LARGE TIME ASYMPTOTICS FOR PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS WITHOUT FOURIER ANALYSIS: APPLICATION TO THE NONLINEARLY DAMPED P-SYSTEM

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Abstract. A new framework to obtain time-decay estimates for partially dissipative hyperbolic systems set on the real line is investigated. Under the classical Shizuta-Kawashima (SK) stability condition, it is known that the solutions of these systems decay exponentially in time for high frequencies and polynomially for low ones. This allows to derive a sharp description of the space-time decay of solutions for large time. However, such analysis relies heavily on the use of the Fourier transform that we avoid here to prove new asymptotic results in the linear and nonlinear settings.

First, inspired by the *hyperbolic hypocoercivity* approach developed by Beauchard and Zuazua in [2], we recover the natural time-decay estimates under the Kalman rank condition, without employing Fourier analysis and without assuming L^1-type conditions on the initial data. Then, combining this new approach with space-weighted estimates, we establish enhanced time-decay rates under algebraic weighted integrability conditions on the initial data. Moreover, this method enables us to prove new results that cannot be easily obtained through Fourier analysis, namely, logarithmic time-decay rates for the solution of the nonlinearly damped p-system.

1. Introduction

In this paper, we study the long time behavior of partially dissipative hyperbolic systems which take the form

$$\partial_t U + A(U)\partial_x U = -BU, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

where $U = U(t,x) \in \mathbb{R}^n$ $(n \geq 2)$ is the unknown, A is a smooth matrix-valued symmetric function, and B is a $n \times n$ matrix. System (1.1) typically governs non-equilibrium processes in physics for media with hyperbolic response, and also arises in the numerical simulation of conservation laws by relaxation schemes (see [16, 31, 38] and references therein).

Here we assume that (1.1) has a *partially dissipative structure*: the matrix B takes the form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \tag{1.2}$$

with D a $n_2 \times n_2$ matrix $(1 \leq n_2 \leq n)$ satisfying the *strong dissipativity condition*: there exists a constant $\kappa > 0$ such that

$$(DX, X) \geq \kappa |X|^2, \qquad \forall X \in \mathbb{R}^{n_2}. \tag{1.3}$$

As an application, we will consider the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\lambda \rho u, \end{cases} \tag{1.4}$$

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where $\rho = \rho(x,t) \geq 0$ denotes the fluid density function, $u = u(x,t) \in \mathbb{R}$ stands for the fluid velocity, $P(\rho)$ is the pressure function, and the friction coefficient $\lambda > 0$ is assumed to be constant. For the pressure law $P(\rho) = A\rho^\gamma$, with $A > 0$ and $\gamma > 1$, a standard symmetrization procedure (see [1, Chapter 4, p.171-172]) allows to rewrite System (1.4) into the symmetric form (1.1):

$$\begin{cases} \partial_t c + u\partial_x c + \dfrac{\gamma - 1}{2} c\partial_x u = 0, \\ \partial_t u + u\partial_x u + \dfrac{\gamma - 1}{2} c\partial_x c = -\lambda u, \end{cases} \tag{1.5}$$

where $c = \sqrt{\frac{\partial P(\rho)}{\partial \rho}}$ corresponds to the sound speed. System (1.4) describes compressible gas flows passing through porous media and can be interpreted as a relaxation approximation (as $\lambda \to \infty$ and under a diffusive scaling, see [26, 5, 6, 36]) of the porous medium equation describing fluid flow, heat transfer or diffusion [30].

The partially dissipative nature of (1.1)-(1.2) does not play a role when studying its local well-posedness [22, 28], but is crucial to justify global well-posedness and time-asymptotic results. For $B = 0$, (1.1) reduces to a system of hyperbolic conservation laws, and it is well-known that for smooth initial data, there exists local-in-time solutions to (1.1)([17, 22, 28]) that may develop singularities (shock waves) in finite time ([9, 19]). On the other hand, when $\text{rank}(B) = n$, Li [21] justified the existence of global-in-time solution that are exponentially damped. In our partially dissipative setting (1.2), at first sight, the dissipative term BU does not provide enough time-decay information to justify the global well-posedness since $\text{rank}(B) < n$ implies that the dissipation does not affect every components of the solution.

The key ingredient to avoid the formation of shock waves and establish global-in-time results is to capture the interactions between the dissipative matrix B and the hyperbolic matrix $\bar{A} = A(\bar{U})$, corresponding to the linearisation of $A(U)$ around a constant equilibrium $\bar{U}$ satisfying $B\bar{U} = 0$. It is well-known that if the couple $(\bar{A}, B)$ satisfies the Kalman rank condition (cf. Lemma 2.1), then one can justify the existence and uniqueness of global-in-time solutions being small perturbations of constant equilibria [29, 37]. Additionally, it is known that the high frequencies of the solutions are exponentially damped while at low frequencies the solutions behave like the heat equation (cf. [2]). For this reason, it is very common to study partially dissipative systems by means of Fourier analysis.

