\|^2_{L^2} + \|\nabla Z\|_{L^\infty}\|Z_{2,q}\|^2_{L^2}.$$

Plugging all the above inequalities in (44), we end up with (82). □

4.1.2. Cross estimates

Remember that for all $q \in \mathbb{Z}$, we have

$$\frac{d}{dt}\mathcal{I}_q + \frac{2^q}{2}\sum_{k=1}^{n-1}\varepsilon_k\int_{\mathbb{R}^d}|NM^k_\omega\widehat{Z}_q|^2\,d\xi \le \frac{2^{-q}\kappa_0}{2}\|NZ_q\|^2_{L^2} + C\|\dot\Delta_q G\|_{L^2}\|Z_q\|_{L^2}. \tag{84}$$

In our new regularity context, we have to add up $2^q\mathcal{I}_q$ if $q \le 0$ (resp. $2^{-q}\mathcal{I}_q$ if $q > 0$) to $\mathcal{L}_q$, then to multiply by $2^{q\frac d2}$ (resp. $2^{q(\frac d2+1)}$). This amounts to bounding $\|G\|^\ell_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}$ and $\|G\|^h_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}$. To this end, we have to refine the estimates (45), (46) and (47) taking the particular structure of the coefficients of the system into account.

As a first, we see that (21) and Proposition 5.3 ensure that

$$\|G_3\|^\ell_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|G_3\|^h_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} \lesssim \|G_3\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} \lesssim \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}. \tag{85}$$

Next, we have, thanks to Propositions 5.2 and 5.4,

$$\|G_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\|G_2\|^{h}_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\lesssim\|G_2\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\lesssim\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.\tag{86}$$

In order to improve the estimate for G_1, we use that $\bar{A}_0^{-1}G_1$ is the sum for $j=1$ to d of

$$\begin{pmatrix}\big((\widetilde{A}^0_{1,1}(V))^{-1}\widetilde{A}^j_{1,1}(V)-(\bar{A}^0_{1,1})^{-1}\bar{A}^j_{1,1}\big)\partial_jZ_1+\big((\widetilde{A}^0_{1,1}(V))^{-1}\widetilde{A}^j_{1,2}(V)-(\bar{A}^0_{1,1})^{-1}\bar{A}^j_{1,2}\big)\partial_jZ_2\\ \big((\widetilde{A}^0_{2,2}(V))^{-1}\widetilde{A}^j_{2,1}(V)-(\bar{A}^0_{2,2})^{-1}\bar{A}^j_{2,1}\big)\partial_jZ_1+\big((\widetilde{A}^0_{2,2}(V))^{-1}\widetilde{A}^j_{2,2}(V)-(\bar{A}^0_{2,2})^{-1}\bar{A}^j_{2,2}\big)\partial_jZ_2\end{pmatrix}.$$

Hence, owing to the structure conditions of Theorem 2.3, we just have

$$\|G_1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\lesssim\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.$$

Together with (85) and (86), we can conclude that

$$\|G\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\|G\|^{h}_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\lesssim\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}.\tag{87}$$

4.1.3. Provisional assessment

Let $\mathcal{L}'\triangleq\sum_{q<0}2^{q\frac d2}\sqrt{\mathcal{L}_q}+\sum_{q\geq0}2^{q(\frac d2+1)}\sqrt{\mathcal{L}_q}$. Putting together Inequalities (81), (82), (84) and (87), using Lemma 5.1 and discarding the redundant terms, we end up with

$$\begin{aligned}\mathcal{L}'(t)+\kappa_0\int_0^t\big(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2+2}_{2,1}}+\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\big)\leq\mathcal{L}'(0)+C\int_0^t\big(\|(W,\nabla Z_2)\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\big)\mathcal{L}'\\ +C\int_0^t\big(\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}+\|Z\|^2_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\big)\cdot\end{aligned}\tag{88}$$

To close the estimates, we need to exhibit the L^1-in-time integrability of W and ∇Z_2 in $\dot{\mathbb{B}}^{\frac d2}_{2,1}$ and the L^2-in-time integrability of Z_2 in $\dot{\mathbb{B}}^{\frac d2}_{2,1}$. This will be a consequence of the bounds of the next paragraph.

