LARGE TIME ASYMPTOTICS FOR PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS WITHOUT FOURIER ANALYSIS

TIMOTHÉE CRIN-BARAT*, LING-YUN SHOU, AND ENRIQUE ZUAZUA

ABSTRACT. A new framework to obtain time-decay estimates for partially dissipative hyperbolic systems set on the real line is investigated. Under the celebrated Shizuta-Kawashima (SK) condition, it is known that the solutions of these systems decay exponentially in time for high frequencies and polynomially for low ones. This allows to derive a sharp description of the space-time decay of solutions for large time. However, such analysis relies heavily on the use of the Fourier transform that we avoid here to obtain new asymptotic results in the linear and nonlinear settings.

First, inspired by the *hyperbolic hypocoercivity* approach developed by Beauchard and Zuazua in [1], we recover the natural time-decay estimates under the Kalman rank condition, without employing Fourier analysis and without assuming L^1-type conditions on the initial data. Then, combining this new approach with space-weighted estimates, we establish enhanced time-decay rates under weighted integrability conditions on the initial data. Moreover, this method enables us to prove new results that cannot be easily obtained through Fourier analysis. We demonstrate this by deriving logarithmic decay rates for the solution of the nonlinearly damped p-system under weighted integrability properties on the initial data.

1. Introduction

In this paper, we study the long time behavior of partially dissipative hyperbolic systems, which take the form

$$\partial_t U + A(U)\partial_x U = -BU, \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+, \tag{1.1}$$

where $U = U(t,x) \in \mathbb{R}^n$ $(n \geq 1)$ is the unknown, A is a smooth matrix-valued symmetric function, and B is a $n\times n$ matrix. System (1.1) typically governs non-equilibrium processes in physics for media with hyperbolic response, and also arises in the numerical simulation of conservation laws by relaxation schemes (see [14, 26, 34] and references therein).

Here we assume that (1.1) has a *partially dissipative structure*: the matrix B takes the form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \tag{1.2}$$

with D a $n_2\times n_2$ matrix $(1 \leq n_2 \leq n)$ satisfying the *strong dissipativity condition*: there exists a constant $\kappa > 0$ such that

$$(DX, X) \geq \kappa |X|^2, \qquad \forall X \in \mathbb{R}^{n_2}. \tag{1.3}$$

Such formulation includes the compressible Euler equation with linear damping:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\lambda\rho u, \end{cases} \tag{1.4}$$

2020 *Mathematics Subject Classification.* 35Q35; 76N10.

Key words and phrases. Partially dissipative systems, nonlinear damping, asymptotic estimates, hypocoercivity, Lyapunov functional,

* Corresponding author: timotheecrinbarat@gmail.com.

where $\rho = \rho(x,t) \geq 0$ denotes the fluid density function, $u = u(x,t) \in \mathbb{R}$ stands for the fluid velocity, $P(\rho)$ is the pressure function, and the friction coefficient $\lambda > 0$ is assumed to be a constant. It describes compressible gas flows passing through porous media and can be interpreted as a relaxation approximation (as $\lambda \to \infty$) of the porous medium equation describing fluid flow, heat transfer or diffusion [25].

The partially dissipative nature of (1.1)-(1.2) does not play a role when studying its local well-posedness [18, 23], but is crucial to justify global well-posedness and time-asymptotic results. For $B = 0$, (1.1) reduces to systems of general hyperbolic conservation laws and it is well-known that for smooth initial data, it admits solutions (cf. [15, 18, 23]) that may develop singularities (shock waves) in finite time (e.g., see [7, 17]).

The key idea is that the dissipation from B, even if it is only partial, can ensure the stability of the system. Indeed, it is well-known that if the couple (A, B) satisfies the Kalman rank condition, then the high frequencies are exponentially damped while at low frequencies the solution behaves like that of the heat equation. For this reason, it is very common to study partially dissipative systems by means of Fourier analysis.

Our goal here is to study the impact of the partially dissipative structure on the asymptotic behavior of (1.1), without employing the Fourier transform. There are multiple reasons for this: Fourier analysis is not easily applied to equations set on bounded domains, it can make it harder to extract useful properties from nonlinear terms, and it is not well-suited to analyse numerical schemes with non-uniform meshes. Therefore, in order to obtain new results in these contexts, we develop a method to study the long-time behavior of partially dissipative systems without frequency-based tools. It allows us to recover the natural decay estimates for linear partially dissipative systems and, coupled with a weighted-estimates approach, to derive enhanced decay rates under weighted integrability conditions on the initial data. Concerning nonlinear systems, we first establish, via a perturbation argument, asymptotic results for the compressible Euler system (1.4). Then, our method allows us to prove new results that cannot be easily obtained through Fourier analysis. We prove logarithmic decay rates for the solution of the nonlinearly damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho = -u|u|^{r-1}, \end{cases} \tag{1.5}$$

with $1 < r < 3$. This system may be viewed as a simplified version, in the case of nonlinear damping, of the Euler system (1.4), so understanding its properties is a first step towards a better understanding of the latter.

The paper is organized as follows. Section 2 presents the challenges of analysing partially dissipative systems without Fourier analysis and states our main results. In Section 3, we prove natural time-decay estimates for linear systems without Fourier analysis and without additional L^1-type regularity assumption on the initial data (cf. Theorem 2.2). These estimates are further improved in Sections 4-5 under additional space-weighted conditions on the initial data (cf. Theorems 2.4 and 2.5). Section 6 is devoted to the analysis of the p-system (1.5), while the nonlinear Euler system (1.4) is studied in Section 7 (cf. Theorem 7.1). Section 8 presents additional results and comments on possible extensions (multi-dimensional setting, numerical analysis) of our methods. A short litterature review on the study of System (1.1) with Fourier-based tools and technical lemmas are relegated to the Appendix.

2. Stability conditions and main results

2.1. **Stability conditions for the linearized model.** Linearizing (1.1) around a constant equilibrium $\bar{U} \in \ker(B)$, the associated linear Cauchy problem reads

$$\begin{cases} \partial_t U + A\partial_x U + BU = 0 \\ \qquad U_0(x,t) = U_0(x), \end{cases} \tag{2.1}$$

where A is a constant symmetric matrix, and B is a constant matrix satisfying (1.2) and the dissipativity condition (1.3). To highlight the partially dissipative structure of (2.1), we decompose $U = (U_1, U_2)$ with $U_1 \in \mathbb{R}^{n_1}$ and $U_2 \in \mathbb{R}^{n_2}$. The couple (U_1, U_2) satisfies

$$\begin{cases} \partial_t U_1 + A_{1,1}\partial_x U_1 + A_{1,2}\partial_x U_2 = 0, \\ \partial_t U_2 + A_{2,1}\partial_x U_1 + A_{2,2}\partial_x U_2 = -DU_2, \end{cases} \quad \text{where } A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}. \tag{2.2}$$

The first difficulty related to partial dissipation is that standard energy estimates leads to an obvious lack of coercivity. Indeed, using the symmetry of A and the condition (1.3), one has

$$\frac{1}{2}\frac{d}{dt}\|(U_1, U_2)(t)\|_{L^2}^2 + \kappa\|U_2(t)\|_{L^2}^2 \leq 0 \tag{2.3}$$

which does not provide any time-decay information on U_1. To ensure that the partial dissipation is sufficient to justify the time-decay of the whole solution, a crucial tool is to take into account the hypocoercive nature of these partially dissipative systems [1]. The idea behind Villani's hypocoercivity theory [27], in our setting, is that: *there may be dissipative mechanisms hidden in the interaction between the hyperbolic matrix A and the partially dissipative matrix B that allows to recover time-decay information for components not directly dissipated.*

Let us have a closer look at this phenomenon on a toy-model, the linearly damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0 \\ \partial_t u + \partial_x \rho = -u. \end{cases} \tag{2.4}$$

Again, standard energy estimates leads to

$$\frac{1}{2}\frac{d}{dt}\|(\rho, u)(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 = 0. \tag{2.5}$$

To overcome the lack of coercivity, one can consider the Lyapunov functional:

$$\mathcal{L}_1(t) = \|(\rho, u, \partial_x \rho, \partial_x u)(t)\|_{L^2} + \int_{\mathbb{R}} u\partial_x \rho \, dx. \tag{2.6}$$

Differentiating it in time and employing Cauchy-Schwarz's and Young's inequalities, one easily reaches

$$\frac{d}{dt}\mathcal{L}_1(t) + \|(u, \partial_x u)(t)\|_{L^2}^2 + \|\partial_x \rho(t)\|_{L^2}^2 \leq 0. \tag{2.7}$$

And since $\mathcal{L}_1(t) \sim \|(\rho, u, \partial_x \rho, \partial_x u)\|_{L^2}^2$, one can now hope to recover the asymptotic behavior of the solution.

Nevertheless, some difficulties remain. One the one hand, in (2.7), the time-decay information on ρ and u are at different levels of regularity and therefore we do not expect to recover the same decay for both components, since the Poincaré inequality is not available in the full space $\mathbb{R}$. And on the other hand, it is not clear, without Fourier analysis, how to recover decay estimates directly from (2.7). With the Fourier transform, one obtains that the low frequencies decay polynomially (as the solution of the heat equation) and the high ones exponentially, see Section 9.1 for more information about this. Since we wish to avoid frequency-based tools, we expect to recover only the weakest behavior of the two regimes: the polynomial decay. And, to recover decay from estimates of the type (2.7), inspired by the work of Hérau and Nier [13, 12] and Porretta and Zuazua [22], we add suitable time-weights in the Lyapunov functional $\mathcal{L}_1$ (2.6).

Concerning general partially dissipative systems of the form (2.1), it is a generalization of the functional (2.6) that we will consider to derive the desired asymptotic behavior. But to make it work, we need to impose an additional stability condition on the matrices A and B. In the work of Beauchard and Zuazua [1], concerning the *hyperbolic hypocoercivity* (hypocoercivity of hyperbolic systems), the authors construct functionals generating dissipation for the whole solution by using the interactions between A and B. The following lemma is central in their argument.

Lemma 2.1 (Lemma 1 in [1])**.** *The following assertions are equivalent:*

- (A, B) *satisfies the Kalman rank condition* (K)*: the Kalman matrix*

$$\mathcal{K} := \begin{pmatrix} B \\ BA \\ \dots \\ BA^{n-1} \end{pmatrix}$$

  has the rank n.
- (A, B) *satisfies the Shizuta-Kawashima* (SK) *condition:*

$$\ker(B) \cap \{\textit{eigenvectors of } A\} = \{0\}.$$

- (A, B) *satisfies the stability condition* (SC): *for any $f \in L^2(\mathbb{R})$,*

$$\|f\|_{L^2_K} := \sum_{k=1}^{n-1} \|BA^k f\|^2_{L^2} \quad \textit{is a norm.}$$

In one space dimension, this condition is equivalent to the non-existence of plane wave solutions propagating in the characteristic directions and ensures the decay [1]. In higher dimensions, a generalisation of the (SK) condition exists and is a sufficient condition for decay, but it is unlikely to hold, refer to [1] for further details. Inspired by this approach (recalled in the Appendix), a generalisation of the toy-model functional (2.6) is the Lyapunov functional:

$$\mathcal{L}(t)_2 \triangleq \|U(t)\|^2_{H^1} + \Re \sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1}U \cdot BA^k \partial_x U)_{L^2}. \tag{2.8}$$

Now, if (A, B) satisfies the Kalman rank condition and if the coefficients ε_k are suitably small, differentiating in time (2.8), one obtains

$$\|U\|_{\dot{H}^1_K} := \sum_{k=1}^{n-1} \|BA^k \partial_x U\|^2_{L^2} < \infty.$$

Thus, we can conclude the time-decay of the whole solution in $\dot{H}^1$ since, thanks to Lemma 2.1, it is clear that under the Kalman rank condition one has

$$\|\cdot\|_{\dot{H}^1_K} \sim \|\cdot\|_{\dot{H}^1}.$$

2.2. **Main results for the linear system.** We are now in position to state our first result concerning the asymptotics of linear hyperbolic partially dissipative systems (2.1).

Theorem 2.2. *Assume that (A, B) satisfies the Kalman rank condition and let $U_0 \in H^1(\mathbb{R})$. Then the solution U to System* (2.1) *satisfies, for all $t > 0$,*

$$\|U(t)\|^2_{H^1} + \int_0^t (\|\partial_x U_1(\tau)\|^2_{L^2} + \|U_2(\tau)\|^2_{H^1}) d\tau \leq C\|U_0\|^2_{H^1}, \tag{2.9}$$

and

$$\|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}} \|U_0\|_{H^1}, \tag{2.10}$$

where $C > 0$ is a constant independent of time.

Remark 2.3 (On Theorem 2.2). Some remarks are in order.

- The proof of this result does not depend on frequency-based tools. This contrasts widely the results obtained in the literature for these systems (see Section 9.1) and allow us to analyse nonlinear systems, for example, the nonlinear compressible Euler system with linear damping (1.4) by assuming the $H^2(\mathbb{R})$-smallness of the initial data, cf. Theorem 7.1 and Section 7.
- As one may notice, we do not recover any decay for the non-directly damped component U_1 at the energetic level $L^2(\mathbb{R})$ but only in $\dot{H}^1(\mathbb{R})$. This comes from the partially dissipative aspect of the system which forces to work at a different regularity level for U_1 and U_2. This can be seen directly in the damped equation of U_2: $\partial_t U_2 + DU_2 + A_{2,2}\partial_x U_2 = -A_{2,1}\partial_x U_1$ which essentially implies that $\partial_x U_1$ has the same decay as U_2.
- Compared to the classical results in the literature (see Proposition 9.2), we are able to recover time-decay estimates without assuming L^1-type assumption on the initial data.
- As discussed in Section 8, we expect our method to provide time-decay estimates at a discrete level, by constructing numerical schemes preserving the hypocoercivity nature of our systems.

The aim of the next two results is to improve the time-decay rates, in particular to recover time-decay for the component U_1 in $L^2(\mathbb{R})$. To that matter, by virtue of a new space-weighted energy method and the Caffarelli-Kohn-Nirenberg inequality, we prove a result for initial data satisfying weighted Lebesgue integrability properties.

Theorem 2.4. *Let $\frac{1}{2} < \mu \leq 1$, the assumptions of Theorem 2.2 be in force, and U be the global solution to System (2.1). Assume furthermore $A_{1,1} = 0$ and (1.3) with $\kappa \geq \kappa_0$ for some positive constant κ_0. Additionally, the initial data U_0 satisfies*

$$\| |x|^{\mu} U_0 \|_{H^1} < \infty. \tag{2.11}$$

Then the solution U of System (2.1) satisfies, $\forall t > 0$,

$$\begin{cases} \|U_1(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}} X_0, \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}-\frac{1}{2}} X_0, \end{cases} \tag{2.12}$$

where $X_0 := \|(1+|x|^{\mu})U_0\|_{H^1}$, and $C > 0$ is a constant independent of time.

The next time-decay result is obtained by assuming that the system (2.1) can be formulated as an extended damped wave system. Indeed, the unknown

$$W(x,t) = \int_{-\infty}^{x} U_1(y,t)dy,$$

satisfies the damped wave system

$$\partial_t^2 W - A_{1,2}A_{2,1}\partial_x^2 W + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t\partial_x W + A_{1,2}DA_{1,2}^{-1}\partial_t W = 0$$

provided that $A_{1,1} = 0$. In order to take advantage of the techniques for damped wave equations and relax the weighted assumption (2.11), we need to impose additional conditions on the matrix A. Our result is stated as follows.

Theorem 2.5. *Let $\frac{1}{2} < \mu \leq 1$, the assumptions of Theorem 2.2 be in force, and U be the global solution to System (2.1). In addition, assume that $A_{1,1} = 0$, $A_{1,2}$ is invertible, D is symmetric, and $A_{1,2}A_{2,1}$ satisfies the following strong dissipative condition: there exists a constant $\kappa_1 > 0$ such that*

$$(A_{1,2}A_{2,1}X, X) \geq \kappa_1 |X|^2, \qquad X \in \mathbb{R}^{n_2}, \qquad \text{if} \quad \mu = 1, \tag{2.13}$$

$$\kappa_1|X|^2 \le (A_{1,2}A_{2,1}X, X) \le |X|^2, \quad X \in \mathbb{R}^{n_2}, \qquad \kappa_1 \le 1, \qquad \textit{if} \ \ \frac{1}{2} < \mu < 1. \tag{2.14}$$

Assume furthermore that the initial data U_0 satisfies

$$\||x|^{\mu} U_{1,0}\|_{L^2} + \||x|^{\mu-\frac{1}{2}} U_{2,0}\|_{L^2} < \infty, \tag{2.15}$$

and that the following compatibility condition holds:

$$\partial_t U_1|_{t=0} = -A_{1,2}\partial_x U_{2,0}. \tag{2.16}$$

Then the solution U to System (2.1) subject to the initial data U_0 satisfies, $\forall t > 0$,

$$\begin{cases} \|U_1(t)\|_{L^2} \le C(1+t)^{-\mu+\frac{1}{2}} Y_0, \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \le C(1+t)^{-\mu} Y_0, \end{cases} \tag{2.17}$$

where $Y_0 := \|U_0\|_{H^1} + \||x|^{\mu} U_{1,0}\|_{L^2} + \||x|^{\mu-\frac{1}{2}} U_{2,0}\|_{L^2}$, and $C > 0$ is a constant independent of time.

Remark 2.6 (On Theorems 2.4 and 2.5). Some comments are in order.

- Theorems 2.4 and 2.5 imply that under some space-weighted assumptions on the initial data and structural conditions on the system, the solution can achieve faster time-decay rates compared with the classical decay rates stated in Proposition 9.2.
- In the context of fluid mechanics, the condition $A_{1,1} = 0$ is satisfied up to a Galilean change of frame.
- The L^2-decay rate $(1+t)^{-\frac{\mu}{2}}$ of U_1 in Theorem 2.4 is consistent with the optimal time-decay rate of the heat equation with the initial data in $\{f|\ \text{s.t.}\ |x|^{\mu} f \in L^2(\mathbb{R})\}$.

