PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS AND APPLICATION TO FLUID MECHANICS

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Presentation of the model

In this work, we consider the n-component hyperbolic system of balance laws in dimension d, which are partial differential equations of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \sum_{j=1}^{d} \frac{\partial F_j(w)}{\partial x_j} = \begin{pmatrix} 0 \\ q(w) \end{pmatrix} \tag{1}$$

where $d, n \in \mathbb{N}^*$, $w_1 =: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{n_1}$, $w_2 =: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{n_2}$ with $0 \in \mathbb{R}^{n_1}, q(w) \in \mathbb{R}^{n_2}, n_1, n_2 \in \mathbb{N}, n_1 + n_2 = n$ and q, F_j are smooth functions. One of the main application of our results is the following **compressible Euler system for isentropic flows with damping**

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0 \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u) + \nabla P + \rho u = 0 \end{cases}$$

Assumptions

Let's now state the conditions that will be needed to obtain our results. We will use the following Theorem which was proved in [2] and [4]

Theorem 1 The following statements are equivalent :

1. The system (1) has an entropy.
2. The system (1) is equivalent to the following system

$$A^0(V)\partial_t V + \sum_{j=1}^{d} A^j(V)\partial_{x_j} V = H(V) \tag{2}$$

where for $\mathcal{M} = \{\phi \in \mathbb{R}^n \; \langle \phi, Q(w) \rangle = 0$ pour tout $w \in \mathcal{O}_w\}$ we have

(a) the matrix $A^0(V)$ is symmetric, positive definite and diagonal by blocks associated with the orthogonal decomposition $\mathbb{R}^n = \mathcal{M} \bigoplus \mathcal{M}^\perp$;

(b) the matrices $A^j(V)$ are symmetric ;

(c) $H(V) = 0$ if $V \in \mathcal{M}$.

(d) Let $\bar{V} \in \mathcal{M}$ fixed, then $H(V) = -LV + r(V)$ où $L := L(\bar{V}), r(V) \in \mathcal{M}^\perp$ pour tout $V \in \mathcal{O}_V$ and

$$|r(V)| \leq C|V - \bar{V}||V_2|$$

Furthermore we need an additional hypothesis on (2), the classical (SK) condition that was first introduced in [3]. To define it we must introduce the linearised system of (2) around $\bar{V}$:

$$A^0(\bar{V})\partial_t V + \sum_{j=1}^{d} A^j(\bar{V})\partial_{x_j} V + LV = 0.$$

Applying the Fourier transform, we get

$$A^0(\bar{V})\widehat{\partial_t V} + i|\xi|A_{\bar{V}}(\omega)\widehat{\partial_{x_j} V} + L\widehat{V} = 0$$

with $A_{\bar{V}}(\omega) = \displaystyle\sum_{j=1}^{d} \omega_j A^j(\bar{V})$ and $\omega = \dfrac{\xi}{|\xi|} \in \mathcal{S}^{d-1}$.

Definition The system (2) verifies the (SK) condition, also called stability condition, in $\bar{V} \in \mathcal{M}$ if the linearised system of (2) around $\bar{V}$ verifies

$$\forall \omega \in \mathcal{S}^{d-1} \quad \ker(L) \cap \{\text{eigenvectors of } A_{\bar{V}}(\omega)\} = \{0\}$$

This condition will be essential to recover the dissipation on all the components of the solution.

We are now able to state our global existence for small initial data result in a critical framework.

Main Result

Theorem 2 Let $d \geq 2$ and $\bar{V} \in \mathcal{M}$. We suppose that the system (2) verifies the (SK) condition in $\bar{V}$. We define $Z = V - \bar{V}$ and assume that $Z_0 \in \dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ and $J_0 = C(\|L\|)$. There exists a constant $c > 0$ such that if

$$\|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq c$$

then (2) has a unique global solution $V \in E_2$ i.e.

$$Z \in \mathcal{C}_b(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z_1^\ell \in L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z_2^\ell \in L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}), \quad Z^h \in L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}).$$

Moreover, for all $T \geq 0$, we have the following a priori estimate :

$$X_2(T) \lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$$

where $X_2(T) = \|Z\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})} + \|Z\|^h_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z_1\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z_2\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z\|^h_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}$

Idea of proof : A 1D Toy-model

Let's consider the following Toy-model :

$$\begin{cases} \partial_t u_j + \partial_x v_j + \partial_x(\dot{\Delta}_j(vu)) = 0 \\ \partial_t v_j + \partial_x u_j + v\partial_x v_j + v_j = R_j \end{cases}$$

where $z_j := \dot{\Delta}_j z$ for all $z \in \mathcal{S}'(\mathbb{R}^d)$, $\dot{\Delta}_j$ correspond to the frequency localisation blocks of the Littlewood-Paley Theory and $R_j := [v, \dot{\Delta}_j]v$ is a commutator. The idea is that **the classical energy estimates are not enough to get dissipation on both** u **and** v. In order to recover the dissipation on the first component we need to do some computation that will take into account the coupling of the two equations. In order to do that we define two Lyapunov functions, one for the low frequencies and one for the high frequencies.