4.1.4. Bounds for the damped mode

With the notations we used to prove (58), remember that

$$\frac12\frac{d}{dt}\|W_q\|^2_{L^2_{\bar{A}^0_{2,2}}}+\kappa_0\|W_q\|^2_{L^2}\leq\big(\|\dot{\Delta}_qh_1\|_{L^2}+C\|\dot{\Delta}_qh_2\|_{L^2}+C\|\dot{\Delta}_qh_3\|_{L^2}\big)\|W_q\|_{L^2}.\tag{89}$$

From Lemma 4.1, Propositions 5.2, 5.3 and 5.4 and, since

$$\partial_tQ(Z)=D_{Z_1}Q(Z)\partial_tZ_1+D_{Z_2}Q(Z)\partial_tZ_2,$$

we readily get

$$\begin{aligned}\|h_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|h_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\lesssim\|h_1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\lesssim\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|W\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}},\\ \|h_3\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|h_3\|^{\ell}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\lesssim\|h_3\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\lesssim\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|W\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.\end{aligned}$$

For bounding h_2, we need to refine the decomposition we did in the previous section. More precisely, we now write that for all $j\in\{1,\cdots,d\}$,

$$\begin{aligned}\partial_t(A^j_{2,1}(V)\partial_jZ_1+A^j_{2,2}(V)\partial_jZ_2)=D_{V_1}A^j_{2,1}(V)\partial_tZ_1\partial_jZ_1+D_{V_2}A^j_{2,1}(V)\partial_tZ_2\partial_jZ_1\\ +A^j_{2,1}(\bar{V})\partial_t\partial_jZ_1+\big(A^j_{2,1}(V)-A^j_{2,1}(\bar{V})\big)\partial_t\partial_jZ_1\\ +D_VA^j_{2,2}(V)\partial_tZ\partial_jZ_2+A^j_{2,2}(\bar{V})\partial_t\partial_jZ_2+\big(A^j_{2,2}(V)-A^j_{2,2}(\bar{V})\big)\partial_t\partial_jZ_2.\end{aligned}$$

Since $A^j_{2,1}$ is linear with respect to V_2, we get after using (80) and Lemma 4.1 that

$$\|\dot\Delta_q h_2\|_{L^2}\lesssim\|\nabla^2 Z_{2,q}\|_{L^2}+\|\nabla W_q\|_{L^2}+c_q2^{-q\frac d2}\Big(\|\nabla Z\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}$$
$$+\|W\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|(Z,\nabla Z)\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\Big)\cdot$$

Hence, reverting to (89), using Lemma 5.1 and keeping the notation (62), we end up for $\sigma\in[d/2,d/2+1]$ with

$$\mathcal W^\sigma(t)+\kappa_0\int_0^t\|W\|^\ell_{\dot{\mathbb B}^\sigma_{2,1}}\leq\mathcal W^\sigma(0)+C\int_0^t\Big(\|(\nabla^2Z_2,\nabla W)\|^\ell_{\dot{\mathbb B}^\sigma_{2,1}}$$
$$+\|Z_2\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|\nabla Z\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|W\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|(Z,\nabla Z)\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\Big)\cdot\qquad(90)$$

In order to compare W with Z_2, one can use the decomposition:

$$W-Z_2={L_2}^{-1}\Big(\sum_{j=1}^d\Big(\bar A^j_{2,1}\partial_jZ_1+\big(\widetilde A^j_{2,1}(V)-\bar A^j_{2,1}\big)\partial_jZ_1+\bar A^j_{2,2}\partial_jZ_2+\big(A^j_{2,2}(V)-\bar A^j_{2,2}\big)\partial_jZ_2\Big)-Q(Z)\Big)$$

which implies that

$$\|W-Z_2\|^h_{\dot{\mathbb B}^{\frac d2}_{2,1}}\lesssim\|\nabla Z\|^h_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z_1\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}\qquad(91)$$

and that, for all $s\geq d/2$,

$$\|W-Z_2\|^\ell_{\dot{\mathbb B}^s_{2,1}}\lesssim\|\nabla Z\|^\ell_{\dot{\mathbb B}^s_{2,1}}+\|Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z_1\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|\nabla Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}\cdot\qquad(92)$$

4.1.5. Closure of the estimates

As in the previous section, if we set

$$\widetilde{\mathcal L}'\triangleq\mathcal L'+\varepsilon\mathcal W^{\frac d2+1}+\varepsilon'\mathcal W^{\frac d2}\quad\text{and}\quad\widetilde{\mathcal H}'\triangleq\|Z\|^\ell_{\dot{\mathbb B}^{\frac d2+2}_{2,1}}+\|Z\|^h_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}+\varepsilon\|W\|^\ell_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}+\varepsilon'\|W\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}}$$

with suitable ε and ε', then putting together (88) and (90) yields

$$\widetilde{\mathcal L}'(t)+\kappa_0\int_0^t\widetilde{\mathcal H}'\leq\widetilde{\mathcal L}'(0)+C\int_0^t\big(\|Z_2\|_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}+\|W\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\big)\mathcal L'$$
$$+C\int_0^t\big(\|Z_2\|^2_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z_2\|_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}+\|Z\|^2_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\big)\cdot\qquad(93)$$