In Section 7, using a perturbation argument, we extend the results from Theorem 2.4 and 2.5 to a nonlinear setting and obtain a results for the compressible Euler system (1.4). Such result can be extended to general partially dissipative systems (1.1) under structural conditions, cf.[5].

Adapting the method used to prove these theorems, we are able to study the long time behavior of a purely nonlinearly dissipated system.

2.3. **Asymptotics for the nonlinearly damped p-system.** We are now interested in the asymptotic behavior of the Cauchy problem of the non-linearly damped p-system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + |v|^{r-1} v = 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), \qquad x \in \mathbb{R}, \end{cases} \tag{2.18}$$

with some constant $1 < r < 3$. For this system, to the best of our knowledge, there are no any results in the literature concerning the large-time asymptotic behavior of the solutions. In contrast with the linear damping setting, using the Fourier transform is hopeless when trying to recover beneficial properties from the nonlinear damping term $|v|^{r-1}v$ as it would involve convolution products.

Therefore, we use a method close to the one developed for Theorem 2.5 and employ weighted integrability properties. Formally defining w such that $u = \partial_x w$ and $v = \partial_t w$, one recovers the nonlinearly damped wave equation

$$\partial_t^2 w - \partial_x^2 w + |\partial_t w|^{r-1} \partial_t w = 0. \tag{2.19}$$

Generalizing the approach developed by Mochizuki and Motai in [20] pertaining to the study of (2.19) on the real line, we obtain the following result.

Theorem 2.7. *Let $1 < r < 3$, $q > 0$, and assume*

$$\partial_t u|_{t=0} = -\partial_x v_0, \qquad (u_0, v_0) \in H^1(\mathbb{R}), \qquad u_0 \in L^1(\mathbb{R}), \qquad \log^q(1+|x|)(u_0, v_0) \in L^2(\mathbb{R}).$$

Then for all $t > 0$, the solution (u, v) of (2.18) associated to the initial data (u_0, v_0) satisfies

$$\|(u,v)(t)\|_{H^1}^2 + \int_0^t \left(\|v(\tau)\|_{L^{r+1}}^{r+1} + \|(\partial_x v^{\frac{r+1}{2}}, \partial_x u^{\frac{r+1}{2}})(\tau)\|_{L^2}^2\right) d\tau \leq C\|(u_0, v_0)\|_{H^1}^2, \tag{2.20}$$

and

$$\|(u,v)(t)\|_{L^2} \leq \frac{C}{\log^q(1+t)}, \tag{2.21}$$

where $C > 0$ is a constant independent of time.

Remark 2.8 (On Theorem 2.7). Some remarks are in order.

- The restrictions on the index r are similar to the one assumed for the nonlinearly damped wave equation. Indeed, the solution to the nonlinearly damped wave equation decays in time under $1 < r < 1 + \frac{d}{2}$, where d corresponds to the dimension, and does not decay if $r > 1 + \frac{d}{2}$ (cf. [9, 21, 20] and references therein).
- One difficulty here is that we consider (2.18) on the real line $\mathbb{R}$, and therefore Poincaré-type inequalities may not be available. To compensate this, we employ weighted integrability conditions.
- Under stronger weighted integrability conditions on the initial data, it is possible to show the polynomial time-decay rates of solutions to (2.18) (cf. [20, 28]).

3. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by employing pure energy arguments and avoid using the Fourier transform. As explained in the previous section, we build a functional by adding time-weights in (2.8). It reads

$$\mathcal{L}(t) := \|U(t)\|_{H^1}^2 + ct\|\partial_x U(t)\|_{L^2}^2 + \mathcal{I}(t), \tag{3.1}$$

where the corrector term $\mathcal{I}(t)$ is defined by

$$\mathcal{I}(t) := \sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1}U, BA^k \partial_x U)_{L^2}, \tag{3.2}$$

and the constants c, ε_i, $i = 1, 2, ..., k-1$, will be determined later.

3.1. Time-derivative of $\mathcal{L}$.

3.1.1. *Energy part.* Standard energy estimates for (2.2) lead to

$$\frac{d}{dt}\|U(t)\|_{L^2}^2 + 2(DU_2, U_2)_{L^2} = 0,$$
$$\frac{d}{dt}\|\partial_x U(t)\|_{L^2}^2 + 2(D\partial_x U_2, \partial_x U_2)_{L^2} = 0,$$
$$c\frac{d}{dt}(t\|\partial_x U(t)\|_{L^2}^2) + 2c(D\partial_x U_2, \partial_x U_2)_{L^2} = c\|\partial_x U(t)\|_{L^2}^2,$$

which, together with the strong dissipativity condition (1.3) of D, gives

$$\frac{d}{dt}\big(\mathcal{L}(t) - \mathcal{I}(t)\big) + 2\kappa\|U_2(t)\|_{L^2}^2 + 2\kappa(1+ct)\|\partial_x U_2(t)\|_{L^2}^2 \leq c\|\partial_x U(t)\|_{L^2}^2. \tag{3.3}$$

Here we miss the dissipation estimate of $\partial_x U$ in $L^2(\mathbb{R})$ to ensure the negativity of $\frac{d}{dt}\mathcal{L}(t)$. It is recovered in the next section thanks to the corrector term $\mathcal{I}(t)$ in the Lyapunov functional (3.1).

3.1.2. *Estimates of the corrector:* Differentiating $\mathcal{I}(t)$ in time, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t)+\sum_{k=1}^{n-1}\varepsilon_k\|BA^k\partial_x U(t)\|_{L^2}^2 =& -\sum_{k=1}^{n-1}\varepsilon_k(BA^{k-1}BU,BA^k\partial_x U)_{L^2}\\ &-\sum_{k=1}^{n-1}\varepsilon_k(BA^{k-1}U,BA^kB\partial_x U)_{L^2}\\ &-\sum_{k=1}^{n-1}\varepsilon_k(BA^{k-1}U,BA^{k+1}\partial_{xx}^2 U)_{L^2}.\end{aligned}$$

Thanks to Lemma 2.1, the second term on the left-hand side of (3.4) leads to time-decay information for $\partial_x U_1$. To absorb the remainder terms, we proceed as in [1, 5, 8] with some adaptations due to the lack of Fourier analysis. It is given by the following lemma.

Lemma 3.1 (Time-derivative of $\mathcal{I}$)**.** *For any positive constant ε_0, there exists a small positive constant sequence $\{\varepsilon_k\}_{k=1,\dots,n-1}$ such that*

$$\frac{d}{dt}\mathcal{I}(t)+\frac{1}{2}\sum_{k=1}^{n-1}\varepsilon_k\|BA^k\partial_x U(t)\|_{L^2}^2\leq \varepsilon_0\|U_2(t)\|_{L^2}^2+\varepsilon_0\|\partial_x U_2(t)\|_{L^2}^2. \tag{3.4}$$

Proof. To begin with, we fix some positive constant ε_0 and estimate the terms in the right-hand side of (3.4) as follows.

- The terms $\mathcal{I}_k^1:=\varepsilon_k(BA^{k-1}BU,BA^k\partial_x U)$ with $k\in\{1,\cdots,n-1\}$: due to $BU=DU_2$ and the fact that the matrices A, D are bounded, we obtain
$$\begin{aligned}|\mathcal{I}_k^1|&\leq C\varepsilon_k\|DU_2(t)\|_{L^2}\|BA^k\partial_x U(t)\|_{L^2}\\ &\leq\frac{\varepsilon_0}{4n}\|U_2(t)\|_{L^2}^2+\frac{C\varepsilon_k^2}{\varepsilon_0}\|BA^k\partial_x U(t)\|_{L^2}^2.\end{aligned}$$
- The term $\mathcal{I}_1^2:=\varepsilon_1(BU,BAB\partial_x U)_{L^2}$: one has
$$\begin{aligned}|\mathcal{I}_1^2|&\leq C\varepsilon_1\|DU_2(t)\|_{L^2}\|D\partial_x U_2(t)\|_{L^2}\\ &\leq\frac{\varepsilon_0}{4n}\|U_2(t)\|_{L^2}^2+\frac{C\varepsilon_1^2}{\varepsilon_0}\|\partial_x U_2(t)\|_{L^2}^2.\end{aligned}$$
- The terms $\mathcal{I}_k^2:=\varepsilon_k(BA^{k-1}U,BA^kB\partial_x U)_{L^2}$ with $k\in\{2,\cdots,n-1\}$: we deduce after integrating by part that
$$\begin{aligned}|\mathcal{I}_k^2|&=\varepsilon_k|(BA^{k-1}\partial_x U,BA^kBU)_{L^2}|\\ &\leq C\varepsilon_k\|BA^{k-1}\partial_x U(t)\|_{L^2}\|BU(t)\|_{L^2}\\ &\leq\frac{\varepsilon_0}{4n}\|U_2(t)\|_{L^2}^2+\frac{C\varepsilon_{k-1}^2}{\varepsilon_0}\|BA^{k-1}\partial_x U(t)\|_{L^2}^2.\end{aligned}$$
- The terms $\mathcal{I}_k^3:=\varepsilon_k(BA^{k-1}U,BA^{k+1}\partial_{xx}^2 U)_{L^2}$ with $k\in\{1,\cdots,n-2\}$: a similar argument yields
$$\begin{aligned}|\mathcal{I}_k^3|&=\varepsilon_k|(BA^{k-1}\partial_x U,BA^{k+1}\partial_x U)_{L^2}|\\ &\leq\frac{\varepsilon_{k-1}}{8}\|BA^{k-1}\partial_x U(t)\|_{L^2}^2+\frac{C\varepsilon_k^2}{\varepsilon_{k-1}}\|BA^{k+1}\partial_x U(t)\|_{L^2}^2.\end{aligned}$$
- The term $\mathcal{I}_{n-1}^3:=\varepsilon_{n-1}(BA^{n-2}U,BA^n\partial_{xx}^2 U)_{L^2}$: we observe that, owing to the Cayley-Hamilton theorem, there exist coefficients c_ω^j $(j=1,2,...,n-1)$ (that are uniformly bounded on $\mathbb{S}^{d-1}$) such that
$$A^n=\sum_{j=0}^{n-1}c^jA^j. \tag{3.5}$$

Consequently, one gets

$$
\begin{aligned}
|\mathcal{I}_{n-1}^3| &\le \varepsilon_{n-1}\sum_{j=0}^{n-1}\|BA^{n-2}\partial_x U(t)\|_{L^2}\|BA^j\partial_x U(t)\|_{L^2}\\
&\le \frac{\varepsilon_{n-2}}{8}\|BA^{n-2}\partial_x U(t)\|_{L^2}^2+\sum_{j=1}^{n-1}\frac{C\varepsilon_{n-1}^2}{\varepsilon_{n-2}}\|BA^j\partial_x U(t)\|_{L^2}^2+\frac{C\varepsilon_{n-1}^2}{\varepsilon_{n-2}}\|\partial_x U_2(t)\|_{L^2}^2.
\end{aligned}
$$

In order to absorb the right-hand side terms of I_k^1 and I_k^2 by the left-hand side of (3.4), we take the constant ε_k small enough so that

$$
C\varepsilon_1^2\le\frac{\varepsilon_0^2}{8},\qquad C\varepsilon_k^2\le\frac{\varepsilon_k\varepsilon_0}{8},\qquad k=1,2,...,n-1. \tag{3.6}
$$

To handle the above estimates of $\mathcal{I}_k^3$ with $k=1,2,...,n-2$, one may let

$$
C\varepsilon_k^2\le\frac{1}{8}\varepsilon_{k-1}\varepsilon_{k+1},\qquad k=1,2,...,n-2. \tag{3.7}
$$

In addition, to handle the term $\mathcal{I}_{n-1}^3$, we assume

$$
C\varepsilon_{n-1}^2\le\frac{1}{8}\varepsilon_j\varepsilon_{n-2},\qquad j=0,\cdots,n-1. \tag{3.8}
$$

Clearly, the inequality (3.4) holds if we find $\varepsilon_1,\cdots,\varepsilon_{n-1}$ fulfilling (3.6) and (3.8). As in [1], one can take $\varepsilon_k=\varepsilon^{m_k}$ with some suitable small constant $\varepsilon\le\varepsilon_0$ and $m_1,\cdots,m_{n-1}$ satisfying for some $\delta>0$ (that can be taken arbitrarily small):

$$
m_k>1,\quad m_k\ge\frac{m_{k-1}+m_{k+1}}{2}+\delta\quad\text{and}\quad m_{n-1}\ge\frac{m_k+m_{n-2}}{2}+\delta,\quad k=1,\cdots,n-2.
$$

This concludes the proof of Lemma 3.1. □

3.2. **Decay for $\partial_x U$.** We first fix suitable small $\varepsilon_k, k-1,2,...,n-1$, to promise (3.4) and

$$
\mathcal{L}(t)\sim\|U(t)\|_{H^1}^2+ct\|\partial_x U(t)\|_{L^2}^2. \tag{3.9}
$$

Combining the Lyapunov inequality (3.3), the estimate (3.4) of the corrector term and Lemma 2.1 together, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}(t)+2\kappa\|U_2(t)\|_{L^2}^2+2\kappa(1+ct)\|\partial_x U_2(t)\|_{L^2}^2+\frac{\varepsilon_*}{C_K}\|\partial_x U(t)\|_{L^2}^2\\
&\le c\|\partial_x U(t)\|_{L^2}^2+\varepsilon_0\|U_2(t)\|_{L^2}^2+\varepsilon_0\|\partial_x U_2(t)\|^2,
\end{aligned} \tag{3.10}
$$

with $\varepsilon_*:=\min_{1\le k\le i-1}\varepsilon_k$ and $C_K>0$ a constant depending only on (A,B) and n. In order to derive the coerciveness of the left-hand side of (3.10), we adjust the coefficients appropriately as

$$
0<c<\frac{\varepsilon_*}{2C_K},\qquad 0<\varepsilon_0<\frac{\kappa}{2}
$$

such that

$$
\frac{d}{dt}\mathcal{L}(t)+\frac{3\kappa}{2}\|U_2(t)\|_{L^2}^2+\kappa(\frac{1}{2}+ct)\|\partial_x U_2(t)\|_{L^2}^2+\frac{\varepsilon_*}{2C_K}\|\partial_x U(t)\|_{L^2}^2\le 0, \tag{3.11}
$$

Therefore, by (3.9) and (3.11), we show

$$
\|U(t)\|_{L^2}+(1+t^{\frac{1}{2}})\|\partial_x U(t)\|_{L^2}\le C\|U_0\|_{H^1}. \tag{3.12}
$$

3.3. **Decay for U_2.** Taking the inner product of $(2.2)_2$ with U_2 and using the property (1.3), we get

$$\frac{d}{dt}\|U_2(t)\|_{L^2}^2 + 2\kappa\|U_2(t)\|_{L^2}^2 \lesssim \|\partial_x U(t)\|_{L^2}\|U_2(t)\|_{L^2}. \tag{3.13}$$

Dividing the above inequality (3.13) by $\sqrt{\|U_2(t)\|_{L^2}^2+\varepsilon}$, employing Grönwall's inequality and then letting $\varepsilon \to 0$, we have

$$\|U_2(t)\|_{L^2} \lesssim e^{-\kappa t}\|U_{2,0}\|_{L^2} + \int_0^t e^{-\kappa(t-\tau)}\|\partial_x U(\tau)\|_{L^2}dx. \tag{3.14}$$

with some constant $\kappa > 0$. Together with the time-decay estimates (3.12) of $\partial_x U$, this leads to

$$\|U_2(t)\|_{L^2} \le e^{-\kappa t}\|U_{2,0}\|_{L^2} + \int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-\frac12}d\tau\|U_0\|_{H^1} \lesssim (1+t)^{-\frac12}\|U_0\|_{H^1},$$

which concludes the proof of Theorem 2.2.

4. Faster time decay for partially dissipative hyperbolic systems: space-weighted energy method

The purpose of this section is to capture the time-decay rates of U_1 in $L^2(\mathbb{R})$. Our method allows to recover strong decay rates compared to the $(1+t)^{-\frac14}$ decay rates obtained in in the work [19] of Matsumura and Nishida who assumed the $L^1(\mathbb{R})$-regularity of the initial data . To avoid the use of the Fourier transform, we will rely on a weighted Lebesgue integrability on the initial data and perform weighted estimates by taking advantage of the Caffarelli-Kohn-Nirenberg inequality.