Our goal here is to study the asymptotic behavior of these partially dissipative structure, without employing the Fourier transform. There are multiple reasons for this: Fourier analysis is not easily applied to equations set on bounded domains, it can make it harder to extract useful properties from nonlinear terms and to handle space-dependent matrices, and it is not well-suited to analyse numerical schemes with non-uniform meshes. Therefore, in order to obtain new results in these contexts, we develop a method to study the long-time behavior of partially dissipative systems without frequency-based tools.

First, we recover the natural decay estimates for a linear version of (1.1)-(1.2) and, coupled with a weighted energy approach, we derive enhanced decay rates under weighted integrability conditions on the initial data. Concerning nonlinear systems, we establish asymptotic results for the compressible Euler system (1.4), via a perturbation argument. Then, our method allows us to prove a new result that cannot be easily obtained through Fourier analysis. We prove logarithmic time-decay rates for the solution of the nonlinearly damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho = -u|u|^{r-1}, \end{cases} \tag{1.6}$$

with $1 < r < 3$. For small velocities ($|u| \ll c$, where c is the sound speed), System (1.6) can be used to model gas-networks, cf. [20, eq.(1.2) p.2]. Moreover, System (1.6) can be viewed as a simplified version of the nonlinearly damped compressible Euler system, so understanding its properties is a first step towards a better understanding of the latter.

The paper is organized as follows. Section 2 presents necessary stability conditions to analyse one-dimensional partially dissipative systems and states our main results. In Section 3, we prove natural time-decay estimates for linear systems without Fourier analysis and without additional L^1-type regularity assumption on the initial data (cf. Theorem 2.2). These estimates are further improved in Sections 4-5 under additional space-weighted conditions on the initial data (cf. Theorems 2.4 and 2.5). Section 6 is devoted to the analysis of the p-system (1.6), while the nonlinear Euler system (1.4) is studied in Section 7 (cf. Theorem 7.1). Section 8 presents additional results and comments on possible extensions (multi-dimensional setting, numerical analysis) of our methods. A short litterature review on the study of System (1.1) with Fourier-based tools and technical lemmas are relegated to Appendix.

2. Stability conditions and main results

2.1. **Stability conditions for the linearized model.** Linearizing (1.1) around a constant equilibrium $\bar{U} \in \ker(B)$, the associated Cauchy problem reads

$$\begin{cases} \partial_t U + A\partial_x U + BU = 0, \\ U_0(x,t) = U_0(x), \end{cases} \tag{2.1}$$

where $A = A(\bar{U})$ is a constant symmetric matrix, and B is a constant matrix satisfying (1.2) and the dissipativity condition (1.3). To highlight the partially dissipative structure of (2.1), we decompose $U = (U_1, U_2)$ with $U_1 \in \mathbb{R}^{n_1}$ and $U_2 \in \mathbb{R}^{n_2}$. The couple (U_1, U_2) satisfies

$$\begin{cases} \partial_t U_1 + A_{1,1}\partial_x U_1 + A_{1,2}\partial_x U_2 = 0, \\ \partial_t U_2 + A_{2,1}\partial_x U_1 + A_{2,2}\partial_x U_2 = -DU_2, \end{cases} \quad \text{where } A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}. \tag{2.2}$$

The first difficulty related to partial dissipation is that standard energy estimates lead to an obvious lack of coercivity. Indeed, using the symmetry of A and the condition (1.3), one has

$$\frac{1}{2}\frac{d}{dt}\|(U_1,U_2)(t)\|_{L^2}^2 + \kappa\|U_2(t)\|_{L^2}^2 \leq 0, \tag{2.3}$$

which does not provide any time-decay information on U_1. To ensure that the partial dissipation is sufficient to justify the time-decay of the whole solution, a crucial tool is to take into account the hypocoercive nature of these partially dissipative systems. The idea behind Villani's hypocoercivity theory [32], in our setting, is that: *there may be dissipative mechanisms hidden in the interaction between the hyperbolic matrix A and the partially dissipative matrix B that allows to recover time-decay information for the non-directly dissipated component U_1.*

Let us have a closer look at this phenomenon on a toy-model, namely, the linearly damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho = -u. \end{cases} \tag{2.4}$$

Again, standard energy estimates leads to

$$\frac{1}{2}\frac{d}{dt}\|(\rho,u)(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 = 0.$$

To overcome the lack of coercivity, one can consider the Lyapunov functional:

$$\mathcal{L}_1(t) = \|(\rho,u,\partial_x\rho,\partial_x u)(t)\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} u\partial_x\rho\, dx. \tag{2.5}$$

Differentiating (2.5) in time and employing Cauchy-Schwarz's and Young's inequalities, one easily reaches

$$\frac{d}{dt}\mathcal{L}_1(t)+\|(u,\partial_x u)(t)\|_{L^2}^2+\|\partial_x\rho(t)\|_{L^2}^2\leq 0, \tag{2.6}$$

and since $\mathcal{L}_1(t)\sim\|(\rho,u,\partial_x\rho,\partial_x u)(t)\|_{L^2}^2$, one can now hope to recover the asymptotic behavior of the solution to System (2.4) as we recovered time-decay information for ρ.