Note that we have

$$\big(\|Z_2\|^h_{\dot{\mathbb B}^{\frac d2}_{2,1}}\big)^2\lesssim\big(\|Z_2\|^h_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\big)^2\lesssim\|Z\|^h_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\mathcal L'$$

and

$$\begin{aligned}\|Z\|^2_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}+\|Z_2\|^h_{\dot{\mathbb B}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}&\lesssim\|Z\|^2_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\\&\lesssim\|Z\|^\ell_{\dot{\mathbb B}^{\frac d2+2}_{2,1}}\|Z\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\big(\|Z\|^h_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\big)^2\\&\lesssim\big(\|Z\|^\ell_{\dot{\mathbb B}^{\frac d2+2}_{2,1}}+\|Z\|^h_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\big)\mathcal L'.\end{aligned}\qquad(94)$$

Hence, using also (91), and (92) with $s=d/2+1$, we see that (93) becomes just

$$\widetilde{\mathcal L}'(t)+\kappa_0\int_0^t\widetilde{\mathcal H}'\leq\widetilde{\mathcal L}'(0)+C\int_0^t\widetilde{\mathcal H}'\mathcal L'+\int_0^t\|Z_2\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}}\big(\|Z_2\|^\ell_{\dot{\mathbb B}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb B}^{\frac d2+1}_{2,1}}\big)\cdot$$

To handle the last integral, let us write that, by virtue of (92) with $s=d/2$, we have

$$(\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}})^2 \lesssim \big(\|W\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\big)^2 + \|Z_2\|^4_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \big(\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}\big)^2 + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z_1\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\nabla Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.$$

The last three terms of the right-hand side may be bounded (owing to (94) and to (28)) by $\widetilde{\mathcal{H}}'\mathcal{L}'$, and we have

$$\big(\|W\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\big)^2 \lesssim \widetilde{\mathcal{H}}'\widetilde{\mathcal{L}}'$$

$$\|Z_2\|^4_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} \lesssim \|Z\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} \lesssim (\mathcal{L}')^2\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.$$

Finally, we have

$$\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} \lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}}.$$

Hence there exists a constant C (that may depend on ε and ε' but not on the solution) such that for all $t\in[0,T]$, we have

$$\widetilde{\mathcal{L}}'(t) + \kappa_0\int_0^t \widetilde{\mathcal{H}}' \le \widetilde{\mathcal{L}}'(0) + C\int_0^t \widetilde{\mathcal{H}}'\widetilde{\mathcal{L}}' + C\int_0^t (\mathcal{L}' + (\mathcal{L}')^2)\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}.$$

Then, one can conclude exactly has in the previous section that if $\widetilde{\mathcal{L}}'(0)$ (or, equivalently, $\mathcal{Z}'_0$) is small enough, then $\widetilde{\mathcal{L}}'$ is a Lyapunov functional such that for some (new) positive real numbers κ_0 and C,

$$\widetilde{\mathcal{L}}'(t) + \kappa_0\int_0^t \widetilde{\mathcal{H}}' \le \widetilde{\mathcal{L}}'(0) + C\widetilde{\mathcal{L}}'(0)\int_0^t \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}. \tag{95}$$

Furthermore, (92) with $s=d/2$ ensures that

$$\begin{aligned}\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})} \lesssim{}& \|W\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \|\nabla Z\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \|Z_2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|\nabla Z\|_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\\ &+\|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|\nabla Z_2\|_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \|Z_2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|Z_2\|^h_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \|Z_2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}.\end{aligned}$$

Inequality (95) combined with (28) and an obvious interpolation inequality thus yields

$$\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})} \lesssim \mathcal{Z}'(0). \tag{96}$$

Similarly, using again (92) but with $s=d/2+1$, we see that

$$\begin{aligned}\|Z_2\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} \lesssim{}& \|W\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \|\nabla Z\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \|Z_2\|_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|\nabla Z\|_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\\ &+\|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|Z_2\|^h_{L^1_T(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|Z_2\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})}.\end{aligned}$$

In light of (28), the last term may be absorbed by the left-hand side and all the other terms may be bounded either through (95) or through (96).

From this point, the rest of the proof of this theorem essentially follows the lines of the previous section. □

4.2. The isentropic compressible Euler System with damping.

We consider

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho u) = 0,\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla(P(\rho)) + \lambda\rho R(u) = 0,\end{cases} \tag{97}$$

where λ is positive parameter, $P=P(\rho)$ is a (smooth) pressure function and $R=R(u)$ is a (smooth) drag term such that $R-\mathrm{Id}$ is at least quadratic with respect to u.