Lemma 4.1. *Let U be the solution to the Cauchy problem* (2.1) *given by Theorem 2.2. If in addition to* (2.10)*, there exists a constant $\widetilde{X}_0$ such that*

$$\widetilde{X}_0 := \sup_{t>0}\||x|^\mu U_1(t)\|_{L^2} < \infty, \qquad \frac12 < \mu \le 1, \tag{4.1}$$

then for all $t > 0$, the following time-decay estimates hold:

$$\begin{cases} \|U_1(t)\|_{L^2} \le C(1+t)^{-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}), \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \le C(1+t)^{-\frac{\mu}{2}-\frac12}(\widetilde{X}_0 + \|U_0\|_{H^1}), \end{cases}$$

Proof. We aim prove that the dissipation in the inequality (3.11) can control the Lyapunov functional $\mathcal{L}(t)$ in a suitable sense so as to apply Lemma 9.5. Indeed, inspired by the interpolation arguments in [10, 29], we deduce from the Caffarelli-Kohn-Nirenberg inequality (9.5) that

$$\|U_1(t)\|_{L^2} \lesssim \||x|^\mu U_1(t)\|_{L^2}^{\frac{1}{1+\mu}}\|\partial_x U_1(t)\|_{L^2}^{\frac{\mu}{1+\mu}}, \tag{4.2}$$

which together with (4.1) yields

$$\|\partial_x U_1(t)\|_{L^2}^2 \gtrsim \widetilde{X}_0^{-\frac{2}{\mu}}\|U_1(t)\|_{L^2}^{2+\frac{2}{\mu}}. \tag{4.3}$$

On the other hand, due to (2.10), one has

$$\|U_2(t)\|_{L^2}^2 \gtrsim \|U_0\|_{H^1}^{-\frac{2}{\mu}}\|U_2(t)\|_{L^2}^{2+\frac{2}{\mu}}, \qquad \|\partial_x U(t)\|_{L^2}^2 \gtrsim \|U_0\|_{H^1}^{-\frac{2}{\mu}}\|\partial_x U(t)\|_{L^2}^{2+\frac{2}{\mu}}. \tag{4.4}$$

Thus, it follows from (3.11) and (4.3)-(4.4) that

$$\frac{d}{dt}(\mathcal{L}_*(t) + ct\|\partial_x U(t)\|_{L^2}^2) + (\widetilde{X}_0 + \|U_0\|_{H^1})^{-\frac{2}{\mu}}\mathcal{L}_*(t)^{2+\frac{2}{\mu}} + \|\partial_x U(t)\|_{L^2}^2 \lesssim 0, \tag{4.5}$$

with

$$\mathcal{L}_*(t) := \mathcal{L}(t) - ct\|\partial_x U(t)\|_{L^2}^2 \sim \|U(t)\|_{H^1}^2.$$

Since the constant c can be sufficiently small, employing Lemma 9.5 to the differential inequality (4.5), we conclude that

$$\|U(t)\|_{H^1} \lesssim (1+t)^{-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}), \qquad \|\partial_x U(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}). \tag{4.6}$$

Finally, the combination of (2.10), (3.14) and (4.6) leads to

$$\begin{aligned}\|U_2(t)\|_{L^2} &\leq e^{-\kappa t}\|U_{2,0}\|_{L^2} + (\widetilde{X}_0 + \|U_0\|_{H^1})\int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-\frac{1}{2}-\frac{\mu}{2}}d\tau \\ &\lesssim (1+t)^{-\frac{1}{2}-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}).\end{aligned}$$

The proof of Lemma 4.1 is complete.

□

The weighted Lebesgue integrability (4.1) is achieved in the following lemma.

Lemma 4.2. *Let U be the solution to the Cauchy problem (2.1) supplemented with the initial data U_0. Then under the assumptions of Theorem 2.4, for all $t > 0$, it holds that*

$$\||x|^\mu U(t)\|_{H^1}^2 + \int_0^t \||x|^\mu \partial_x U(\tau)\|_{L^2}^2 d\tau \leq C X_0^2, \tag{4.7}$$

with $X_0 := \|(1+|x|^\mu)U_0\|_{H^1}$.

Proof. We will perform space-weighted estimates and take advantage of the Caffarelli–Kohn–Nirenberg inequality (9.6). The proof is split into three steps.

- *Step 1: Space-weighted estimates of U.*

We first perform weighted estimates of U_2. In what follows, $C > 0$ denotes a suitable large constant independent of time and the matrix B. Taking the $L^2(\mathbb{R})$-inner of $(2.2)_2$ with $|x|^{2\mu}U_2$ and using Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\||x|^\mu U_2(t)\|_{L^2}^2 + 2\kappa\||x|^\mu U_2(t)\|_{L^2}^2 \\ &= 2\int_{\mathbb{R}}(-|x|^{2\mu}A_{2,2}\partial_x U_2 U_2 - |x|^{2\mu}A_{2,1}\partial_x U_1 U_2)dx \\ &\leq \frac{\kappa}{2}\||x|^\mu U_2(t)\|_{L^2}^2 + \frac{C}{\kappa}\||x|^\mu \partial_x U(t)\|_{L^2}^2.\end{aligned} \tag{4.8}$$

Before estimating U_1, we apply the Caffarelli-Kohn-Nirenberg inequality (9.5) to obtain

$$\||x|^{\mu-1}U_1(t)\|_{L^2} \leq \frac{(2\mu-1)}{2}\||x|^\mu \partial_x U_1(t)\|_{L^2}. \tag{4.9}$$

Thence, it follows from $(2.2)_2$ with $A_{1,1} = 0$ and (4.9) that

$$\begin{aligned}\frac{d}{dt}\||x|^\mu U_1(t)\|_{L^2}^2 &= 2\int_{\mathbb{R}}|x|^{2\mu}A_{1,2}\partial_x U_2 U_1 dx \\ &= 2\int_{\mathbb{R}}(\partial_x|x|^{2\mu}A_{1,2}U_2U_1 + |x|^{2\mu}A_{1,2}U_2\partial_x U_1)dx \\ &\leq C\||x|^\mu U_2(t)\|_{L^2}(\||x|^{\mu-1}U_1(t)\|_{L^2} + \||x|^\mu\partial_x U_1(t)\|_{L^2}) \\ &\leq \frac{\kappa}{2}\||x|^\mu U_2(t)\|_{L^2}^2 + \frac{C}{\kappa}\||x|^\mu \partial_x U_1(t)\|_{L^2}^2.\end{aligned}$$

This together with (2.10) and (4.8) yields

$$\begin{aligned}&\||x|^\mu U(t)\|_{L^2}^2 + \kappa\int_0^t \||x|^\mu U_2(\tau)\|_{L^2}^2 d\tau \\ &\leq \||x|^\mu U_0\|_{L^2}^2 + \frac{C}{\kappa}\int_0^t \||x|^\mu \partial_x U_1(\tau)\|_{L^2}^2 d\tau.\end{aligned} \tag{4.10}$$

In addition, we are going to obtain the weighted estimate of BU which are needed in estimates. From (2.2), we have

$$\frac{d}{dt}\||x|^\mu BU(t)\|_{L^2}^2+2(|x|^\mu BBU,|x|^\mu BU)_{L^2}=-2(|x|^\mu BA\partial_x U,|x|^\mu BU)_{L^2}.$$

One can show

$$2(|x|^\mu BBU,|x|^\mu BU)_{L^2}=2(|x|^\mu DDU_2,|x|^\mu DU_2)_{L^2}\geq 2\kappa\||x|^\mu DU_2\|_{L^2}^2.$$

and

$$2(|x|^\mu BA\partial_x U,|x|^\mu BU)_{L^2}\leq \kappa\||x|^\mu DU_2\|_{L^2}^2+\frac{C}{\kappa}\||x|^\mu BA\partial_x U(t)\|_{L^2}^2.$$

Thus, it holds that

$$\||x|^\mu BU(t)\|_{L^2}^2+\kappa\||x|^\mu DU_2\|_{L^2}^2\leq \||x|^\mu BU_0\|_{L^2}^2+\frac{C}{\kappa}\||x|^\mu BA\partial_x U(t)\|_{L^2}^2. \tag{4.11}$$

- *Step 2: Space-weighted estimates of $\partial_x U$.*

Differentiating (1.1) with respect to x and taking the $L^2(\mathbb{R})$-inner of the resulting system with $|x|^{2\mu}\partial_x U$, we get

$$\frac{d}{dt}\||x|^\mu \partial_x U(t)\|_{L^2}^2+2\kappa\||x|^\mu\partial_x U_2(t)\|_{L^2}^2=4\mu\int_{\mathbb{R}}|x|^{2\mu-1}A\partial_x U\partial_x U dx.$$

For $\mu=1$ and for a constant $\eta\in(0,1)$ to be chosen later, one has

$$4\int_{\mathbb{R}}|x|A\partial_x U\partial_x U dx\leq C\||x|\partial_x U(t)\|_{L^2}\|\partial_x U(t)\|_{L^2}\leq \eta\||x|\partial_x U(t)\|_{L^2}^2+\frac{C}{\eta}\|\partial_x U(t)\|_{L^2}^2.$$

As for $\frac{1}{2}<\mu<1$, it also holds that

$$\begin{aligned}4\mu\int_{\mathbb{R}}|x|^{2\mu-1}A\partial_x U\partial_x U dx&\leq C\||x|^{\mu-1}\partial_x U(t)\|_{L^2}\||x|^\mu\partial_x U(t)\|_{L^2}\\&\leq C(\eta\||x|^\mu\partial_x U(t)\|_{L^2}+\frac{1}{\eta}\|\partial_x U(t)\|_{L^2})\||x|^\mu\partial_x U(t)\|_{L^2}\\&\leq C\eta\||x|^\mu\partial_x U(t)\|_{L^2}^2+\frac{C}{\eta^3}\|\partial_x U(t)\|_{L^2}^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\||x|^\mu\partial_x U(t)\|_{L^2}^2+2\kappa\int_0^t\||x|^\mu\partial_x U_2(\tau)\|_{L^2}^2d\tau\\&\quad\leq \||x|^\mu\partial_x U_0\|_{L^2}^2+C\eta\int_0^t\||x|^\mu\partial_x U(\tau)\|_{L^2}^2d\tau+\frac{C}{\eta}\int_0^t\|\partial_x U(\tau)\|_{L^2}^2d\tau\\&\quad\leq \||x|^\mu\partial_x U_0\|_{L^2}^2+\frac{C}{\eta^3}\|U_0\|_{H^1}^2+C\eta\int_0^t\||x|^\mu\partial_x U(\tau)\|_{L^2}^2d\tau,\end{aligned} \tag{4.12}$$

where one has used (2.10).

- *Step 3: Cross estimates.*

We are ready to close the $H^1(\mathbb{R})$-estimate of U with the weight $|x|^\mu$. It is sufficient to estimate $\int_0^t\sum_{k=1}^{n-1}\|BA^k\partial_x(xU)(\tau)\|_{L^2}^2d\tau$ by Lemma 2.1. To this matter, let the constants $\widetilde{\varepsilon}_i$, $i=1,2,...,k-1$ to be chosen later. By direct computations on system (2.1) with the weight $|x|^\mu$, we infer

$$\frac{d}{dt}\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^k\partial_x U)_{L^2}+\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2=\mathcal{R}, \tag{4.13}$$

with

$$\mathcal{R} := -\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}BU,|x|^\mu BA^k\partial_x U)_{L^2}$$
$$-\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^{k+1}\partial_{xx}^2 U)_{L^2}-\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^kB\partial_x U)_{L^2}.$$

We use similar arguments in the proof of Lemma 3.1 to control the reminder term W:

- The terms $\widetilde{\mathcal{I}}_k^1 := \widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}BU,|x|^\mu BA^k\partial_x U)$ with $k\in\{1,\cdots,n-1\}$: one has
$$|\widetilde{\mathcal{I}}_k^1|\leq C\||x|^\mu DU_2(t)\|_{L^2}^2+\widetilde{\varepsilon}_k^2\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2.$$
- The term $\widetilde{\mathcal{I}}_1^2 := \widetilde{\varepsilon}_1(|x|^\mu BU,|x|^\mu BAB\partial_x U)$: One gets
$$\begin{aligned}|\widetilde{\mathcal{I}}_1^2|&\leq C\widetilde{\varepsilon}_1\||x|^\mu DU_2(t)\|_{L^2}\||x|^\mu D\partial_x U_2(t)\|_{L^2}\\&\leq C\||x|^\mu DU_2(t)\|_{L^2}^2+\widetilde{\varepsilon}_1^2\||x|^\mu D\partial_x U_2(t)\|_{L^2}^2.\end{aligned}$$
- The terms $\widetilde{\mathcal{I}}_k^2 := \widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^kB\partial_x U)_{L^2}$ with $k\in\{2,\cdots,n-1\}$: by integration by parts and the Caffarelli-Kohn-Nirenberg inequality (9.5), there holds that
$$\begin{aligned}|\widetilde{\mathcal{I}}_k^2|&\leq\widetilde{\varepsilon}_k|(|x|^\mu BA^{k-1}\partial_x U,|x|^\mu BA^kBU)_{L^2}|+2\mu\widetilde{\varepsilon}_k|(|x|^{\mu-1}BA^{k-1}U,|x|^\mu BA^kBU)_{L^2}|\\&\leq C\widetilde{\varepsilon}_k(\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}+\||x|^{\mu-1}BA^{k-1}U(t)\|_{L^2})\||x|^\mu BU(t)\|_{L^2}\\&\leq C\widetilde{\varepsilon}_k\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}\||x|^\mu DU_2(t)\|_{L^2}\\&\leq C\||x|^\mu DU_2(t)\|_{L^2}^2+C\widetilde{\varepsilon}_{k-1}^2\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}^2.\end{aligned}$$
- The terms $\widetilde{\mathcal{I}}_k^3 := \widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^{k+1}\partial_{xx}^2 U)_{L^2}$ with $k\in\{1,\cdots,n-2\}$: similar argument yields
$$\begin{aligned}|\widetilde{\mathcal{I}}_k^3|&\leq\widetilde{\varepsilon}_k|(|x|^\mu BA^{k-1}\partial_x U,|x|^\mu BA^{k+1}\partial_x U)_{L^2}|\\&\quad+2\mu\widetilde{\varepsilon}_k|(|x|^{\mu-1}BA^{k-1}U,|x|^\mu BA^{k+1}\partial_x U)_{L^2}|\\&\leq\frac{\widetilde{\varepsilon}_{k-1}}{8}\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}^2+\frac{C\widetilde{\varepsilon}_k^2}{\widetilde{\varepsilon}_{k-1}}\||x|^\mu BA^{k+1}\partial_x U(t)\|_{L^2}^2.\end{aligned}$$
- The term $\widetilde{\mathcal{I}}_{n-1}^2 := \widetilde{\varepsilon}_{n-1}(|x|^\mu BA^{n-2}U,|x|^\mu BA^n\partial_{xx}^2 U)_{L^2}$: one obtains from (3.5) and the Caffarelli-Kohn-Nirenberg inequality (9.5) that
$$\begin{aligned}|\widetilde{\mathcal{I}}_{n-1}^3|&\leq\widetilde{\varepsilon}_{n-1}\sum_{j=0}^{n-1}\||x|^\mu BA^{n-2}\partial_x U(t)\|_{L^2}\||x|^\mu BA^j\partial_x U(t)\|_{L^2}\\&\quad+2\mu\widetilde{\varepsilon}_{n-1}\sum_{j=0}^{n-1}\||x|^{\mu-1}BA^{n-2}U(t)\|_{L^2}\||x|^\mu BA^j\partial_x U(t)\|_{L^2}\\&\leq\frac{\widetilde{\varepsilon}_{n-2}}{8}\||x|^\mu BA^{n-2}\partial_x U(t)\|_{L^2}^2+\sum_{j=1}^{n-1}\frac{C\widetilde{\varepsilon}_{n-1}^2}{\widetilde{\varepsilon}_{n-2}}\||x|^\mu BA^j\partial_x U(t)\|_{L^2}^2\\&\quad+\frac{C\widetilde{\varepsilon}_{n-1}^2}{\widetilde{\varepsilon}_{n-2}}\||x|^\mu D\partial_x U_2(t)\|_{L^2}^2.\end{aligned}$$

Therefore, we are able to find a small positive constant sequence $\{\widetilde{\varepsilon}_k\}_{k=1,\dots,n-1}$ satisfying

$$\begin{cases}\widetilde{\varepsilon}_k\leq\frac{1}{8}, & k=1,2,...,n-1,\\ 8C\widetilde{\varepsilon}_k^2\leq\widetilde{\varepsilon}_{k-1}\widetilde{\varepsilon}_{k+1}, & k=1,2,...,n-2,\\ 8C\widetilde{\varepsilon}_{n-1}^2\leq\widetilde{\varepsilon}_{n-2}\widetilde{\varepsilon}_j, & j=1,\cdots,n-1.\end{cases}\tag{4.14}$$

such that the term $\mathcal{R}$ can be bounded by

$$\mathcal{R} \leq \frac{1}{2}\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2 + C\|D|x|^\mu U_2(t)\|_{L^2}^2 + C\|D|x|^\mu\partial_x U_2(t)\|_{L^2}^2.$$

This together with (4.11) and (4.13) leads to

$$\begin{aligned}&\frac{1}{2}\int_0^t\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2\\&\leq -\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^k\partial_x U)_{L^2}|_0^t + C\int_0^t\||x|^\mu D\partial_x U_2(\tau)\|_{L^2}^2 d\tau + C\||x|^\mu BU_0\|_{L^2}^2\\&\quad+\frac{C}{\kappa}\int_0^t\||x|^\mu BA\partial_x U(\tau)\|_{L^2}^2.\end{aligned}$$

Since $\widetilde{\varepsilon}_k$ and C are independent of κ and B, one chooses $\kappa \geq \frac{4C}{\varepsilon_1}$ and make use of the estimates (4.10) and (4.12) such that

$$\begin{aligned}&\frac{1}{4}\int_0^t\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2\\&\leq \sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^k\partial_x U)_{L^2}|_0^t + C\int_0^t\||x|^\mu D\partial_x U_2(\tau)\|_{L^2}^2 d\tau + C\||x|^\mu BU_0\|_{L^2}^2,\end{aligned}$$

from which and Lemma 2.1 we infer

$$\begin{aligned}&\int_0^t\||x|^\mu\partial_x U(\tau)\|_{L^2}^2 d\tau\\&\lesssim \||x|^\mu\partial_x U_0\|_{L^2}^2 + \eta^{\frac{1}{2}}\||x|^\mu U(t)\|_{L^2}^2 + \frac{1}{\eta^{\frac{1}{2}}}\||x|^\mu\partial_x U(t)\|_{L^2} + \int_0^t\||x|^\mu\partial_x U_2(\tau)\|_{L^2}^2 d\tau.\end{aligned}\tag{4.15}$$

Combining the estimates (4.10), (4.12) and (4.15) together and choosing η suitable small, we obtain (4.7). □

Proof of Theorem 2.4: Let the assumptions of Theorem 2.4 hold, and U be the solution to the Cauchy problem (2.1) subject to the initial data U_0. Then by virtue of Lemma 4.2, the $L^2(\mathbb{R})$-norm of $|x|^\mu U_1$ is uniformly bounded with respect to $t>0$. Therefore, we are able to employ Lemma (4.1) to show the time-decay estimate (2.12) which completes the proof of Theorem 2.4.