$$\mathcal{L}^\ell_j := \left\|(u_j, v_j)\right\|^2_{L^2} + 2\varepsilon \int_{\mathbb{R}} v_j \partial_x u_j \quad \text{if } 2^j \leq 1.$$

$$\mathcal{L}^h_j := \left\|(\partial_x u_j, \partial_x v_j)\right\|^2_{L^2} + 2\varepsilon \int_{\mathbb{R}} v_j \partial_x u_j \quad \text{if } 2^j > 1$$

where ε is a small positive constant. After differentiating the Lyapunov in time, we get, for the linearised system,

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}^\ell_j + \varepsilon 2^{2j}\mathcal{L}^\ell_j \leq 0 \text{ and } \frac{1}{2}\frac{d}{dt}\mathcal{L}^h_j + \varepsilon\mathcal{L}^h_j \leq 0.$$

By integrating in time on [0,T], summing those two inequalites, doing a classical procedure and using the fact that for ε small enough we have $\frac{1}{2}\mathcal{L}^\ell_j \leq \left\|(u_j, v_j)\right\|^2_{L^2} \leq 2\mathcal{L}^\ell_j$, we get

$$\|(u,v)\|^h_{L^\infty_T(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|(u,v)\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})} + \|(u,v)\|^h_{L^1_T(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|(u,v)\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1})} \leq X_0(T) + X_2^2(T).$$

But we cannot conclude yet, we need to improve the regularity of v in low frequencies or the bootstrap argument would fail. Indeed we need to show that $v^\ell \in L^1_T(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})$. In order to do that we will apply ∂_x to the second equation

$$\partial_t\partial_x v_j + \partial_x v_j + \partial_x(v\partial_x v_j) = -\partial^2_{xx}u_j + R'_j. \tag{3}$$

With the previous computations we know that we have $u^\ell \in L^1_T(\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1})$ so the right-hand side of (3) will be absorbed in low frequencies. We obtain

$$\|\partial_x v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^T \|\partial_x v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|\partial_x v_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^T \left\|\partial^2_{xx}u\right\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^T \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

Therefore we get $X_2(T) \lesssim X_0 + X_2(T)^2$ and since we assumed that X_0 is small, we can conclude with a bootstrap argument.

Idea of proof : The general case

For the general case we will follow the same scheme. Using an idea from [1], **the Lyapunov function will have additional low-order terms** :

$$\mathcal{L} = \|Z\|^2_{L^2} + \min\left(1, 2^{2j}\right) K_{\bar{V}}$$

where $K_{\bar{V}}(\omega) := \sum_{k=1}^{n} \varepsilon_K \left(A_{\bar{V}}(\omega)^{*k} L^* L A_{\bar{V}}(\omega)^{k-1} - A_{\bar{V}}(\omega)^{*k-1} L^* L A_{\bar{V}}(\omega)^{k}\right)$.
The function $K_{\bar{V}}$ is called the compensating function and has a strong link with the (SK) condition, indeed from [4] we have the following proposition :
Proposition The following assertions are equivalent :

1. The system (2) verifies the (SK) condition in $\bar{V} \in \mathcal{M}$;
2. It exists a matrix $n \times n$ $K(\omega)$ with $\omega \in \mathcal{S}^{d-1}$ verifying the following properties
   (a) $K(-\omega) = -K(\omega)$ $\forall \omega \in \mathcal{S}^{d-1}$;
   (b) $K(\omega)A^0(\bar{V})$ is skew-symmetric $\forall \omega \in \mathcal{S}^{d-1}$;
   (c) $[K(\omega)A_{\bar{V}}(\omega)]' + L$ is positive definite $\forall \omega \in \mathcal{S}^{d-1}$ where $[X]'$ is the symmetric part of the matrix X.

By derivating in time this Lyapunov, choosing ε_K such that $\mathcal{L} \sim \|Z\|^2_{L^2}$ and using the properties on $K(\omega)$ we can obtain $X_2(T) \leq X_0 + X_2^2(T)$ and thus conclude.

Fluid mechanics

The Toy-model we considered previously is exactly the compressible Euler system with damping in 1D with the special pressure $P(\rho) = \frac{\rho^2}{2}$.
For a gamma law pressure $P(\rho) = \rho^\gamma$ with $\gamma > 1$, by considering the sound speed $c(\rho) = \sqrt{P'(\rho)}$ and defining $m = \frac{2}{\gamma-1}(c(\rho) - \bar{c})$, the system becomes

$$\begin{cases} \partial_t m + \bar{c}\,\text{div}u + u.\nabla m + \frac{\gamma-1}{2}m\,\text{div}u = 0 \\ \partial_t u + \bar{c}\,\nabla m + u.\nabla u + \frac{\gamma-1}{2}m\nabla m + u = 0. \end{cases}$$

This system verifies the condition of **Theorem 2**. With a more complex symmetrisation it is also possible to get a result for P such that $P'(\bar{\rho}) > 0$, for$\bar{\rho} > 0$.

Extension

For $p \in [2, 4]$, we are also able to obtain a Theorem in L^p-based Besov space for the Toy-model in low frequencies. Here is the space in which we get our global existence for small initial data result.

$$u^\ell \in \mathcal{C}_b(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1_T(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}) \quad v^\ell \in \mathcal{C}_b(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1_T(\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}) \quad (u,v)^h \in \mathcal{C}_b(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}) \cap L^1_T(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})$$

We expect to get a L^p result for the damped compressible Euler system in 3D. Moreover, we obtain optimal decay results without additionnal smallness assumption.

References

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