We are concerned with the global existence issue in the neighborhood of $(\rho, u) = (1, 0)$, if

$$P'(\rho) > 0 \quad \text{for} \quad \rho \quad \text{close to} \quad 1 \quad \text{and} \quad P'(1) = 1. \tag{98}$$

Considering the new unknown $n(\rho) = \displaystyle\int_1^\rho \frac{P'(s)}{s}\,ds$, we can rewrite (97) under the form

$$\begin{cases} \partial_t n + u \cdot \nabla n + \operatorname{div} u + G(n)\operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \nabla n + \lambda R(u) = 0, \end{cases} \tag{99}$$

where $G(n)$ is defined by the relation[5] $G(n(\rho)) = P'(\rho) - 1$.

In order to state our global existence for (99), we need to introduce the following notations:

$$z^{\ell,\lambda} \triangleq \sum_{2^q \le \lambda} \dot\Delta_q z, \qquad z^{h,\lambda} \triangleq \sum_{2^q > \lambda} \dot\Delta_q z,$$

$$\|z\|^{\ell,\lambda}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{2^q \le \lambda} 2^{qs} \|\dot\Delta_q z\|_{L^2} \quad \text{and} \quad \|z\|^{h,\lambda}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{2^q > \lambda} 2^{qs} \|\dot\Delta_q z\|_{L^2}.$$

Theorem 4.1. *Let (n_0, u_0) be in $\dot{\mathbb{B}}^{\frac d2}_{2,1} \cap \dot{\mathbb{B}}^{\frac d2+1}_{2,1}$ and $\varepsilon > 0$. Then, there exist two positive constants c and C depending only on G and on d, such that if*

$$\|(n_0, u_0)\|^{\ell,\lambda}_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \lambda^{-1} \|(n_0, u_0)\|^{h,\lambda}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} \le c,$$

then System (99) supplemented with initial data (n_0, u_0) admits a unique global-in-time solution (n, u) in the space defined by

$$\begin{aligned} &(n, u) \in \mathcal{C}_b(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac d2}_{2,1} \cap \dot{\mathbb{B}}^{\frac d2+1}_{2,1}), \quad (n^{h,\lambda}, u^{h,\lambda}) \in L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac d2+1}_{2,1}), \quad n^{\ell,\lambda} \in L^1(\mathbb{R}_+, \dot{\mathbb{B}}^{\frac d2+2}_{2,1}), \\ &u^{\ell,\lambda} \in L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac d2+1}_{2,1}), \quad u \in L^2(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac d2}_{2,1}) \quad \text{and} \quad \nabla n + \lambda R(u) \in L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac d2}_{2,1}). \end{aligned}$$

Moreover we have the following a priori estimate:

$$\mathcal{Z}_\lambda(t) \lesssim \|(n_0, u_0)\|^{\ell,\lambda}_{\dot{\mathbb{B}}^{\frac d2}_{2,1}} + \lambda^{-1} \|(n_0, u_0)\|^{h,\lambda}_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} \quad \text{for all } t \ge 0 \tag{100}$$

where

$$\begin{aligned} \mathcal{Z}_\lambda(t) \triangleq\; & \|(n, u)\|^{\ell,\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \lambda^{-1} \|(n, u)\|^{h,\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} \\ &+ \lambda^{-1} \|n\|^{\ell,\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac d2+2}_{2,1})} + \|(n, u)\|^{h,\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \|u\|^{\ell,\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac d2+1}_{2,1})} + \lambda^{1/2} \|u\|^{\ell,\lambda}_{L^2_t(\dot{\mathbb{B}}^{\frac d2}_{2,1})} + \|\nabla n + \lambda R(u)\|^{\ell,\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac d2}_{2,1})}. \end{aligned}$$

If, furthermore, (n_0, u_0) belongs to $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in]-d/2, d/2]$, then the solution (n, u) satisfies (25) and the decay estimates mentioned at the end of Theorem 2.3 hold true.

Proof. Performing the rescaling

$$(n, u)(t, x) \triangleq (\widetilde n, \widetilde u)(\lambda t, \lambda x)$$

reduces the proof to $\lambda = 1$ (rescaling back will give the desired dependency with respect to λ in the above statement).

Then, the whole result is a corollary of Theorem 2.3 with n (resp. u) playing the role of Z_1 (resp. Z_2) provided System (99) satisfies the structure assumptions that are required there.

[5] Observe that $\rho \mapsto n(\rho)$ is a smooth diffeomorphism from a neighborhood of 1 to a neighborhood of 0.