5. Faster time decay for partially dissipative hyperbolic systems: wave formulation method

The key ingredient for the proof of Theorem 2.5 is the introduction of the unknown

$$W(x,t) = \int_{-\infty}^x U_1(y,t)dy,\tag{5.1}$$

which satisfies the following damped wave formulation

$$\partial_t^2 W - A_{1,2}A_{2,1}\partial_x^2 W + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t\partial_x W + A_{1,2}DA_{1,2}^{-1}\partial_t W = 0.\tag{5.2}$$

Indeed, since $A_{1,1}=0$ in this section, integrating $(2.2)_1$ over $(-\infty,x)$, we obtain

$$\partial_t W + A_{1,2}U_2 = 0.\tag{5.3}$$

And after differentiating in time the above system and making use of $(2.2)_2$, we get

$$\partial_{tt}^2 W - A_{1,2}A_{2,1}\partial_x U_1 - A_{1,2}A_{2,2}\partial_x U_2 - A_{1,2}DU_2 = 0.\tag{5.4}$$

Combining (5.3)-(5.4) together, we have (5.2).

Note that $\partial_x W = U_1$ and $|\partial_t W| \sim |U_2|$, and therefore we are able to exhibit the $L^2(\mathbb{R})$-decay of U as long as we establish the decay estimates of the basic wave energy $\|(\partial_t W, \partial_x W)(t)\|_{L^2}^2$. Indeed, the solution W to the damped wave formulation (5.2) satisfies the following time-space weighted estimates.

Lemma 5.1. *Let W be defined by (5.1). Then under the assumptions of Theorem 2.5, for all $t > 0$, we have*

$$\begin{aligned}&\int_{\mathbb{R}} \big((1+t+|x|)^{2\mu-1}(|\partial_t W|^2 + |\partial_x W|^2) + (1+t+|x|)^{2\mu-2}|W|^2\big)dx \\ &\quad + \int_0^t \int_{\mathbb{R}} \big((1+t+|x|)^{2\mu-1}|\partial_x W|^2 + (1+t+|x|)^{2\mu-2}|\partial_x W|^2\big)dxd\tau \le CY_0^2,\end{aligned} \tag{5.5}$$

with $Y_0 := \|U_0\|_{H^1} + \||x|^{\mu}U_{1,0}\|_{L^2} + \||x|^{\mu-\frac{1}{2}}U_{2,0}\|_{L^2}$.

Proof. The proof is split into the cases $\mu = 1$ and $\frac{1}{2} < \mu < 1$ separately.

- **Case 1:** $\mu = 1$.

We are going to estimate (5.2) with the weight $(1+t+|x|)$. Multiplying (5.2) with $(1+t+|x|)\partial_t W$ and integrating it by parts, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(1+t+|x|)(|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W \partial_x W)dx \\ &\quad + \int_{\mathbb{R}} \big((1+t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W \partial_t W - \frac{1}{2}A_{1,2}A_{2,1}\partial_x W \partial_x W\big)dx \\ &= \int_{\mathbb{R}} \big(\frac{1}{2}|\partial_t W|^2 - \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W \partial_t W + \frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W \partial_t W\big)dx,\end{aligned} \tag{5.6}$$

where we used

$$\begin{aligned}&\int_{\mathbb{R}} (1+t+|x|)(W\partial_t W - A_{1,2}A_{2,1}\partial_x^2 W \partial_t W)dx \\ &\quad = \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(1+t+|x|)(|W|^2 + A_{1,2}A_{2,1}\partial_x W \partial_x W)dx \\ &\qquad - \frac{1}{2}\int_{\mathbb{R}} (|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W \partial_x W)dx + \int_{\mathbb{R}} \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W \partial_t W dx,\end{aligned}$$

and

$$\int_{\mathbb{R}} (1+t+|x|)A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t \partial_x W \partial_t W dx = -\frac{1}{2}\int_{\mathbb{R}} \frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W \partial_t W dx.$$

In addition, taking the inner product of (5.2) with W, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} \big(W\partial_t W + \frac{1}{2}A_{1,2}DA_{1,2}^{-1}|W|^2\big)dx + \int_{\mathbb{R}} A_{1,2}A_{2,1}\partial_x W \partial_x W dx \\ &= \int_{\mathbb{R}} \big(|\partial_t W|^2 + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W \partial_x W\big)dx.\end{aligned} \tag{5.7}$$

We now define

$$\begin{aligned}\mathcal{W}(t) : &= \int_{\mathbb{R}} \frac{1}{2}(1+t+|x|)(|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W \partial_x W)dx \\ &\quad + \int_{\mathbb{R}} \big(W\partial_t W + \frac{1}{2}A_{1,2}DA_{1,2}^{-1}|W|^2\big)dx, \\ \mathcal{H}(t) : &= \int_{\mathbb{R}} (1+t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W \partial_t W dx + \int_{\mathbb{R}} \frac{1}{2}A_{1,2}A_{2,1}\partial_x W \partial_x W dx.\end{aligned}$$

It thence follows from (5.6)-(5.7) that

$$\begin{aligned}&\frac{d}{dt}\mathcal{W}(t)+\mathcal{H}(t)\\&=\int_{\mathbb{R}}\big(\frac{3}{2}|\partial_t W|^2-\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W+\frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W+A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_x W\big)dx.\end{aligned}$$

Notice that $A_{1,2}DA_{1,2}^{-1}$ satisfies (1.3) since D is positive definite and the eigenvalues of $A_{1,2}DA_{1,2}^{-1}$ and D are the same. By the strong dissipation conditions (1.3) and (2.13), we have

$$\begin{aligned}\mathcal{W}(t)&\geq\int_{\mathbb{R}}\big((1+t+|x|)(\frac{1}{2}|\partial_t W|^2+\kappa_1|\partial_x W|^2)+\kappa|W|^2-C|\partial_t W|^2\big)dx,\\\mathcal{W}(t)&\leq\int_{\mathbb{R}}\big((1+t+|x|)(\frac{1}{2}|\partial_t W|^2+C|\partial_x W|^2)+C|W|^2+C|\partial_t W|^2\big)dx,\end{aligned}$$

and

$$\mathcal{H}(t)\geq\int_0^t\int_{\mathbb{R}}\big(\kappa(1+t+|x|)|\partial_t W|^2+\frac{\kappa_1}{2}|\partial_x W|^2\big)dxd\tau.$$

Since $|\partial_t W|\sim|U_2|$ due to (5.3) and the fact that $A_{1,2}$ is invertible, one also gains

$$\begin{aligned}&\int_{\mathbb{R}}\big(\frac{3}{2}|\partial_t W|^2-\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W+\frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W+A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_x W\big)dx\\&\quad\leq\frac{\kappa_1}{4}\int_{\mathbb{R}}|\partial_x W|^2dx+C\int_{\mathbb{R}}|U_2|^2dx.\end{aligned}$$

It therefore follows that

$$\begin{aligned}&\int_{\mathbb{R}}\big((1+t+|x|)(\frac{1}{2}|\partial_t W|^2+\kappa_1|\partial_x W|^2)+\kappa|W|^2\big)dx\\&\quad+\int_0^t\int_{\mathbb{R}}\big(\kappa(1+t+|x|)|\partial_t W|^2+\frac{\kappa_1}{4}|\partial_x W|^2\big)dxd\tau\\&\leq\mathcal{W}(0)+C\int_{\mathbb{R}}|\partial_t W|^2dx+C\int_0^t\int_{\mathbb{R}}|\partial_t W|^2dxd\tau.\end{aligned}\tag{5.8}$$

Under the assumptions (2.15)-(2.16), one deduces from the Caffarelli-Kohn-Nirenberg inequality (9.6) that

$$\mathcal{W}(0)\lesssim\int_{\mathbb{R}}\big((1+|x|)(|(u_0,v_0)|^2+|\int_{-\infty}^x u_0(y)dy|^2\big)dx\lesssim Y_0^2.\tag{5.9}$$

It also holds by $|\partial_t W|\sim|U_2|$ and (2.10) that

$$C\int_{\mathbb{R}}|\partial_t W|^2dx+C\int_0^t\int_{\mathbb{R}}|\partial_t W|^2dxd\tau\lesssim\int_{\mathbb{R}}|U_2|^2dx+C\int_0^t\int_{\mathbb{R}}|U_2|^2dxd\tau\lesssim Y_0^2.\tag{5.10}$$

Inserting (5.9)-(5.10) into (5.8), we get (5.5).

- **Case 2:** $\frac{1}{2}<\mu<1$.

In this case, let $\varphi(s):\mathbb{R}^+\to\mathbb{R}^+$ be a weight function to be determined later. Similarly to the case $\mu=1$, multiplying (5.2) with $\varphi(t+|x|)\partial_t W$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}\varphi(t+|x|)(|\partial_t W|^2+A_{1,2}A_{2,1}\partial_x W\partial_x W)dx\\&\quad+\int_{\mathbb{R}}\big(\varphi(t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W\partial_t W-\frac{1}{2}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W\big)dx\\&=\int_{\mathbb{R}}\varphi'(t+|x|)\big(\frac{1}{2}|\partial_t W|^2+\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W+\frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)dx.\end{aligned}$$

After taking the inner product of (5.2) with $\varphi'(t+|x|)W$, we verify that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\big(\varphi'(t+|x|)\partial_t WW-\varphi''(t+|x|)|W|^2+\frac{1}{2}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}WW\big)dx\\
&\quad+\int_{\mathbb{R}}\big(\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W+\varphi'''(t+|x|)|W|^2\big)dx\\
&\quad-\int_{\mathbb{R}}\big(\varphi'''(t+|x|)+\varphi''(t+|x|)\delta_0(x)\big)A_{1,2}A_{2,1}WWdx\\
&=\int_{\mathbb{R}}\varphi'(t+|x|)|\partial_t W|^2dx.
\end{aligned}$$

Here we have used

$$\begin{aligned}
&\int_{\mathbb{R}}\varphi'(t+|x|)\partial_t^2 WWdx\\
&=\frac{d}{dt}\int_{\mathbb{R}}\varphi'(t+|x|)\partial_t WWdx-\int_{\mathbb{R}}\varphi''(t+|x|)W\partial_t Wdx-\int_{\mathbb{R}}\varphi'(t+|x|)|\partial_t W|^2dx\\
&=\frac{d}{dt}\int_{\mathbb{R}}\big(\varphi'(t+|x|)\partial_t WW-\varphi''(t+|x|)|W|^2\big)dx\\
&\quad+\int_{\mathbb{R}}\varphi'''(t+|x|)|W|^2dx-\int_{\mathbb{R}}\varphi'(t+|x|)|\partial_t W|^2dx,\\
&\int_{\mathbb{R}}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t WWdx\\
&=\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}WWdx-\int_{\mathbb{R}}\varphi''(t+|x|)A_{1,2}DA_{1,2}^{-1}WWdx,
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x^2 WWdx\\
&\quad=\int_{\mathbb{R}}\varphi''(t+|x|)\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x WWdx+\int_{\mathbb{R}}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x Wdx\\
&\quad=-\int_{\mathbb{R}}\big(\varphi'''(t+|x|)+\varphi''(t+|x|)\delta_0(x)\big)A_{1,2}A_{2,1}WWdx\\
&\qquad+\int_{\mathbb{R}}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x Wdx,
\end{aligned}$$

where $\delta_0(x)$ denotes the Dirac function at 0. Therefore, we get the following Lyapunov inequality

$$\begin{aligned}
&\frac{d}{dt}\mathcal{W}_\mu(t)+\mathcal{H}_\mu(t)\\
&=\int_{\mathbb{R}}\varphi'(t+|x|)\big(\frac{3}{2}|\partial_t W|^2+\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W+\frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)dx,
\end{aligned}\tag{5.11}$$

with

$$\begin{aligned}
\mathcal{W}_\mu(t):=&\int_{\mathbb{R}}\frac{1}{2}\varphi(t+|x|)(|\partial_t W|^2+A_{1,2}A_{2,1}\partial_x W\partial_x W)dx\\
&+\int_{\mathbb{R}}\big(\varphi'(t+|x|)\partial_t WW-\varphi''(t+|x|)|W|^2+\frac{1}{2}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}WW\big)dx\\
\mathcal{H}_\mu(t):=&\int_{\mathbb{R}}\big(\varphi(t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W\partial_t W+\frac{1}{2}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W\big)dx,\\
&+\int_{\mathbb{R}}\big(\varphi'''(t+|x|)|W|^2-(\varphi'''(t+|x|)+\varphi''(t+|x|)\delta_0(x))A_{1,2}A_{2,1}WW\big)dx.
\end{aligned}$$

In order the bound $\mathcal{W}_\mu(t)$ and $\mathcal{H}_\mu(t)$, one requires that $\varphi(s)$ satisfies

$$\varphi'\geq 0,\qquad \varphi''\leq 0,\qquad \varphi'''\geq 0,\qquad \frac{1}{4}\varphi(t+|x|)\geq\frac{1}{\kappa_1}\varphi'(t+|x|).\tag{5.12}$$

Indeed, due to the strong dissipation conditions (1.3) and (2.13) and the properties (5.12), there holds that

$$\mathcal{W}_\mu(t) \geq \int_{\mathbb{R}} \big(\frac{1}{4}\varphi(t+|x|)|\partial_t W|^2 + \kappa\varphi(t+|x|)|\partial_x W|^2 + (\frac{\kappa_1}{4}\varphi'(t+|x|) - \varphi''(t+|x|))|W|^2\big)dx,$$
$$\mathcal{W}_\mu(t) \leq \int_{\mathbb{R}} \big(\frac{3}{4}\varphi(t+|x|)|\partial_t W|^2 + \kappa\varphi(t+|x|)|\partial_x W|^2 + (\frac{3\kappa_1}{4}\varphi'(t+|x|) - \varphi''(t+|x|))|W|^2\big)dx,$$

and

$$\begin{aligned}\mathcal{H}_\mu(t) &\geq \int_{\mathbb{R}} \big(\kappa\varphi(t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{2}\varphi'(t+|x|)|\partial_x W|^2\big)dx\\ &\quad + \int_{\mathbb{R}} \varphi'''(t+|x|)(|W|^2 - A_{1,2}A_{2,1}WW)dx - \kappa_1\varphi''(t)|W(0,t)|^2\\ &\geq \int_{\mathbb{R}} \big(\kappa\varphi(t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{2}\varphi'(t+|x|)|\partial_x W|^2\big)dx,\end{aligned}$$

where we have used the condition (2.14). In addition, one has

$$\begin{aligned}&\int_{\mathbb{R}} \varphi'(t+|x|)\big(\frac{3}{2}|\partial_t W|^2 + \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W + \frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)dx\\ &\quad\leq \int_{\mathbb{R}} \big(C\varphi'(t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{4}\varphi'(t+|x|)|\partial_x W|^2\big)dx.\end{aligned}$$

which can be controlled by the side of (5.11) provided that we impose

$$C\varphi'(1+t+|x|) \leq \frac{\kappa}{2}\varphi(t+|x|). \tag{5.13}$$

In addition, under the assumptions (2.15)-(2.16), one needs $\mu > \frac{1}{2}$ and

$$\varphi(s) \sim (1+s)^{2\mu-1}, \qquad \varphi'(s) \sim (1+s)^{2\mu-2}, \tag{5.14}$$

so as to bound the initial energy $\mathcal{W}(0)$ by Y_0^2 in terms of (2.15) and the Caffarelli-Kohn-Nirenberg inequality (9.6). One can show that the function

$$\varphi(s) = (a+s)^{2\mu-1} \quad \text{with } \frac{1}{2} < \mu < 1 \text{ and some constant } a > \max\{\frac{4}{\kappa_1}, \frac{2C}{\kappa}\} \tag{5.15}$$

fulfills the conditions (5.12) and (5.13). Therefore, integrating (5.11) over $[0,t]$, we have (5.5).

□

Proof of Theorem 2.5: Let the assumptions of Theorem 2.5 hold, and U be the solution to the Cauchy problem (2.1) subject to the initial data U_0. Multiplying the Lyapunov inequality (3.11) by $t^{2\mu-1}$ for $\frac{1}{2} < \mu \leq 1$, we obtain

$$\begin{aligned}&\frac{d}{dt}(t^{2\mu-1}\mathcal{L}(t)) + \frac{3\kappa}{2}t^{2\mu-1}\|U_2(t)\|_{L^2}^2 + \kappa t^{2\mu-1}(\frac{1}{2}+ct)\|\partial_x U_2(t)\|_{L^2}^2 + \frac{\varepsilon_*}{2C_K}t^{2\mu-1}\|\partial_x U(t)\|_{L^2}^2\\ &\leq (2\mu-1)t^{2\mu-2}\mathcal{L}(t).\end{aligned}$$

This together with $t^{2\mu-1}\mathcal{L}(t)|_{t=0} = 0$ for $\mu > \frac{1}{2}$ gives rise to

$$\begin{aligned}&t^{2\mu-1}\|U(t)\|_{L^2}^2 + (t^{2\mu-1} + ct^{2\mu-2})\|\partial_x U(t)\|_{L^2}^2\\ &\quad + \int_0^t \big(\tau^{2\mu-1}\|U_2(\tau)\|_{L^2}^2 + (\tau^{2\mu-1} + c\tau^{2\mu})\|\partial_x U_2(\tau)\|_{L^2}^2 + \tau^{2\mu-1}\|\partial_x U(\tau)\|_{L^2}^2\big)d\tau\\ &\lesssim \int_0^t \big(\tau^{2\mu-2}\|U(\tau)\|_{L^2}^2 + (\tau^{2\mu-2} + c\tau^{2\mu-1})\|\partial_x U(\tau)\|_{L^2}^2\big)d\tau.\end{aligned} \tag{5.16}$$

For $\mu = 1$, by (2.10), one has

$$\int_0^t \tau^{2\mu-2}\|\partial_x U(\tau)\|_{L^2}^2 d\tau = \int_0^t \|\partial_x U(\tau)\|_{L^2}^2 d\tau \lesssim Y_0^2. \tag{5.17}$$

In the case $\frac{1}{2} < \mu < 1$, it also holds by (2.10) that

$$\begin{aligned}&\int_0^t \tau^{2\mu-2}\|U_2(\tau)\|_{L^2}^2 d\tau \\ &\lesssim Y_0^2 \int_0^t \tau^{2\mu-2}(1+\tau)^{-1} d\tau \\ &\lesssim Y_0^2 \int_0^1 \tau^{2\mu-1} d\tau + Y_0^2 \int_1^t (1+\tau)^{2\mu-3} d\tau \lesssim Y_0^2.\end{aligned} \tag{5.18}$$

In addition, it follows from the estimates (5.5) obtained in Lemma 5.1 and $\partial_x W = U_1$ that

$$\int_0^t \int_{\mathbb{R}} \tau^{2\mu-2}|U|^2 dx d\tau \lesssim Y_0^2. \tag{5.19}$$

Substituting (5.17)-(5.19) into (5.16) and noticing the fact that the constant c is suitable small, we derive

$$\|U(t)\|_{L^2} \lesssim Y_0(1+t)^{-\mu+\frac{1}{2}}, \qquad \|\partial_x U(t)\|_{L^2} \lesssim Y_0(1+t)^{-\mu}. \tag{5.20}$$

Finally, we apply (3.14) and (5.20) to infer

$$\|U_2(t)\|_{L^2} \lesssim e^{-\kappa t}\|U_{2,0}\|_{L^2} + Y_0 \int_0^t e^{-\kappa(t-\tau)}\|\partial_x U(\tau)\|_{L^2} dx \lesssim (1+t)^{-\mu} Y_0.$$

The proof of Theorem 2.5 is complete.

6. Time decay for the p-system with nonlinear damping

In this section, we establish the logarithmic time-decay rates of the nonlinearly damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + |v|^{r-1}v = 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ (u,v)(x,0) = (u_0, v_0)(x), \qquad x \in \mathbb{R}, \end{cases} \tag{6.1}$$

with some constant $1 < r < 3$.

6.1. Dissipation estimates.

- Basic L^2-estimates:

First of all, the L^2 norm of the solution is conserved as

$$\frac{d}{dt}\|(u,v)(t)\|_{L^2}^2 + 2\|v(t)\|_{L^{r+1}}^{r+1} = 0. \tag{6.2}$$

This gives

$$\|(u,v)(t)\|_{L^2}^2 + 2\int_0^t \|v(\tau)\|_{L^{r+1}}^{r+1} d\tau = \|(u_0,v_0)\|_{L^2}^2. \tag{6.3}$$

- Basic H^1-estimates

From direct energy estimates in (6.1), we get

$$\frac{d}{dt}\|(\partial_x u, \partial_x v)(t)\|_{L^2}^2 + 2\int_{\mathbb{R}} \partial_x(|v|^{r-1}v)\partial_x v = 0. \tag{6.4}$$

One has

$$\int_{\mathbb{R}} \partial_x(|v|^{r-1}v)\partial_x v dx = \int_{\mathbb{R}} \partial_x|v|^{r-1} v \partial_x v dx + \int_{\mathbb{R}} |v|^{r-1}(\partial_x v)^2 dx = r\int_{\mathbb{R}} |v|(\partial_x v)^2 dx,$$

- Dissipation of $\partial_x u$ with weights.

Multiplying $(6.1)_2$ with $|v|^{r-1}\partial_x u$, we infer

$$\int_{\mathbb{R}} \partial_t v|v|^{r-1}\partial_x u dx + \int_{\mathbb{R}} |v|^{r-1}|\partial_x u|^2 dx - \int_{\mathbb{R}} |v|^{2r-2} v\partial_x u dx = 0. \tag{6.5}$$

From $(6.1)_1$, one also has

$$\int_{\mathbb{R}} \frac{1}{r}|v|^{r-1} v\partial_x\partial_t u dx + \int_{\mathbb{R}} \frac{1}{r}|v|^{r-1} v\partial_x^2 v dx = 0. \tag{6.6}$$

By (6.5)-(6.6), $\partial_t v|v|^{r-1} = \frac{1}{r}\partial_t(|v|^{r-1}v)$ and integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{r}|v|^{r-1} v\partial_x u dx + \int_{\mathbb{R}} |v|^{r-1}|\partial_x u|^2 dx - \int_{\mathbb{R}} |v|^{2r-2} v\partial_x u dx - \int_{\mathbb{R}} |v|^{r-1}|\partial_x v|^2 dx = 0. \tag{6.7}$$

Defining

$$\begin{aligned}
\mathcal{W}_*(t) &:= \|(u, v, \partial_x u, \partial_x v)(t)\|_{L^2}^2 + \varepsilon \int \frac{1}{r}|v|^{r-1} v\partial_x u dx,\\
\mathcal{H}_*(t) &:= 2\int_0^t \int_{\mathbb{R}} |v|(|v|^2 + |\partial_x v|^2) dx d\tau\\
&\quad + \varepsilon\Big(\int_{\mathbb{R}} |v|^{r-1}|\partial_x u|^2 dx - \int_{\mathbb{R}} |v|^{2r-2} v\partial_x u dx - \int_{\mathbb{R}} |v|^{r-1}|\partial_x v|^2 dx\Big),
\end{aligned}$$

we obtain from (6.3), (6.4) and (6.7) that

$$\frac{d}{dt}\mathcal{W}_*(t) + \mathcal{H}_*(t) = 0. \tag{6.8}$$

Since due to (6.3) and (6.4),

$$\|v\|_{L^\infty_{t,x}} \lesssim \|v\|_{L^\infty_t(L^2)}^{\frac{1}{2}} \|\partial_x v\|_{L^\infty_{t,x}}^{\frac{1}{2}} \lesssim 1,$$

we are able to choose a suitable small constant $\varepsilon > 0$ such that

$$\mathcal{W}_*(t) \sim \|(u, v, \partial_x u, \partial_x v)(t)\|_{L^2}^2, \qquad \mathcal{H}_*(t) \gtrsim \int_{\mathbb{R}} |v|^{r-1}(|v|^2 + |\partial_x v|^2 + |\partial_x u|^2) dx_1. \tag{6.9}$$

Integrating (6.8) over $[0, t]$ and making use of (6.9), we have

$$\|(u, v)(t)\|_{H^1}^2 + \int_0^t (\|v(\tau)\|_{L^{r+1}}^{r+1} + \|(\partial_x v^{\frac{r+1}{2}}, \partial_x u^{\frac{r+1}{2}})(\tau)\|_{L^2}^2) d\tau \lesssim \|(u_0, v_0)\|_{H^1}^2. \tag{6.10}$$

6.2. **Wave formulation.** Differentiating (6.1) with respect to t, we rewrite (6.1) by two non-classical damped wave equations

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + r|v|^{r-1}\partial_t u = 0,\\ \partial_t^2 v - \partial_x^2 v + r|v|^{r-1}\partial_t v = 0. \end{cases} \tag{6.11}$$

Then it follows by the equation $(6.1)_1$ that

$$v = -\partial_t \int_{-\infty}^x u(y, t) dy, \tag{6.12}$$

from which one has the following fact

$$r|v|^{r-1}\partial_t u = r\big|\partial_t \int_{-\infty}^x u(y,t)dy\big|^{r-1}\partial_x\partial_t \int_{-\infty}^x u(y,t)dy = \partial_x\big(\big|\partial_t \int_{-\infty}^x u(y,t)dy\big|^{r-1}\partial_t \int_{-\infty}^x u(y,t)dy\big),$$

Thus, integrating the equation $(6.11)_1$ over $(-\infty, x)$, we obtain the classical wave equation with nonlinear damping:

$$\partial_t^2 w - \partial_x^2 w + |\partial_t w|^r \partial_t w = 0, \tag{6.13}$$

where the new unknown w is defined by

$$w := \int_{-\infty}^x u(y, t) dy.$$

The basic energy of (6.13) is conserved as follows:

$$\frac{1}{2}\frac{d}{dt}\|(\partial_t w, \partial_x w)(t)\|_{L^2}^2 + \|\partial_t w(t)\|_{L^{r+1}}^{r+1} = 0.$$

From $(6.1)_1$, it is easy to see

$$\|\partial_t w(t)\|_{L^2}^2 = \|v(t)\|_{L^2}^2, \qquad \|\partial_x w(t)\|_{L^2}^2 = \|u(t)\|_{L^2}^2. \tag{6.14}$$

Therefore, once we get the decay rate of $\|(\partial_t w, \partial_x w)(t)\|_{L^2}^2$, then the $L^2(\mathbb{R})$-decay of (u, v) follows.

6.3. **Time-decay rate.** In this section, we adapt the method developed by Mochizuki and Motai [20] to show the logarithmic time-decay rate of energy to the wave equation (6.13).

Let the weight functions $\varphi_1(s), \varphi_2(s)$ for $s \geq 0$ to be determined later. We first take the inner product of (6.13) with $\varphi_1(t+|x|)\partial_t w$ and obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)dx \\ &\quad + \int_{\mathbb{R}} \varphi_1(t+|x|)|\partial_t w|^r dx - \int_{\mathbb{R}} \frac{1}{2}\varphi_1'(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)dx = 0. \end{aligned} \tag{6.15}$$

In addition, multiplying (6.13) with $\varphi_1'(t+|x|)w$ and intergrating by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} (\varphi_1'(t+|x|)w\partial_t w - \frac{1}{2}\varphi_1''(t+|x|)|w|^2)dx + \int_{\mathbb{R}} \varphi_1'(t+|x|)(|\partial_x w|^2 - |\partial_t w|^2)dx \\ &\quad + \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^{r-1}\partial_t w w dx - \frac{1}{2}\varphi_1''(t)|w(0,t)|^2 = 0. \end{aligned} \tag{6.16}$$

Here we have used the facts

$$\begin{aligned} &\int_{\mathbb{R}} \varphi_1'(t+|x|)w\partial_t^2 w dx \\ &= \frac{d}{dt}\int_{\mathbb{R}} \varphi_1'(t+|x|)w\partial_t w dx - \int_{\mathbb{R}} \varphi_1''(t+|x|)w\partial_t w dx - \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^2 dx \\ &= \frac{d}{dt}\int_{\mathbb{R}} \varphi_1'(t+|x|)w\partial_t w dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \varphi_1''(t+|x|)|w|^2 dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \varphi_1'''(t+|x|)|w|^2 dx - \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^2 dx, \end{aligned}$$

and

$$\begin{aligned} &-\int_{\mathbb{R}} \varphi_1'(t+|x|)w\partial_x^2 w dx \\ &= \int_{\mathbb{R}} \varphi_1''(t+|x|)\frac{x}{|x|}w\partial_x w dx + \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_x w|^2 dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} \varphi_1'''(t+|x|)\left|\frac{x}{|x|}\right|^2|w|^2 dx - \int_{\mathbb{R}} \varphi_1''(t+|x|)\delta_0(x)|w|^2 dx + \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_x w|^2 dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} \varphi_1'''(t+|x|)|w|^2 dx - \varphi_1''(t)|w(0,t)|^2 + \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_x w|^2 dx, \end{aligned}$$

where $\delta_0(x)$ denotes the Dirac function at 0. In addition, we have

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi_2(t+|x|)|w|^{r+1}dx - \int_{\mathbb{R}} \varphi_2'(t+|x|)|w|^{r+1}dx + (r+1)\int_{\mathbb{R}} \varphi_2(t+|x|)|w|^{r-1}w\partial_t w dx = 0. \tag{6.17}$$

For some small constant $\eta > 0$ to be chosen later, define

$$\begin{aligned} \mathcal{L}_\varphi(t) := &\int_{\mathbb{R}} \frac{1}{2}\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)dx \\ &+ \eta\int_{\mathbb{R}} (\varphi_1'(t+|x|)w\partial_t w - \frac{1}{2}\varphi_1''(t+|x|)|w|^2 + \varphi_2(t+|x|)|w|^{r+1}dx)dx. \end{aligned}$$

Thus, from (6.15), (6.16) and (6.17), we get the following Lyapunov inequality

$$
\begin{aligned}
&\mathcal{L}_\varphi(t) + \int_0^t\int_{\mathbb{R}} \big(\varphi_1(\tau+|x|)|\partial_t w|^r + \frac{\eta}{2}\varphi_1'(\tau+|x|)|\partial_x w|^2 - \eta\varphi_2'(\tau+|x|)|w|^{r+1}\big)dx\\
&\quad - \eta\int_0^t \varphi_1''(\tau)|w(0,\tau)|^2 d\tau\\
&= \mathcal{L}_\varphi(0) + \frac{3}{2}\int_0^t\int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^2 dxd\tau - \int_0^t\int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^{r-1}\partial_t w w dxd\tau\\
&\quad - (r+1)\int_0^t\int_{\mathbb{R}} \varphi_2(\tau+|x|)|w|^{r-1}w\partial_t w dxd\tau.
\end{aligned} \tag{6.18}
$$

In order to control $\mathcal{L}_\varphi(t)$ and derive the desired dissipation estimates, we require

$$
\varphi_1>0,\qquad \varphi_1'>0,\qquad \varphi_1''<0,\qquad |\varphi_1'|^2\le C\varphi_1|\varphi_1''|,\qquad \varphi_2>0,\qquad \varphi_2'<0. \tag{6.19}
$$

Indeed, under the condition (6.19), one has

$$
\int_{\mathbb{R}} \varphi_1'(t+|x|)|w\partial_t w|dx| \le C\int_{\mathbb{R}} \varphi_1(t+|x|)|\partial_t w|^2 dx - \frac{1}{4}\int_{\mathbb{R}} \varphi_1''(t+|x|)|w|^2 dx, \tag{6.20}
$$

which implies

$$
\mathcal{L}_\varphi(t) \ge \int_{\mathbb{R}} (\frac{1}{2} - C\eta)\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)dx - \frac{1}{4}\varphi_1''(t+|x|)|w|^2 + \varphi_2(t+|x|)|w|^{r+1}dx\big)dx. \tag{6.21}
$$

We estimate the nonlinear terms on the right-hand side of (6.18) as follows. First, a use of Young's inequality leads to

$$
\begin{aligned}
&\frac{3}{2}\int_0^t\int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^2 dxd\tau\\
&\quad\le \frac{3}{2}\Big(\int_0^t\int_{\mathbb{R}} \varphi_1(t+|x|)|\partial_t w|^{r+1}dxd\tau\Big)^{\frac{2}{r+1}}\Big(\int_0^t\int_{\mathbb{R}} \frac{|\varphi_1'(t+|x|)|^{\frac{r+1}{r-1}}}{\varphi_1(t+|x|)^{\frac{2}{r-1}}}dxd\tau\Big)^{\frac{r-1}{r+1}}\\
&\quad\le \frac{1}{4}\int_0^t\int_{\mathbb{R}} \varphi_1(\tau+|x|)|\partial_t w|^{r+1}dxd\tau + C\int_0^t\int_{\mathbb{R}} \frac{|\varphi_1'(t+|x|)|^{\frac{r+1}{r-1}}}{\varphi_1(t+|x|)^{\frac{2}{r-1}}}dxd\tau.
\end{aligned} \tag{6.22}
$$

Similarly, we infer

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} \varphi_1'(\tau+|x|)|\partial_t w|^r|w|dxd\tau\\
&\quad\le \frac{1}{4}\int_0^t\int_{\mathbb{R}} \varphi_1(\tau+|x|)|\partial_t w|^{r+1}dxd\tau + C\int_0^t\int_{\mathbb{R}} \frac{|\varphi_1'(t+|x|)|^{r+1}}{\varphi_1(\tau+|x|)^r}|w|^{r+1}dxd\tau.
\end{aligned} \tag{6.23}
$$

and

$$
\begin{aligned}
&\int_0^t\int_R \varphi_2(\tau+|x|)|w|^r|\partial_t w|dxd\tau\\
&\quad\le \frac{1}{4}\int_0^t\int_{\mathbb{R}} \varphi_1(\tau+|x|)|\partial_t w|^{r+1}dxd\tau + C\int_0^t\int_{\mathbb{R}} \frac{|\varphi_2(t+|x|)|^{r+1}}{\varphi_1(\tau+|x|)^r}|w|^{r+1}dxd\tau.
\end{aligned} \tag{6.24}
$$

Then it follows from (6.20),(6.21), (6.22), (6.23) and (6.24) that

$$
\begin{aligned}
&\int_{\mathbb{R}} (\frac{1}{4}\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2 - \frac{1}{4}\varphi_1''(t+|x|)|w|^2 + \varphi_2(t+|x|)|w|^{r+1}dx\big)dx\\
&\quad + \int_0^t\int_{\mathbb{R}} (\frac{1}{2}\varphi_1(\tau+|x|)|\partial_t w|^r + \frac{1}{2}\varphi_1'(\tau+|x|)|\partial_x w|^2 + \mathcal{C}_1(t+|x|)|w|^{r+1}\big)dxd\tau\\
&\le \mathcal{L}_\varphi(0) + C\int_0^t\int_{\mathbb{R}} \mathcal{C}_2(t+|x|)dxd\tau.
\end{aligned} \tag{6.25}
$$

with

$$\mathcal{C}_1(s) := -\varphi_2'(s) - \frac{C(|\varphi_1'(s)|^{r+1} + |\varphi_2(s)|^{r+1})}{\varphi_1(s)^r}, \qquad \mathcal{C}_2(s) := \frac{|\varphi_1'(s)|^{\frac{r+1}{r-1}}}{\varphi_1(s)^{\frac{2}{r-1}}}.$$

Therefore, one needs to choose φ', φ'' such that

$$\mathcal{C}_1(s) > 0, \qquad \int_0^\infty \int_{\mathbb{R}} \mathcal{C}_2(t + |x|) dx d\tau < \infty. \tag{6.26}$$