Now, one can take as a symmetrizer the matrix $\begin{pmatrix}(1+G(n))^{-1} & 0\\ 0 & I_d\end{pmatrix}$ where the first diagonal block is of size 1×1 and the second one, of size $d\times d$. The blocks of type $A^j_{1,1}$ and $A^j_{2,1}$ depend only (and linearly) on u. Finally, the damped mode (in the case $\lambda=1$) is $W=u+\nabla n+u\cdot\nabla u+R(u)$. Now, by virtue of (100),

$$\|W-(R(u)+\nabla n)\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\lesssim\|u\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|\nabla u\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\lesssim\Big(\|(n_0,u_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|(n_0,u_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\Big)^2,$$

hence $R(u)+\nabla n$ satisfies the same estimates as W, which completes the proof. □

Acknowledgments:

The authors are indebted to the anonymous referee for his/her suggestions that contributed to improve the present article.

5. Appendix

Here we gather a few technical results that have been used repeatedly in the paper.

The first one is the justification that one may choose arbitrarily small positive real numbers $\varepsilon_0,\cdots,\varepsilon_{n-1}$ so that, whenever $\widehat{Z}$ satisfies (11), Inequality (14) holds true. As a start, we fix some positive ε_0. Then, we explain how to bound the terms in the right-hand side of (13).

- Terms $\mathcal{I}^1_k\triangleq\big(NM^{k-1}_\omega N\widehat{Z}\cdot NM^k_\omega\widehat{Z}\big)$ with $k\in\{1,\cdots,n-1\}$.

  Since matrices M_ω are bounded on $\mathbb{S}^{d-1}$, we may write

  $$\varepsilon_k|\mathcal{I}^1_k|\lesssim\varepsilon_k|N\widehat{Z}||NM^k_\omega\widehat{Z}|\leq\frac{\varepsilon_0}{4n\rho}|N\widehat{Z}|^2+C\rho\frac{\varepsilon_k^2}{\varepsilon_0}|NM^k_\omega\widehat{Z}|^2.$$

- Terms $\varepsilon_k\big(NM^{k-1}_\omega\widehat{Z}\cdot NM^k_\omega N\widehat{Z}\big)$ with $k\in\{2,\cdots,n-1\}$ may be bounded similarly.

- We have $\varepsilon_1|\big(N\widehat{Z}\cdot NM_\omega N\widehat{Z}\big)|\leq C\varepsilon_1|N\widehat{Z}|^2$.

- Terms $\mathcal{I}^2_k\triangleq\rho\big(NM^{k-1}_\omega\widehat{Z}\cdot NM^{k+1}_\omega\widehat{Z}\big)$ with $k\in\{1,\cdots,n-2\}$. We have

  $$\begin{aligned}\varepsilon_k|\mathcal{I}^2_k|&\leq\varepsilon_k\rho|NM^{k-1}_\omega\widehat{Z}|\,|NM^{k+1}_\omega\widehat{Z}|\\&\leq\frac{\varepsilon_{k-1}}{4}\rho|NM^{k-1}_\omega\widehat{Z}|^2+C\rho\frac{\varepsilon_k^2}{\varepsilon_{k-1}}|NM^{k+1}_\omega\widehat{Z}|^2.\end{aligned}$$

  As we want the two terms to be absorbed by the left-hand side of (13), we take ε_k so that

  $$4C\varepsilon_k^2\leq\varepsilon_{k-1}\varepsilon_{k+1}.\tag{101}$$

  We keep in mind that ε_0 has been set to κ_0 (but can be taken smaller if needed).

- Term $\mathcal{I}^2_{n-1}\triangleq\varepsilon_{n-1}\rho\big(NM^{n-2}_\omega\widehat{Z}\cdot NM^n_\omega\widehat{Z}\big)$. We start with the observation that, owing to Cayley-Hamilton theorem, there exist coefficients c^j_ω (that are uniformly bounded on $\mathbb{S}^{d-1}$, such that

  $$M^n_\omega=\sum_{j=0}^{n-1}c^j_\omega M^j_\omega.$$

  Consequently, one may write

  $$\begin{aligned}|\mathcal{I}^2_{n-1}|&\lesssim\varepsilon_{n-1}\rho\sum_{j=0}^{n-1}|NM^{n-2}_\omega\widehat{Z}||NM^j_\omega\widehat{Z}|\\&\leq\frac{C\varepsilon^2_{n-1}\rho}{\varepsilon_j}|NM^j_\omega\widehat{Z}|^2+\frac{\varepsilon_j\rho}{4}|NM^j_\omega\widehat{Z}|^2.\end{aligned}$$