We choose the functions

$$\varphi_1(s) = \log^q(a+s), \qquad \varphi_2(s) = \frac{\log^{q-r+1}(a+s)}{|a+s|^r} \quad \text{for all } q > 0 \text{ and suitable large constant } a,$$

which fulfills the conditions (6.19) and (6.26). Indeed, for suitable large $a > 0$, it easy to verify that

$$\varphi_1'(s) = \frac{q \log^{q-1}(a+s)}{a+s} > 0, \quad \varphi_1''(s) = -\frac{q\log^{q-2}(a+s)(\log(a+s) - 1 + q)}{|a+s|^2} < 0,$$

$$|\varphi_1'|^2 \le \frac{1}{4}\varphi_1|\varphi_1''|, \qquad \varphi_2'(s) = -\frac{\log(a+s)^{q-r}(r\log(a+s) - q + r - 1)}{|a+s|^{r+1}} < 0,$$

and

$$\begin{aligned}
\mathcal{C}_1(s) &= \frac{\log(a+s)^{q-r}(r\log(a+s) - q + r - 1) - Cq^{r+1}\log^{q-r-1}(a+s)}{|a+s|^{r+1}} - \frac{C\log^{q-r^2+1}(a+s)}{|a+s|^{r+r^2}} \\
&\ge \frac{\log(a+s)^{q-r}\big(r\log(a) - q + r - 1 - Cq^{r+1}\log^{-1}(a) - C\log^{r-r^2+1}(a)\big)a^{-r^2+1}}{|a+s|^{r+1}} > 0.
\end{aligned}$$

For the second term, one has

$$\int_0^\infty \int_{\mathbb{R}} \mathcal{C}_2(t+|x|)dxd\tau = \int_0^\infty \int_{\mathbb{R}} \frac{\log^{q-\frac{r+1}{r-1}}(a+s)}{|a+s|^{\frac{r+1}{r-1}}} dxd\tau < \infty,$$

due to $\frac{r+1}{r-1} > 2$, i.e., $1 < r < 3$. To bound the initial energy $\mathcal{L}_\varphi(0)$, we notice that

$$\begin{aligned}
\mathcal{L}_\varphi(0) &\le C\int_{\mathbb{R}} \log^q(1+|x|)(|v_0|^2 + |u_0|^2)dx \\
&\quad + C\int_{\mathbb{R}} \frac{\log^{q-2}(1+|x|)}{(1+|x|)^2}dx\|u_0\|_{L^1}^2 + C\int_{\mathbb{R}} \frac{\log^{q-r-1}(1+|x|)}{(1+|x|)^r}dx\|u_0\|_{L^1}^{r+1} \le C.
\end{aligned}$$

Therefore, we substitute $\varphi_1(s) = \log(a+s)^q$ into (6.25) to have

$$\|(u,v)(t)\|_{L^2} = \|(\partial_t w, \partial_x w)(t)\|_{L^2} \le \frac{C}{\log^{\frac{q}{2}}(1+t)}, \qquad q > 0, \qquad 1 < r < 3. \tag{6.27}$$

7. Time decay for the damped compressible Euler equations

7.1. **Asymptotics for the damped compressible Euler equations.** In this subsection, we apply the methods developed in Theorems 2.4 and 2.5 to some concrete nonlinear partially dissipative hyperbolic equations: the damped compressible Euler equations (1.4) with the pressure function $P(\rho)$ satisfying

$$P(\rho) \in C^\infty(\mathbb{R}), \qquad P'(\rho) > 0. \tag{7.1}$$

For example, one can choose the γ-law $P(\rho) = \rho^\gamma$ where the adiabatic exponent $\gamma > 1$ corresponds to the isentropic flow and $\gamma = 1$ corresponds to the isothermal flow.

We now establish the long time behavior of System (1.4) as follows.

Theorem 7.1. *Let (7.1) hold, and $\bar\rho > 0$ be a given constant. Assume that the initial data (ρ_0, v_0) fulfills $(\rho_0 - \bar\rho, v_0) \in H^2(\mathbb{R})$ and*

$$\|(\rho_0 - \bar\rho, v_0)\|_{H^2} \leq \delta_0, \tag{7.2}$$

where $\delta_0 > 0$ is a suitable small constant. Then the System (1.4) admits a unique global solution (ρ, v) which satisfies $(\rho - \bar\rho, v) \in C(\mathbb{R}_+; H^2(\mathbb{R}))$ and

$$\|(\rho - \bar\rho, v)(t)\|_{H^2}^2 + \int_0^t (\|\partial_x(\rho - \bar\rho)(\tau)\|_{H^1}^2 + \|v(\tau)\|_{H^2}^2) d\tau \leq C\|(\rho_0 - \bar\rho, v_0)\|_{H^2}^2, \tag{7.3}$$

and

$$\|v(t)\|_{L^2} + \|\partial_x(\rho - \bar\rho, v)(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}, \tag{7.4}$$

for all $t > 0$, where $C > 0$ is a universal constant.

Furthermore, for $\frac{1}{2} \leq \mu \leq 1$, the following statements hold:

- $(\mu = 1)$. *Assume, in addition to (7.2), that $|x|(\rho_0 - \bar\rho) \in L^2(\mathbb{R})$, $|x|^{\frac{1}{2}} v_0 \in L^2(\mathbb{R})$ and $\partial_t \rho|_{t=0} = -\partial_x(\rho_0 v_0)$, then for all $t > 0$, the solution (ρ, v) satisfies*

$$\begin{cases} \|(\rho - \bar\rho)(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}, \\ \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar\rho, v)(t)\|_{L^2} \leq C(1+t)^{-1}. \end{cases} \tag{7.5}$$

- $(\frac{1}{2} < \mu < 1)$. *In addition to (7.2), assume $P'(\bar\rho) \leq 1$, $|x|^{\mu}(\rho_0 - \bar\rho) \in L^2(\mathbb{R})$, $|x|^{\frac{\mu}{2}} v_0 \in L^2(\mathbb{R})$, and $\partial_t \rho|_{t=0} = -\partial_x(\rho_0 v_0)$. Then for all $t > 0$, there holds that*

$$\begin{cases} \|(\rho - \bar\rho)(t)\|_{L^2} \leq C(1+t)^{-\mu+\frac{1}{2}}, \\ \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar\rho, v)(t)\|_{L^2} \leq (1+t)^{-\mu}. \end{cases} \tag{7.6}$$

  Moreover, without the condition $P'(\bar\rho) \leq 1$, if there exists a constant $\lambda_0 > 0$ such that the friction coefficient κ satisfies $\lambda \geq \lambda_0$, and (ρ_0, v_0) satisfies the stronger condition $|x|^{\mu}(\rho_0 - \bar\rho, v_0) \in H^1(\mathbb{R})$, then for all $t > 0$, the solution (ρ, v) satisfies the faster time-decay rates as follows

$$\begin{cases} \|(\rho - \bar\rho)(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}}, \\ \|\partial_x(\rho - \bar\rho, v)(t)\|_{L^2} + \|v(t)\|_{L^2} \leq (1+t)^{-\frac{\mu}{2}-\frac{1}{2}}. \end{cases} \tag{7.7}$$

7.2. **Global existence and time-decay for higher-order derivatives.** To prove the global existence of System (1.4), we establish global a-priori estimates as follows.

Lemma 7.2. (*A priori estimates*) *Let (ρ, v) be the solution to System (1.4) on $[0, T]$ for any given time $T > 0$. Define*

$$\begin{aligned} X(t) := {} & \|(\rho - \bar\rho, v)(t)\|_{H^2}^2 + t\|v(t)\|_{L^2}^2 + t\|\partial_x(\rho - \bar\rho, v)(t)\|_{L^2}^2 \\ & + \int_0^t (\|\partial_x(\rho - \bar\rho)(t)\|_{H^1}^2 + \|v(\tau)\|_{H^2}^2 + \tau\|\partial_x v(t)\|_{L^2}^2) d\tau. \end{aligned} \tag{7.8}$$

If (ρ, u) satisfies

$$\|(\rho - \bar\rho, v)(t)\|_{H^2} << 1, \qquad 0 < t < T, \tag{7.9}$$

then there exists a generic constant $C_0 > 0$ such that

$$X(t) \leq C_0\|(\rho_0 - \bar\rho, v_0)\|_{H^2}^2, \qquad 0 < t < T. \tag{7.10}$$

Proof. We employ similar arguments as used in Subsection 3.1. Denote

$$n := \rho - \bar\rho.$$

Then (n, v) solves

$$\begin{cases} \partial_t n + \bar\rho\partial_x v = F_1 := -\partial_x(nv), \\ \partial_t v + \dfrac{P'(\bar\rho)}{\bar\rho}\partial_x n + \lambda v = F_2 := -v\partial_x v - \big(\dfrac{P'(\bar\rho+n)}{\bar\rho+n} - \dfrac{P'(\bar\rho)}{\bar\rho}\big)\partial_x n, \\ (n, v)|_{t=0} = (\rho_0 - \bar\rho, v_0). \end{cases} \tag{7.11}$$

By direct computations on (7.14), one has the H^1-estimate

$$\begin{aligned} &\frac{d}{dt}\big(\|n(t)\|_{H^1}^2 + \frac{\bar\rho^2}{P'(\bar\rho)}\|v(t)\|_{H^1}^2\big) + \frac{2\bar\rho^2\lambda}{P'(\bar\rho)}\|v(t)\|_{H^1}^2 \\ &\le 2\|F_1(t)\|_{H^1}\|n(t)\|_{H^1} + \frac{2\bar\rho^2}{P'(\bar\rho)}\|F_2(t)\|_{H^1}\|v(t)\|_{H^1}. \end{aligned} \tag{7.12}$$

In addition, we need to perform additional estimates with time weight as follows

$$\begin{aligned} &\frac{d}{dt}\big(t\|\partial_x n(t)\|_{L^2}^2 + \frac{\bar\rho^2}{P'(\bar\rho)}t\|\partial_x v(t)\|_{L^2}^2\big) + \frac{2\bar\rho^2\lambda}{P'(\bar\rho)}t\|\partial_x v(t)\|_{L^2}^2 \\ &\le \|\partial_x n(t)\|_{L^2}^2 + \frac{\bar\rho^2}{P'(\bar\rho)}\|\partial_x v(t)\|_{L^2}^2 \\ &\quad + 2t\|\partial_x F_1(t)\|_{L^2}\|\partial_x n(t)\|_{L^2} + \frac{2\bar\rho^2}{P'(\bar\rho)}t\|\partial_x F_2(t)\|_{L^2}\|\partial_x v(t)\|_{L^2}. \end{aligned} \tag{7.13}$$

Different from linear analysis, the H^2-estimate is needed for estimating nonlinear terms. To this matter, we write the equations of $(\partial_x^2 n, \partial_x^2 v)$ by

$$\begin{cases} \partial_t\partial_x^2 n + v\cdot\partial_x^3 n + \bar\rho\partial_x^3 v + n\partial_x^3 v = R_1, \\ \partial_t\partial_x^2 v + v\cdot\partial_x^3 v + \dfrac{P'(\bar\rho+n)}{\bar\rho+n}\partial_x^2 n + \lambda\partial_x^2 u = R_2. \end{cases} \tag{7.14}$$

where $R_1 := [v, \partial_x^3]n + [n, \partial_x^2]\partial_x v$ and $R_2 := [v, \partial_x^2]\partial_x n + [\frac{P'(\bar\rho+n)}{\bar\rho+n}, \partial_x^2]\partial_x n$ By L^2-energy estimates for (7.14), it holds that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}\big(|\partial_x^2 n|^2 + \frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2\big)dx + 2\lambda\int_{\mathbb{R}}\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2 dx \\ &\le \|\partial_t\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L^\infty_{t,x}}\|\partial_x^2 v\|_{L^2}^2 + 2\|R_1\|_{L^2}\|\partial_x^2 n\|_{L^2} + 2\|\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L^\infty}\|R_2\|_{L^2}\|\partial_x^2 v\|_{L^2}. \end{aligned} \tag{7.15}$$

Here we used some weight function to overcome the loss of derivatives in F_1 and F_2. To capture the dissipation of n, from (7.11) we have the following cross estimates

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}(v\partial_x n + \partial_x v\partial_x^2 n)dx + \frac{P'(\bar\rho)}{\bar\rho}\|\partial_x n(t)\|_{H^1}^2 - \|\partial_x v(t)\|_{H^1}^2 + \lambda\int_{\mathbb{R}}(v\partial_x n + \partial_x v\partial_x^2 n)dx \\ &\le \|F_1(t)\|_{H^1}\|\partial_x v(t)\|_{H^1} + \|F_2(t)\|_{H^1}\|\partial_x n(t)\|_{H^1}. \end{aligned} \tag{7.16}$$

Let $c_1, c_2 \in (0, 1)$ be two constants to be chosen later. Define

$$\begin{aligned} \mathcal{L}_{euler}(t) := &\|n(t)\|_{H^1}^2 + \frac{\bar\rho^2}{P'(\bar\rho)}\|v(t)\|_{H^1}^2 + c_1 t\|\partial_x n(t)\|_{L^2}^2 + \frac{\bar\rho^2}{P'(\bar\rho)}t\|\partial_x v(t)\|_{L^2}^2 \\ &+ \int_{\mathbb{R}}\big(|\partial_x^2 n|^2 + \frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2\big)dx + c_2\int_{\mathbb{R}}(v\partial_x n + \partial_x v\partial_x^2 n)dx. \end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_{euler}(t)=&\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}\|v(t)\|_{H^1}^2+\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}c_1t\|\partial_x v(t)\|_{L^2}^2-c_1\|\partial_x n(t)\|_{L^2}^2-c_1\frac{\bar\rho^2}{P'(\bar\rho)}\|\partial_x v(t)\|_{L^2}^2\\&+\big(2\lambda-\|\partial_t\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L^\infty_{t,x}}\big)\int_{\mathbb{R}}\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2dx\\&+c_2\big(\frac{P'(\bar\rho)}{\bar\rho}\|\partial_x^2 n\|_{L^2}^2-t\|\partial_x^2 v\|_{L^2}^2+\lambda\int_{\mathbb{R}}\partial_x v\partial_x^2 n dx\big).\end{aligned}$$

Due to (7.9), there holds that

$$0<\frac{\bar\rho}{2P'(\bar\rho)}\leq\frac{\bar\rho+n}{P'(\bar\rho+n)}(x,t)\leq\frac{2\bar\rho}{P'(\bar\rho)},\qquad(x,t)\in\mathbb{R}\times(0,T),\tag{7.17}$$

and

$$\|\partial_t\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L^\infty_{t,x}}\leq C\|\partial_t n\|_{L^\infty_{t,x}}\leq C\|v\|_{L^\infty}\|\partial_x n\|_{L^\infty_{t,x}}+C\|\partial_x v\|_{L^\infty_{t,x}}(1+\|n\|_{L^\infty_{t,x}})\leq\lambda.\tag{7.18}$$

Adjusting the coefficients c_1, c_2 suitably and making use of (7.17)-(7.18), we are able to obtain

$$\mathcal{L}_{euler}(t)\sim\|(n,v)(t)\|_{H^2}^2+c_1t\|\partial_x(n,v)(t)\|_{L^2}^2.\tag{7.19}$$

and

$$\mathcal{D}_{euler}(t)\gtrsim\|v(t)\|_{H^2}^2+\|\partial_x n(t)\|_{H^1}^2+t\|\partial_x v(t)\|_{L^2}^2.\tag{7.20}$$

Then, it follows from (7.12)-(7.13), (7.15), (7.16) and (7.17) that

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_{euler}(t)+\mathcal{D}_{euler}(t)\\&\lesssim\|F_1(t)\|_{H^1}\|(n,\partial_x v)(t)\|_{H^1}+\|F_2(t)\|_{H^1}\|(\partial_x n,v)(t)\|_{H^1}\\&\quad+t\|\partial_x F_1(t)\|_{L^2}\|\partial_x n(t)\|_{L^2}+t\|\partial_x F_2(t)\|_{L^2}\|\partial_x v(t)\|_{L^2}\\&\quad+\|R_1\|_{L^2}\|\partial_x^2 n(t)\|_{L^2}+\|R_2\|_{L^2}\|\partial_x^2 v(t)\|_{L^2}.\end{aligned}\tag{7.21}$$

The nonlinear terms on the right-hand side of (7.21) are analyzed as follows. First, standard product laws and (7.9) yield

$$\begin{aligned}&\|F_1(t)\|_{H^1}\|(n,\partial_x v)(t)\|_{H^1}+\|F_2(t)\|_{H^1}\|(\partial_x n,v)(t)\|_{H^1}\\&\lesssim(\|v(t)\|_{H^2}\|\partial_x n\|_{H^1}+\|v(t)\|_{H^1}\|\partial_x v(t)\|_{H^1})\|(\partial_x n,v)(t)\|_{H^1}\\&\quad+\|\big(\frac{P'(\bar\rho+n)}{\bar\rho+n}-\frac{P'(\bar\rho)}{\bar\rho}\big)(t)\|_{H^1}\|\partial_x n(t)\|_{H^1}\|(\partial_x n,v)(t)\|_{H^1}\\&\lesssim\|(n,v)(t)\|_{H^2}(\|v(t)\|_{H^2}^2+\|\partial_x n\|_{H^1}^2)\\&\lesssim o(1)\mathcal{D}_{euler}(t).\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}&t\|\partial_x F_1(t)\|_{L^2}\|\partial_x n(t)\|_{L^2}+t\|\partial_x F_2(t)\|_{L^2}\|\partial_x v(t)\|_{L^2}\\&\lesssim t(\|v(t)\|_{H^1}\|\partial_x n(t)\|_{H^1}+\|n(t)\|_{H^1}\|\partial_x v(t)\|_{H^1})\|\partial_x n(t)\|_{L^2}\\&\quad+t\big(\|v(t)\|_{H^1}\|\partial_x v(t)\|_{H^1}+\|\big(\frac{P'(\bar\rho+n)}{\bar\rho+n}-\frac{P'(\bar\rho)}{\bar\rho}\big)(t)\|_{H^1}\|\partial_x n\|_{H^1}\big)\|\partial_x v(t)\|_{L^2}\\&\lesssim\|(n,v)(t)\|_{H^1}(\|v(t)\|_{H^2}^2+\|\partial_x n(t)\|_{H^1}^2+t\|\partial_x v(t)\|_{L^2}^2+t^2\|\partial_x^2 v(t)\|_{L^2}^2+t\|\partial_x^2 n(t)\|_{L^2}^2)\\&\lesssim o(1)\mathcal{D}_{euler}(t).\end{aligned}$$