Therefore one needs to assume in addition that

$$4C\varepsilon_{n-1}^2 \leq \varepsilon_j\varepsilon_{n-2}, \qquad j = 0, \cdots, n-1. \tag{102}$$

Clearly, one is done if it is possible to find $\varepsilon_1, \cdots, \varepsilon_{n-1}$ fulfilling (101) and (102). According to [2], one can take $\varepsilon_k = \varepsilon^{m_k}$ with ε small enough and $m_1, \cdots, m_{n-1}$ satisfying for some $\delta > 0$ (that can be taken arbitrarily small):

$$m_k \geq \frac{m_{k-1} + m_{k+1}}{2} + \delta \quad \text{and} \quad m_{n-1} \geq \frac{m_k + m_{n-2}}{2} + \delta, \quad k = 1, \cdots, n-2.$$

We often used the following well known result (see e.g. [11] for the proof).

Lemma 5.1. *Let $X : [0,T] \to \mathbb{R}_+$ be a continuous function such that X^2 is differentiable. Assume that there exists a constant $B \geq 0$ and a measurable function $A : [0,T] \to \mathbb{R}_+$ such that*

$$\frac{1}{2}\frac{d}{dt}X^2 + BX^2 \leq AX \quad a.e.\ on\ \ [0,T].$$

Then, for all $t \in [0,T]$, we have

$$X(t) + B\int_0^t X \leq X_0 + \int_0^t A.$$

The following estimates are proved in e.g. [1, Chap. 2].

Proposition 5.1. *The following inequalities hold true:*

- *If $-d/2 < s \leq d/2+1$, then*

$$2^{qs}\left\|[w, \dot{\Delta}_q]\nabla v\right\|_{L^2} \leq Cc_q \left\|\nabla w\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \left\|v\right\|_{\dot{\mathbb{B}}^{s}_{2,1}} \quad with \quad \sum_{q\in\mathbb{Z}} c_q = 1. \tag{103}$$

- *If $-d/2 \leq s < d/2+1$, then*

$$\sup_{q\in\mathbb{Z}} 2^{qs}\|[w, \dot{\Delta}_q]\nabla v\|_{L^2} \leq C\|\nabla w\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|v\|_{\dot{\mathbb{B}}^{s}_{2,\infty}}. \tag{104}$$

The following product laws in Besov spaces have been used several times.

Proposition 5.2. *Let (s,r) be in $]0,\infty[\times[1,\infty]$. Then, $\dot{\mathbb{B}}^s_{2,r} \cap L^\infty$ is an algebra and we have*

$$\|ab\|_{\dot{\mathbb{B}}^s_{2,r}} \leq C\big(\|a\|_{L^\infty}\|b\|_{\dot{\mathbb{B}}^s_{2,r}} + \|a\|_{\dot{\mathbb{B}}^s_{2,r}}\|b\|_{L^\infty}\big). \tag{105}$$

If, furthermore, $-d/2 < s \leq d/2$, then the following inequality holds:

$$\|ab\|_{\dot{\mathbb{B}}^s_{2,1}} \leq C\|a\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|b\|_{\dot{\mathbb{B}}^s_{2,1}}. \tag{106}$$

Finally, if $-d/2 < \sigma_1 \leq d/2$, then the following inequality holds true:

$$\|fg\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|f\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|g\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}. \tag{107}$$

The next proposition can be found in [1, Chap.2].

Proposition 5.3. *Let f be a function in $\mathcal{C}^\infty(\mathbb{R})$ such that $f(0) = 0$. Let $(s_1, s_2) \in]0,\infty[^2$ and $(r_1, r_2) \in [1,\infty]^2$. We assume that $s_1 < d/2$ or that $s_1 = d/2$ and $r_1 = 1$.*

Then, for every real-valued function u in $\dot{\mathbb{B}}^{s_1}_{2,r_1} \cap \dot{\mathbb{B}}^{s_2}_{2,r_2} \cap L^\infty$, the function $f \circ u$ belongs to $\dot{\mathbb{B}}^{s_1}_{2,r_1} \cap \dot{\mathbb{B}}^{s_2}_{2,r_2} \cap L^\infty$ and we have

$$\|f \circ u\|_{\dot{\mathbb{B}}^{s_k}_{2,r_k}} \leq C\left(f', \|u\|_{L^\infty}\right)\|u\|_{\dot{\mathbb{B}}^{s_k}_{2,r_k}} \quad for\ k \in \{1,2\}.$$

As a consequence (see [1, Cor. 2.66]), if g is a $\mathcal{C}^\infty(\mathbb{R})$ function such that $g'(0)=0$. Then, for all u,v in $\dot B^s_{2,1}\cap L^\infty$ with $s>0$, we have

$$\|g(v)-g(u)\|_{\dot B^s_{2,1}}\leq C\Big(\|v-u\|_{L^\infty}\|(u,v)\|_{\dot{\mathbb{B}}^s_{2,1}}+\|v-u\|_{\dot{\mathbb{B}}^s_{2,1}}\|(u,v)\|_{L^\infty}\Big)\cdot \tag{108}$$

We used the following result to estimate the remainder of the dissipative term.

Proposition 5.4. *Assume that* $H(\bar V)=0$. *Let* $Z\triangleq V-\bar V$ *and define*

$$Q(Z)\triangleq \widetilde H(\bar V+Z)+LZ\quad\textit{with}\quad L\triangleq -D_V\widetilde H(\bar V).$$

Assume that $Q(Z_1,0)=0$ *for* Z_1 *in a neighborhood of* 0. *Then, provided* $\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}$ *is sufficiently small, the following inequalities hold true:*

$$\|Q(Z)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\quad \textit{for all } \sigma\in]-d/2,d/2] \tag{109}$$

and, for all $\sigma>d/2$,

$$\|Q(Z)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^\sigma_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\,. \tag{110}$$

Furthermore, if both Z^1 *and* Z^2 *are sufficiently small in* $\dot{\mathbb{B}}^{\frac d2}_{2,1}$ *then we have the following estimate for* $\widetilde Z:=Z^1-Z^2$*:*

$$\|Q(Z^1)-Q(Z^2)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim \|Z^1\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|\widetilde Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}+\|\widetilde Z\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z^2_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\,,\qquad \sigma\in]0,d/2]. \tag{111}$$

Finally, if Q *is at least quadratic with respect to* Z_2 *(that is* $D^2_{V_i,V_j}Q(0)=0$ *for* $(i,j)\neq(2,2)$*), then we have*

$$\|Q(Z)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\qquad \textit{for all } \sigma\in]-d/2,d/2] \tag{112}$$

$$\textit{and}\quad \|Q(Z)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim \|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac d2}_{2,1}}\qquad \textit{for all } \sigma>d/2. \tag{113}$$

Proof. Since $Q(Z_1,0)=0$ for Z_1 close to 0, the mean value formula gives

$$Q(Z_1,Z_2)=\int_0^1 D_{Z_2}Q(Z_1,\tau Z_2)\cdot Z_2\,d\tau.$$

Furthermore, we have $DQ(0)=0$ and thus $D_{Z_2}Q(0)=0$. Hence there exists a smooth function F defined near 0 and such that $D_{Z_2}Q(Z)=F(Z)\cdot Z$. Consequently, as $Q(0)=0$, there exists a smooth function G vanishing at 0, and such that

$$Q(Z_1,Z_2)=G(Z)\cdot Z_2. \tag{114}$$

Hence the first two inequalities of the proposition readily follow from Propositions 5.2 and 5.3 combined with the embedding $\dot{\mathbb{B}}^{\frac d2}_{2,1}\hookrightarrow L^\infty$.

To prove (111), we use the decomposition

$$Q(Z^1)-Q(Z^2)=G(Z^1)\cdot(Z^2_2-Z^1_2)+\big(G(Z^2)-G(Z^1)\big)\cdot Z^2_2,$$

then Propositions 5.2 and 5.3, combined with Corollary 2.66 from [1].

Finally, if Q is quadratic with respect to Z_2 then there exists a quadratic form $\widetilde Q$ and a smooth function F such that $Q(Z)=\widetilde Q(Z_2)F(Z)$, whence

$$Q(Z)=F(0)\widetilde Q(Z_2)+G(Z)\widetilde Q(Z_2)\quad\text{with}\quad G(Z)\triangleq F(Z)-F(0).$$

In the case $\sigma \in]-d/2, d/2]$, we can thus write by virtue of Propositions 5.2 and 5.3 (as $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$ is small),

$$\|Q(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\widetilde{Q}(Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}(1+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}) \lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}$$

while, if $\sigma > d/2$,

$$\|Q(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\widetilde{Q}(Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}(1+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}) + \|Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\|\widetilde{Q}(Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}},$$

whence the last inequality. □

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften.* Springer, Heidelberg, 2011.

[2] K. Beauchard and E. Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Rational Mech. Anal*, 199, 177–227, 2011.

[3] S. Benzoni-Gavage and D. Serre. *Multi-dimensional Hyperbolic Partial Differential Equations : First-order Sytems and Applications.* Oxford Science Publications, New-York, 2007.