From standard commutator estimates, one has

$$\begin{aligned}\|R_1\|_{L^2} &\lesssim \|[v,\partial_x^3]n\|_{L^2} + \|[n,\partial_x^2]\partial_x v\|_{L^2}\\ &\lesssim \|\partial_x^2 v\|_{L^2}\|n\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\|\partial_x^2 n\|_{L^2} + \|\partial_x^2 n\|_{L^2}\|v\|_{L^\infty} + \|\partial_x n\|_{L^\infty}\|\partial_x^2 v\|_{L^2}\\ &\lesssim \|(n,v)(t)\|_{H^2}\|\partial_x^2(n,v)(t)\|_{L^2}.\end{aligned}$$

and similarly,

$$\|R_2\|_{L^2} \lesssim \|[v,\partial_x^2]\partial_x n\|_{L^2} + \|[\frac{P'(\bar\rho+n)}{\bar\rho+n},\partial_x^2]\partial_x n\|_{L^2} \lesssim \|(n,v)(t)\|_{H^2}\|\partial_x^2(n,v)(t)\|_{L^2}.$$

We thus have

$$\begin{aligned}&\|R_1\|_{L^2}\|\partial_x^2 n(t)\|_{L^2} + \|R_2\|_{L^2}\|\partial_x^2 v(t)\|_{L^2}\\ &\quad\lesssim \|(n,v)(t)\|_{H^2}\|\partial_x^2(n,v)(t)\|_{L^2}^2 \lesssim o(1)\mathcal{D}_{euler}(t).\end{aligned}$$

Substituting the above estimates on nonlinear terms into (7.21) and using (7.20), we derive the following Lyapunov inequality

$$\frac{d}{dt}\mathcal{L}_{euler}(t) + \|v(t)\|_{H^2}^2 + \|\partial_x n(t)\|_{H^1}^2 + t\|\partial_x v(t)\|_{L^2}^2 \lesssim 0. \tag{7.22}$$

One then deduces after integrating (7.22) over $[0,t]$ and taking advantage of (7.19) that

$$\begin{aligned}&\|(n,v)(t)\|_{H^2}^2 + t\|\partial_x(n,v)(t)\|_{L^2}^2\\ &\quad+\int_0^t (\|\partial_x(\rho-\bar\rho)(\tau)\|_{H^1}^2 + \|v(\tau)\|_{H^2}^2 + \tau\|\partial_x v(\tau)\|_{L^2}^2)d\tau\\ &\quad\le C\|(\rho_0-\bar\rho,v_0)\|_{H^2}^2.\end{aligned} \tag{7.23}$$

Finally, taking the $L^2(\mathbb{R})$-inner product of $(7.11)_2$ with v, we have

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + 2\lambda\|v(t)\|_{L^2}^2 \le \big(\|v\|_{L^\infty_{t,x}}\|\partial_x v(t)\|_{L^2} + \frac{4P'(\bar\rho)}{\bar\rho}\|\partial_x n(t)\|_{L^2}\big)\|v(t)\|_{L^2}. \tag{7.24}$$

This together with Grönwall's inequality and (7.23) gives rise to

$$\begin{aligned}\|v(t)\|_{L^2} &\le e^{-\lambda t}\|v_0(t)\|_{L^2} + C\int_0^t e^{-\lambda(t-\tau)}\|\partial_x(n,v)(\tau)\|_{L^2}d\tau\\ &\le C\|(\bar\rho_0-\bar\rho,v_0)\|_{H^2}^2(1+t)^{-\frac12}.\end{aligned}$$

The proof of Lemma 7.2 is complete. □

Proof of Theorem 7.1: Let (ρ_0,v_0) satisfy (7.2). According to classical local existence results (for example, cf. [2]), there exists a time $T_0>0$ such that the Cauchy problem of System (1.4) and initial data (ρ_0,v_0) has a unique solution (ρ,v) satisfying $(\rho-\bar\rho,v)\in C([0,T];H^2(\mathbb{R}))$. Then in accordance with a priori estimates (7.10) established in Lemma 7.10 and the standard bootstrap argument, one can extend the solution (ρ,v) globally in time and show the properties (7.3)-(7.4).

The time-decay estimates (7.5), (7.6) and (7.7) will be proved in Lemmas 7.3 and 7.4 in the next subsection.

7.3. Faster time-decay rates.

Lemma 7.3. *Let U be the global solution to the Cauchy problem of System (1.4) and initial data (ρ_0,u_0) given by Subsection 7.2. Assume that there exists a constant $\lambda_0>0$ such that the friction coefficient κ satisfies $\lambda\ge\lambda_0$, and in addition to (7.2), (ρ_0,v_0) satisfies $|x|^\mu(\rho_0-\bar\rho,v_0)\in H^1(\mathbb{R})$ with $\frac12<\mu\le1$. Then it holds that*

$$\|(\rho-\bar\rho,v)(t)\|_{H^2} + t^{\frac12}\|v(t)\|_{L^2}^2 + t^{\frac12}\|\partial_x(\rho-\bar\rho,v)(t)\|_{L^2} \le C(1+t)^{-\frac{\mu}{2}} \qquad \textit{for all } t>0. \tag{7.25}$$

Proof. By virtue of (7.22) and similar arguments as used in Lemma 4.1, the time-decay estimates (7.25) follows provided that we show

$$\sup_{t>0}\||x|^{\mu}(\rho-\bar{\rho})(t)\|_{L^2}<\infty. \tag{7.26}$$

To this end, one needs to perform space-weighed estimates similar to Lemma 4.2. Recall that (n,v) with $n=\rho-\bar{\rho}$ satisfies (7.11). Note that due to (9.6), it holds that

$$\||x|^{\mu-1}n(t)\|_{L^2}\leq\frac{(2\mu-1)}{2}\||x|^{\mu}\partial_x n(t)\|_{L^2}. \tag{7.27}$$

Taking the $L^2(\mathbb{R})$-inner product of $(7.11)_1$ and $(7.11)_2$ with $|x|^{2\mu}n$ and $|x|^{2\mu}v$, respectively, and applying (7.27), we have

$$\begin{aligned}
&\frac{d}{dt}\big(\||x|^{\mu}n(t)\|_{L^2}^2+\frac{\bar{\rho}^2}{P'(\bar{\rho})}\||x|^{\mu}v(t)\|_{L^2}^2\big)+\frac{2\bar{\rho}^2\lambda}{P'(\bar{\rho})}\||x|^{\mu}v(t)\|_{L^2}^2\\
&=\int_{\mathbb{R}}(-\partial_x|x|^{2\mu}nv+\partial_x(|x|^{2\mu}n)nv+\frac{\bar{\rho}^2}{P'(\bar{\rho})}|x|^{2\mu}F_2v)dx\\
&\leq 2\mu\||x|^{\mu-1}n(t)\|_{L^2}\||x|^{\mu}v(t)\|_{L^2}+\|n(t)\|_{L^2}(2\mu\||x|^{\mu-1}n(t)\|_{L^2}+\||x|^{\mu}\partial_x n(t)\|_{L^2})\||x|^{\mu}v(t)\|_{L^2}\\
&\quad+\frac{\bar{\rho}^2}{P'(\bar{\rho})}\||x|^{\mu}F_2(t)\|_{L^2}\||x|^{\mu}v(t)\|_{L^2}\\
&\leq\frac{\bar{\rho}^2\lambda}{P'(\bar{\rho})}\||x|^{\mu}v(t)\|_{L^2}^2+\frac{C}{\lambda}\||x|^{\mu}\partial_x n(t)\|_{L^2}^2\\
&\quad+C\|(n,v)(t)\|_{L^2}\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+C\||x|^{\mu}n(t)\|_{L^2}\|\partial_x n(t)\|_{L^2}\||x|^{\mu}v(t)\|_{L^2}.
\end{aligned} \tag{7.28}$$

Similarly, it also follows that

$$\begin{aligned}
&\frac{d}{dt}\big(\||x|^{\mu}\partial_x n(t)\|_{L^2}^2+\frac{\bar{\rho}^2}{P'(\bar{\rho})}\||x|^{\mu}\partial_x v(t)\|_{L^2}^2\big)+\frac{2\bar{\rho}^2\lambda}{P'(\bar{\rho})}\||x|^{\mu}\partial_x v(t)\|_{L^2}^2\\
&=\int_{\mathbb{R}}(-\partial_x|x|^{2\mu}\partial_x n\partial_x v+|x|^{2\mu}\partial_x F_1\partial_x n+|x|^{2\mu}\partial_x F_2\partial_x v)dx\\
&\leq\frac{\bar{\rho}^2\lambda}{P'(\bar{\rho})}\||x|^{\mu}\partial_x v(t)\|_{L^2}^2+\varepsilon\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\frac{C}{\varepsilon}\|\partial_x(n,v)(t)\|_{L^2}^2\\
&\quad+C\|(n,v)(t)\|_{H^2}\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+C\||x|^{\mu}n(t)\|_{H^1}\|\partial_x n(t)\|_{H^1}\||x|^{\mu}\partial_x v(t)\|_{L^2}.
\end{aligned} \tag{7.29}$$

Here one has used $|\int_{\mathbb{R}}\partial_x|x|^{2\mu}\partial_x n\partial_x v dx|\leq\varepsilon\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\frac{C}{\varepsilon}\|\partial_x(n,v)(t)\|_{L^2}^2$ for any $\varepsilon>0$ due to the fact that $\mu>\frac{1}{2}$. Adding (7.28)-(7.29) together and integrating it over $[0,t]$, we have

$$\begin{aligned}
&\||x|^{\mu}(n,v)(t)\|_{H^1}^2+\lambda\int_0^t\||x|^{\mu}v(\tau)\|_{H^1}^2d\tau\\
&\leq C\||x|^{\mu}(\rho_0-\bar{\rho},v_0)(t)\|_{H^1}+C(\frac{1}{\lambda}+\varepsilon)\int_0^t\||x|^{\mu}\partial_x(n,v)(\tau)\|_{L^2}^2d\tau+\frac{C}{\varepsilon}\int_0^t\|\partial_x(n,v)(\tau)\|_{L^2}^2d\tau\\
&+C\sup_{t>0}\|(n,v)(t)\|_{H^2}\int_0^t\||x|^{\mu}\partial_x(n,v)(\tau)\|_{L^2}^2d\tau\\
&+\big(\int_0^t\|\partial_x n(t)\|_{H^1}^2d\tau\big)^{\frac{1}{2}}\sup_{t>0}\||x|^{\mu}n(t)\|_{H^1}\big(\int_0^t\||x|^{\mu}\partial_x v(\tau)\|_{H^1}^2d\tau\big)^{\frac{1}{2}}.
\end{aligned} \tag{7.30}$$

The combination of (7.2), (7.3) and (7.30) yields

$$
\begin{aligned}
&\||x|^\mu(n,v)(t)\|_{H^1}^2+\lambda\int_0^t\||x|^\mu v(\tau)\|_{H^1}^2d\tau\\
&\le C\||x|^\mu(\rho_0-\bar\rho,v_0)(t)\|_{H^1}^2+\frac{C}{\varepsilon}\|(\rho_0-\bar\rho,v_0)(t)\|_{H^2}^2\\
&\quad+C(\frac{1}{\lambda}+\varepsilon)\int_0^t\||x|^\mu\partial_x(n,v)(\tau)\|_{L^2}^2d\tau+\frac{C}{\varepsilon}\int_0^t\|\partial_x(n,v)(\tau)\|_{L^2}^2d\tau.
\end{aligned}
\tag{7.31}
$$

In addition, the cross term is estimated by

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}|x|^{2\mu}v\partial_x n dx+\frac{P'(\bar\rho)}{\bar\rho}\||x|^\mu\partial_x n\|_{L^2}^2+\int_{\mathbb{R}}(\lambda|x|^{2\mu}v\partial_x n-(|x|^{2\mu}\partial_x n)_x\partial_x v)dx\\
&\le\||x|^\mu F_1(t)\|_{L^2}\||x|^\mu\partial_x v(t)\|_{L^2}+\||x|^\mu F_2(t)\|_{L^2}\||x|^\mu\partial_x n(t)\|_{L^2}\\
&\le C\|(n,v)(t)\|_{L^2}\||x|^\mu\partial_x(n,v)(t)\|_{L^2}^2\\
&\le C\|(n,v)(t)\|_{H^2}\||x|^\mu\partial_x(n,v)(\tau)\|_{L^2}^2,
\end{aligned}
$$

from which, (7.31) and $\|(n,v)(t)\|_{H^2}<<1$ we have

$$
\begin{aligned}
&\||x|^\mu(n,v)(t)\|_{H^1}^2+\lambda\int_0^t\||x|^\mu v(\tau)\|_{H^1}^2d\tau+\int_0^t\||x|^\mu\partial_x n(\tau)\|_{H^1}^2d\tau\\
&\le C\||x|^\mu(\rho_0-\bar\rho,v_0)(t)\|_{H^1}^2+\frac{C}{\varepsilon}\|(\rho_0-\bar\rho,v_0)(t)\|_{H^2}^2\\
&\quad+C(\frac{1}{\lambda}+\varepsilon)\int_0^t\||x|^\mu\partial_x(n,v)(\tau)\|_{L^2}^2d\tau+\frac{C}{\varepsilon}\int_0^t\|\partial_x(n,v)(\tau)\|_{L^2}^2d\tau..
\end{aligned}
\tag{7.32}
$$

Choosing ε sufficiently small and letting λ large enough, we obtain (7.26), and therefore the time-decay estimates (7.25) follows.

□

Lemma 7.4. *Let U be the global solution to the Cauchy problem of System (1.4) and initial data (ρ_0,u_0) given by Subsection 7.2. In addition to (7.2), Assume $P'(\bar\rho)\le1$, $\partial_t\rho|_{t=0}=-\partial_x(\rho_0v_0)$, $|x|^\mu(\rho_0-\bar\rho)\in L^2(\mathbb{R})$ and $|x|^{\mu-\frac12}v_0\in L^2(\mathbb{R})$ with $\frac12<\mu\le1$. Then it holds that*

$$
\|(\rho-\bar\rho,v)(t)\|_{H^2}+t^{\frac12}\|v(t)\|_{L^2}+t^{\frac12}\|\partial_x(\rho-\bar\rho,v)(t)\|_{L^2}\le C(1+t)^{-\mu+\frac12}\qquad \textit{for all } t>0.
\tag{7.33}
$$

Proof. We mention that in order to apply the method in 5, one needs to consider the momentum $m=nv$ instead of the velocity v. Then System (1.4) is rewritten in terms of $(n,m)=(\rho-\bar\rho,nv)$ as

$$
\begin{cases}
\partial_t n+\partial_x m=0,\\
\partial_t m+P'(\bar\rho)\partial_x n+\lambda m=F_3:=-\partial_x\big(\dfrac{m^2}{\bar\rho+n}+P(\bar\rho+n)-P(\bar\rho)-P'(\bar\rho)n\big),\\
(n,m)|_{t=0}=(\rho_0-\bar\rho,\rho_0v_0).
\end{cases}
\tag{7.34}
$$

As in Section 5, we introduce the effect unknown

$$
M(x,t):=\int_{-\infty}^x n(y,t)dy
$$

such that

$$
\partial_t^2M-P'(\bar\rho)\partial_x^2M+\lambda\partial_tM=F_3.
\tag{7.35}
$$

Then following similar computations as done in Lemma 5.1, one can show

$$\begin{aligned}&\int_{\mathbb{R}}\big((1+t+|x|)^{2\mu-1}(|\partial_t M|^2+|\partial_x M|^2)+(1+t+|x|)^{2\mu-2}|M|^2\big)dx\\&\quad+\int_0^t\int_{\mathbb{R}}\big((1+t+|x|)^{2\mu-1}|\partial_t M|^2+(1+t+|x|)^{2\mu-2}|\partial_x M|^2\big)dxd\tau\\&\leq C+C\int_0^t\int_{\mathbb{R}}(1+t+|x|)^{2\mu-1}|F_3|^2dxd\tau.\end{aligned}\tag{7.36}$$

From (7.3) and composite estimates, it is easy to verify that

$$|F_3|\lesssim|m||\partial_x m|+|n||\partial_x n|.$$

It thus follows that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}(1+t+|x|)^{2\mu-1}|F_3|^2dxd\tau\\&\lesssim\int_0^t\|\partial_x(m,n)(\tau)\|_{L^\infty}^2\int_{\mathbb{R}}(1+t+|x|)^{2\mu-1}(|m|^2+|n|^2)dxd\tau\\&\lesssim\int_0^t\|\partial_x(n,v)(\tau)\|_{H^1}^2\int_{\mathbb{R}}(1+t+|x|)^{2\mu-1}(|\partial_t M|^2+|\partial_x M|^2)dxd\tau,\end{aligned}\tag{7.37}$$

where one has used the facts that $\partial_x M=n$ and $\partial_t M=-m$. Inserting the above estimate into (7.36) and using (7.3) and Grönwall's inequality, we get

$$\|(n,v)(t)\|_{L^2}\lesssim(1+t)^{-\mu+\frac{1}{2}},\qquad\int_0^t(1+\tau)^{2\mu-2}\|(n,v)(\tau)\|_{L^2}^2d\tau\leq C.\tag{7.38}$$

Finally, as $\frac{1}{2}<\mu\leq 1$, multiplying the Lyapunov inequality (7.22) with $t^{2\mu-1}$ and integrating it over $[0,t]$, we have

$$\begin{aligned}&t^{2\mu-1}\|(n,v)(t)\|_{L^2}^2+(t^{2\mu-1}+c_1t^{2\mu})\|\partial_x(n,v)(t)\|_{L^2}^2+\int_0^t\tau^{2\mu-1}\|\partial_x(n,v)(\tau)\|_{L^2}^2d\tau\\&\lesssim\int_0^t\big(\tau^{2\mu-2}\|(n,v)(\tau)\|_{L^2}^2+(\tau^{2\mu-2}+c_1\tau^{2\mu-1})\|\partial_x(n,v)(\tau)\|_{L^2}^2\big)d\tau.\end{aligned}\tag{7.39}$$

Similarly to (5.17)-(5.18), one deduces from estimates (7.4) that

$$\int_0^t\tau^{2\mu-2}\|\partial_x(n,v)(\tau)\|_{L^2}^2d\tau\lesssim 1,\tag{7.40}$$

which, together with (7.38), (7.39) and the fact that c_1 is sufficiently small, implies

$$\|\partial_x(n,v)(t)\|_{L^2}^2\lesssim(1+t)^{-\mu},$$

from which and Grönwall's inequality to (7.24) implies

$$\|v(t)\|_{L^2}^2\lesssim(1+t)^{-\mu}.$$

Therefore, the desired time-decay estimates (7.33) hold. □

8. Extensions

8.1. **Numerics.** As an application of the method developed here and inspired by Porretta and Zuazua's paper [22], one can prove that a finite-difference centered approximation for partially dissipative system (2.2) in the whole space preserves the asymptotic properties of the continuous solutions as $t\to\infty$. Such result would highlight that the hyperbolic hypocoercive nature of the system can be preserved at the discrete level. Compared to the results obtained by Porretta and Zuazua concerning the Kolmogorov equation, here we employ hypocoercive properties of rank n, meaning that we need information from the whole Kalman matrix $\mathcal{K} = (B, BA, BA^2, \cdots, BA^{n-1})$ to recover dissipation on all the components. In the case of the Kolmogorov equation, only rank-2 hypocoercivity is involved.

8.2. **Multi-dimensional setting:** In the multi-dimensional setting, one could look at n-component systems in $\mathbb{R}^d$ of the type:

$$\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j \frac{\partial V}{\partial x_j} = BV, \tag{8.1}$$

where the A^j $(j = 0, .., d)$ are symmetric matrices, B is strongly dissipative matrix and the unknown $V = V(t, x) \in \mathbb{R}^n$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$ $(d \geq 2)$.

For these general systems, under the (SK) condition, Beauchard and Zuazua in [1] were able to obtain time-decay rates using hypocoercivity-type arguments. However, the method developed in the present paper does not allow to derive time-decay rates. The issue comes from the appearance of mixed derivatives when differentiating in time the lower order corrector term from the Lyapunov functional.

Nevertheless, it is possible to obtain results under additional structural conditions on the system: it needs to have a structure similar to the damped Euler equation (1.4). Indeed, for the multi-dimensional version of (1.4), straightforward computations show that the Lyapunov functional:

$$\mathcal{L}(t) = \|(\rho - \bar{\rho}, u)(t)\|_{H^1}^2 + ct\|(\nabla\rho, \nabla u)(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} u \cdot \nabla\rho dx$$

allows to directly recover time-decay rates in any dimension.

9. Appendix

9.1. **A short litterature review on partially dissipative systems.** To capture the dissipative structures for large time and overcome the coercivity issue mentioned in (2.3), Shizuta and Kawashima in [24] developed a condition, the well-known stability condition

$$\{\text{eigenvectors of } A\} \cap \operatorname{Ker}(B) = \{0\}. \tag{SK}$$

In one space dimension, this condition is equivalent to the non-existence of plane wave solutions propagating in the characteristic directions and ensures the decay. In higher dimensions, the (SK) condition is a sufficient condition for decay, but is unlikely to hold, refer to [1] for further details. Under the (SK) assumption, the global well-posedness of classical solutions for quasilinear partially dissipative hyperbolic systems near some constant equilibrium state was proved by Yong [33] and Hanouzet-Natalini [11] in Sobolev spaces with high regularity, and then by Xu-Kawashima [30] in critical inhomogenous Besov spaces.

Under the (SK) assumption, the authors in [24] proved the following time-decay theorem for linear problems by using the Fourier transform.

Theorem 9.1 ([24](SK) decay estimate)**.** *Under the condition* (SK) *and for $U_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the solution U of* (2.1) *verifies*

$$\|U^h(t)\|_{L^2} \leq Ce^{-\gamma t}\|U_0^h\|_{L^2}, \tag{9.1}$$

$$\|U^\ell(t)\|_{L^\infty} \leq Ct^{-d/2}\|U_0^\ell\|_{L^1}, \tag{9.2}$$

where C, γ are positive constants depending only on A, B and D, $\widehat{U}^h(\xi, t) := \widehat{U}(\xi, t)\mathbb{1}_{\{|\xi|>1\}}$ and $\widehat{U}^\ell(\xi, t) := \widehat{U}(\xi, t)\mathbb{1}_{\{|\xi|<1\}}$.

We also refer to the papers of Bianchini, Hanouzet and Natalini in [2] and Xu and Kawashima [31, 32] for results concerning the asymptotic behavior of these systems in, respectively, the inhomogeneous Sobolev and Besov frameworks. Essentially, under the (SK) condition, one sees that the low-frequency part of the Green function behaves like the heat kernel and the high frequency-part of the solution decays exponentially.

More recently, Beauchard and Zuazua in [1], aiming to handle multi-dimensional systems, retrieved directly the time-decay estimates by introducing and employing the following Lyapunov functional, written here in the one-dimensional setting:

$$\mathcal{L}(t)^2 \triangleq \|U(t)\|_{L^2}^2 + \min(\xi,\xi^{-1})\Re\sum_{k=1}^{n-1}\varepsilon_k(BA^{k-1}\widehat{U}\cdot BA^k\widehat{U})_{L^2} \tag{9.3}$$

where $\Re$ is the real part operator. Assuming suitable smallness conditions on the coefficients ε_k, one obtains

$$\frac{d}{dt}\mathcal{L}(t)^2 + \min(1,|\xi|^2)\mathcal{L}^2(t) \leq 0, \quad \text{and} \quad \mathcal{L}(t)^2 \sim \|U(t)\|_{L^2}^2.$$

This approach, inspired by the hypocoercivity theory of Villani [27], gives the same result as Theorem 9.1 in a direct way and allows to better describe the asymptotic behavior of solutions when the (SK) condition fails (in the multidimensional setting). Inspired by their method, Crin-Barat and Danchin in [5, 6] obtained global well-posedness and sharp time-decay results in the framework of critical homogeneous Besov spaces for general nonlinear hyperbolic partially dissipative systems. We also refer to [4] for a new approach to deal with the relaxation limit associated to these systems by employing a precise frequency decomposition thanks to the Littlewood-Paley theory.

To be able to compare the results obtained in this paper with the classical ones, we now state time-decay estimates that are obtain in e.g. [1].

Proposition 9.2. *Let $k\geq 1$ and $U\in C(\mathbb{R}_+;H^k(\mathbb{R}))$ be a solution of System (2.1) subject to the initial data $U_0\in H^k(\mathbb{R})\cap L^1(\mathbb{R})$. Then it holds that*

$$\begin{cases} \|U(t)\|_{L^2} \lesssim (1+t)^{-\frac14}\|U_0\|_{H^k\cap L^1}, \\ \|U_2(t)\|_{L^2} \lesssim (1+t)^{-\frac34}\|U_0\|_{H^k\cap L^1}, \\ \|\nabla^i U_1(t)\|_{L^2} \lesssim (1+t)^{-\frac14-\frac{i}{2}}\|U_0\|_{H^k\cap L^1}, \quad i=1,2,...,k, \\ \|\nabla^i U_2(t)\|_{L^2} \lesssim (1+t)^{-\frac34-\frac{i}{2}}\|U_0\|_{H^k\cap L^1}, \quad i=1,2,...,k-1. \end{cases} \tag{9.4}$$

Remark 9.3. The L^1-type assumption on the initial data can be replaced by $\dot{B}^{-\frac12}_{2,\infty}(\mathbb{R})$, cf. [5, 31, 32]. Moreover, these decay rates are optimal in the sense that they follow the ones of the heat equation, which is expected by the low frequencies of the solution.

9.2. **Technical lemmas.**

Lemma 9.4. (*General Caffarelli-Kohn-Nirenberg inequality*)

- ([16]) *For all $h\in\mathcal{C}_c(\mathbb{R})$, it holds that*

$$\||x|^\kappa h\|_{L^r} \leq C\||x|^\alpha\partial_x h\|_{L^p}^\theta\||x|^\beta h\|_{L^q}^{1-\theta}, \tag{9.5}$$

where $1\leq p,q<\infty$, $0<r<\infty$, $0\leq\theta\leq 1$, $\frac1p+\alpha>0$, $\frac1q+\beta>0$, $\frac1r+\kappa>0$ such that

$$\frac1r+\kappa = \theta(\frac1p+\alpha-1)+(1-\theta)(\frac1q+\beta), \qquad \kappa=\theta\sigma+(1-\theta)\beta,$$

with σ satisfying

$$\sigma\leq\alpha \quad \text{if } \theta>0, \qquad \sigma\geq\alpha-1 \quad \text{if } \theta>0 \text{ and } \frac1p+\alpha-1=\frac1r+\kappa.$$

- ([3]) *For all $h \in \mathcal{C}_c(\mathbb{R})$, it holds that*

$$\| |x|^{\mu_1} h \|_{L^p} \leq C_{\mu_1,\mu_2} \| |x|^{\mu_2} \partial_x h \|_{L^2}, \tag{9.6}$$

with $\mu_2 > \frac{1}{2}$, $\mu_2 - 1 \leq \mu_1 \leq \mu_2 - \frac{1}{2}$ and $p = \frac{2}{2(\mu_2-\mu_1)-1}$. If $\mu_1 = \mu_2 - 1$, then $p = 2$ and the best constant in (9.6) is

$$C_{\mu_1,\mu_2} = C_{\mu_2-1,\mu_2} = \frac{|2\mu_2 - 1|}{2}.$$

Lemma 9.5. *Let $T > 0$ be given time, and $E_1(t), E_2(t)$ be two nonnegative and absolutely continuous functions on $[0, T)$. Suppose that*

$$\frac{d}{dt}\big(E_1(t) + ctE_2(t)\big) + a_1 E_1(t)^{1+\frac{1}{\mu}} + a_2 E_2(t) \leq 0, \qquad t \in (0,T), \tag{9.7}$$

where c, a_1, a_2 and μ are constants satisfying

$$a_1, a_2, \mu > 0, \qquad 0 < c < \min\{\frac{a_2}{\mu}, a_2\}.$$

Then it holds that

$$E_1(t) + ctE_2(t) \leq C a_1^{-\mu} t^{-\mu}, \qquad t \in (0,T), \tag{9.8}$$

where $C > 0$ is a constant independent of T and a_1.

Proof. Let the constant p satisfy $\max\{1, \mu\} < p < \frac{a_2}{c}$. Multiplying (9.7) with t^p, we get

$$\frac{d}{dt}\big(t^p E_1(t) + ct^{p+1} E_2(t)\big) + a_1 t^p E_1(t)^{1+\frac{1}{\mu}} + (a_2 - pc) t^p E_2(t) \leq p t^{p-1} E_1(t). \tag{9.9}$$

Noticing $a_2 - pc > 0$ and

$$pt^{p-1}E_1(t) = p\big(t^p E_1(t)^{1+\frac{1}{\mu}}\big)^{\frac{\mu}{1+\mu}} \big(t^{p-\mu-1}\big)^{\frac{1}{1+\mu}} \leq a_1 t^p E_1(t)^{1+\frac{1}{\mu}} + \big(\frac{p}{\mu+1}\big)^{\mu+1} \big(\frac{\mu}{a_1}\big)^{\mu} t^{p-\mu-1},$$

we prove after integrating (9.9) over $[0, t]$ that

$$t^p E_1(t) + ct^{p+1} E_2(t) \leq \big(\frac{p}{\mu+1}\big)^{\mu+1} \big(\frac{\mu}{a_1}\big)^{\mu} \int_0^t \tau^{p-\mu-1} d\tau = \big(\frac{p}{\mu+1}\big)^{\mu+1} \big(\frac{\mu}{a_1}\big)^{\mu} \frac{1}{p-\mu} t^{p-\mu}.$$

This completes the proof of Lemma 9.5. □

Acknowledgments

T. Crin-Barat and E. Zuazua have been funded by the Alexander von Humboldt-Professorship program and the Transregio 154 Project "Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks" of the DFG. E. Zuazua has been funded by the ModConFlex Marie Curie Action, HORIZON-MSCA-2021-DN-01, the COST Action MAT-DYN-NET, and grants PID2020-112617GB-C22 and TED2021-131390B-I00 of MINECO (Spain).

References

[1] K. Beauchard and E. Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Rational Mech. Anal*, 199:177–227, 2011.

[2] S. Bianchini, B. Hanouzet, and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure and Appl. Math.*, 60, 1559-1622, 2007.

[3] F. Catrina and Z.-Q. Wang. On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, pages 0226–0258, 2001.

[4] T. Crin-Barat and R. Danchin. Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit. *Mathematische Annalen*, 2022.

[5] T. Crin-Barat and R. Danchin. Partially dissipative hyperbolic systems in the critical regularity setting : the multi-dimensional case. *Journal de Mathématiques Pures et Appliquées*, 165:1–41, 2022.

[6] T. Crin-Barat and R. Danchin. Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. *Pure and Applied Analysis*, 4(1):85–125, 2022.

[7] C. M. Dafermos. Hyperbolic conservation laws in continuum physics, 3rd edition. *Grundlehren Math. Wiss., vol. 325, Springer, Heidelberg, Dordrecht, London and New York,*, 2010.

[8] R. Danchin. Fourier analysis methods for the compressible Navier-Stokes equations. *in : Giga Y., Novotný A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham*, 2018.

[9] M. Daoulatli. Energy decay rates for solutions of the wave equations with nonlinear damping in exterior domain. *Journal of Differential Equations*, 264, 4260-4302, 2018.

[10] Y. Guo and Y. Wang. Decay of dissipative equations and negative sobolev spaces. *Communications in Partial Differential Equations*, 37, 2165-2208, 2012.

[11] B. Hanouzet and R. Natalini. Global existence of smooth solutions for partially dissi- pative hyperbolic systems with convex entropy. *Arch. Ration. Mech. Anal.*, 169, 89-117, 2003.

[12] F. Hérau. Isotropic hypoellipticity and trend to equilibrium for the fokker-planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.*, 171:151–218, 2004.

[13] F. Hérau. Short and long time behavior of the fokker-planck equation in a confining potential and application. *J. Func. Anal.*, 244:95–118, 2007.

[14] S. Jin and Z. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Commun. Pure Appl. Math.*, 48, 235-276, 1995.

[15] T. Kato. The cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Ration. Mech. Anal.*, 58:181–205, 1975.

[16] R. K. L. Caffarelli and L. Nirenberg. First order interpolation inequality with weights. *Compos. Math.*, 53, 259-275, 1984.

[17] P. Lax. Hyperbolic systems of conservation laws ii. *Commun. Pure Appl. Math.*, (4) 14:537–566, 1957.

[18] A. Majda. Compressible fluid flow and systems of conservation laws in several space variables. *Appl. Math. Sci., vol. 53, Springer-Verlag, Berlin, New York*, 1984.

[19] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20, 67-104, 1980.

[20] K. Mochizuki and T. Motai. On energy decay-nondecay problems for wave equations with nonlinear dissipative term in R^N. *J. Math. Soc. Japan*, 47 (3), 1995.

[21] M. Nakao. Decay of solutions to the cauchy problem for the klein–gordon equation with a localized nonlinear dissipation. *Hokkaido Math.*, 27, 245-271, 2017.

[22] A. Porretta and E. Zuazua. Numerical hypocoercivity for the kolmogorov equation. *Mathematics of computation*, 86(303):97–119, 2017.

[23] D. Serre. *Systèmes de lois de conservation, tome 1.* Diderot editeur, Arts et Sciences, Paris, New-York, Amsterdam, 1996.

[24] S. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14, 249-275, 1985.

[25] J. Vazquez. *The Porous Medium Equation: Mathematical Theory.* Clarendon Press, 2007.

[26] W. Vicenti and C. Kruger. Introduction to physical gas dynamics. *Melbourne: Robert E. Krieger*, 1982.

[27] C. Villani. Hypocoercivity. *Mem. Am. Math. Soc.*, 2010.

[28] K. Wakasa and B. Yordanov. On the energy decay for dissipative nonlinear wave equations in one space dimension. *J. Math. Anal. Appl.*, 455 (2), 1317-1322, 2017.

[29] Z. Xin and J. Xu. Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions. *Journal of Differential Equations*, 274, 543-575, 2021.

[30] J. Xu and S. Kawashima. Global classical solutions for partially dissipative hyperbolic system of balance laws. *Arch. Rational Mech.Anal*, 211, 513–553, 2014.

[31] J. Xu and S. Kawashima. The optimal decay estimates on the framework of Besov spaces for generally dissipative systems. *Arch. Rational Mech. Anal*, 218, 275–315, 2015.

[32] J. Xu and S. Kawashima. Frequency-localization duhamel principle and its application to the optimal decay of dissipative systems in low dimensions. *Journal of Differential Equations*, 261, 2670-2701, 2016.

[33] W.-A. Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Rational Mech. Anal*, 172, 47–266, 2004.

[34] Y. Zeng. Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation. *Arch. Rational Mech. Anal*, 150, 225-279, 2004.

(T. Crin-Barat) Chair for Dynamics, Control, and Numerics (Alexander von Humboldt Professorship), Department of Data Science, Friedrich-Alexander University Erlangen-Nuremberg, Erlangen, Germany

Email address: timothee.crin-barat@fau.de

(L-Y. Shou) Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 211106, P. R. China

Email address: shoulingyun11@gmail.com

(E. Zuazua) Friedrich-Alexander-Universität Erlangen-Nürnberg, Department of Data Science, Chair for Dynamics, Control and Numerics (Alexander von Humboldt Professorship), Cauerstr. 11, 91058 Erlangen, Germany.

Chair of Computational Mathematics, Fundación Deusto, Avenida de las Universidades, 24, 48007 Bilbao, Basque Country, Spain.

Universidad Autónoma de Madrid, Departamento de Matemáticas, Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain.

Email address: enrique.zuazua@fau.de