[4] R. Bianchini and R. Natalini. Nonresonant bilinear forms for partially dissipative hyperbolic systems violating the Shizuta-Kawashima condition. *arXiv:1906.02767*, 2019.

[5] S. Bianchini, B. Hanouzet, and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure and Appl. Math.*, 60, 1559-1622, 2007.

[6] C. Burtea, T. Crin-Barat, and J. Tan. Relaxation limit of a damped Baer-Nunziato model. *arXiv:2109.07746*, 2021.

[7] G. Carbou and B. Hanouzet. Relaxation approximation of Kerr model for the three dimensional initial boundary value problem. *Journal of Hyperbolic Differential Equations*, 6(3):577–614, 2009.

[8] F. Charve and R. Danchin. A global existence result for the compressible Navier-Stokes equations in the critical L^p framework. *Arch. Rational Mech. Anal*, 198, 233-271, 2010.

[9] Q. Chen, C. Miao, and Z. Zhang. Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity. *Comm. Pure Appl. Math.*, 63(9):1173–1224, 2010.

[10] J.-M. Coron. *Control and nonlinearity*, volume 136. Mathematical Surveys and Monographs. American Mathematical Society, 2007.

[11] T. Crin-Barat and R. Danchin. Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. *Pure and Applied Mathematics*, 2021.

[12] T. Crin-Barat and R. Danchin. Global existence result for partially dissipative hyperbolic systems in the L^p framework and application to the relaxation of the damped compressible Euler system. *arXiv:2201.06822*, 2022.

[13] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Inventiones Mathematicae*, 141, 579-614, 2000.

[14] R. Danchin. Fourier analysis methods for the compressible Navier-Stokes equations. *in : Giga Y., Novotný A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham*, 2018.

[15] R. Danchin and L. He. The incompressible limit in L^p type critical spaces. *Math. Ann.*, 366 (3-4), 1365-1402, 2016.

[16] Y. Guo and Y. Wang. Decay of dissipative equations and negative sobolev spaces. *Communications in Partial Differential Equations*, 37, 2165-2208, 2012.

[17] D. Iftimie. Resolution of the Navier-Stokes equations in anisotropic spaces. *Revista Matematica Ibero-americana*, 15(1):1–36, 1999.

[18] S. Kawashima. Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics. *Doctoral Thesis*, 1983.

[19] S. Kawashima and W.-A. Yong. Dissipative structure and entropy for hyperbolic systems of balance laws. *Arch. Rational Mech. Anal.*, 174, 345–364, 2004.

[20] S. Kawashima and W.-A. Yong. Decay estimates for hyperbolic balance laws. *Journal for Analysis and its Applications*, 28, 1–33, 2009.

[21] J. Li, Y. Yu, and W. Zhu. Ill-posedness for the Burgers equation in Sobolev spaces. *arXiv:2103.08352*, 2021.

[22] F. Linares, D. Pilod, and Jean-Claude J.-C. Saut. Dispersive perturbations of Burgers and hyperbolic equations I: Local theory. *SIAM J. Math. Anal.*, 46(2):1505–1537, 2014.

[23] A. Majda. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variable.* Springer, New-York, 1984.

[24] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20, 67-104, 1980.

[25] P. Qu and Y. Wang. Global classical solutions to partially dissipative hyperbolic systems violating the Kawashima condition. *Journal de Mathématiques Pures et Appliquées*, 109:93–146, 2018.

[26] D. Serre. *Systèmes de lois de conservation, tome 1.* Diderot editeur, Arts et Sciences, Paris, New-York, Amsterdam, 1996.

[27] S. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14, 249-275, 1985.

[28] T. Sideris, B. Thomases, and D. Wang. Long time behavior of solutions to the 3D compressible Euler equations with damping. *Comm. Partial Differential Equations*, 28, 795-816, 2003.

[29] C. Villani. Hypocoercivity. *Mem. Am. Math. Soc.*, 2010.

[30] W. Wang and T. Yang. The pointwise estimates of solutions for Euler equations with damping in multi-dimensions. *J Diff. Eqs.*, 173, Issue 2, 410-450, 2001.

[31] Z. Xin and J. Xu. Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions. *Journal of Differential Equations*, 274, 543-575, 2021.

[32] J. Xu and S. Kawashima. Global classical solutions for partially dissipative hyperbolic system of balance laws. *Arch. Rational Mech.Anal*, 211, 513–553, 2014.

[33] J. Xu and S. Kawashima. The optimal decay estimates on the framework of Besov spaces for generally dissipative systems. *Arch. Rational Mech. Anal*, 218, 275–315, 2015.

[34] W.-A Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Rational Mech. Anal*, 172, 47–266, 2004.