Nevertheless, some difficulties remain. One the one hand, in (2.6), the time-decay information on ρ and u are at different levels of regularity, and therefore we do not expect to derive the same decay rates for both components, since Poincaré's inequality is not available in the full space $\mathbb{R}$. On the other hand, it is not clear, without Fourier analysis, how to establish decay estimates directly from (2.6). With the Fourier transform, one can easily show that the low frequencies decay polynomially (as the solutions of the heat equation on $\mathbb{R}$) and the high ones exponentially as in (2.8)-(2.9). Here, as we avoid frequency-based tools, we expect to recover only the weakest behavior of the two regimes: polynomial decay.

Concerning general linear partially dissipative systems of the form (2.1), one has to generalize the functional (2.5) to derive the desired asymptotic behavior. In [2], Beauchard and Zuazua construct functionals generating dissipation for the whole solution under a stability condition on the matrices A and B. The following lemma is central in their argument.

Lemma 2.1 (Lemma 1 in [2]). *The following assertions are equivalent*:

- (A,B) *satisfies the Kalman rank condition* (K)*: the Kalman matrix*

$$\mathcal{K}:=\begin{pmatrix}B\\ BA\\ \cdots\\ BA^{n-1}\end{pmatrix}$$

  has the rank n.
- (A,B) *satisfies the Shizuta-Kawashima* (SK) *condition:*

$$\ker(B)\cap\{\textit{eigenvectors of } A\}=\{0\}.$$

- (A,B) *satisfies the stability condition* (SC)*: For any $y\in\mathbb{C}$,*

$$\Big(\sum_{k=0}^{n-1}|BA^k y|^2\Big)^{\frac12}\ \textit{defines a norm},$$

  and it holds that

$$\sum_{k=0}^{n-1}|BA^k y|^2\sim|y|^2.$$

In the one-dimensional setting, it is shown in [2] that the conditions from Lemma 2.1 are equivalent to the non-existence of travelling wave solutions propagating in the characteristic directions and ensures the time-decay of the solutions. Their proof relies on the time-differentiation of the Lyapunov functional

$$\mathcal{L}_\xi(t):=|\widehat{U}|^2+2\min\Big\{\frac{1}{|\xi|},|\xi|\Big\}\mathrm{Re}\sum_{k=1}^{n-1}\varepsilon_k<B(A\omega)^{k-1}\widehat{U}\cdot B(A\omega)^k\widehat{U}>, \tag{2.7}$$

where $\xi=|\xi|\omega$ and $<\cdot>$ denots the Hermitian scalar product in $\mathbb{C}^n$. For $\varepsilon_0=\frac{\kappa_0}{2}$ and suitably small coefficients ε_k $(k=1,2,...,n-1)$, we deduce from the Fourier transofrm of System (2.1) that

$$\frac{d}{dt}\mathcal{L}_\xi(t)+\min\{1,|\xi|^2\}\sum_{k=0}^{n-1}\varepsilon_k|BA^k\widehat{U}|^2\leq 0\quad\text{and}\quad\mathcal{L}_\xi(t)\sim|\widehat{U}|^2.$$

Then, under the Kalman rank condition for (A, B), Lemma 2.1 implies that

$$|\widehat{U}(\xi,t)|^2 \lesssim |\widehat{U_0}(\xi)|^2 e^{-N_* \min\{1,|\xi|^2\}t} \quad \text{with} \quad N_* := \inf\Big\{\sum_{k=0}^{n-1} \varepsilon_k |BA^k y|^2,\ y \in \mathbb{S}^{n-1}\Big\} > 0.$$

A low-high frequency splitting argument allows to conclude the classical time-decay rates

$$\|U^\ell(t)\|_{L^\infty} \leq Ct^{-1/2}\|U_0\|_{L^1}, \tag{2.8}$$

$$\|U^h(t)\|_{L^2} \leq Ce^{-\gamma t}\|U_0\|_{L^2}, \tag{2.9}$$

where C, γ are positive constants depending only on A, B, and $\widehat{U}^\ell(\xi,t) := \widehat{U}(\xi,t)\mathbb{1}_{\{|\xi|<1\}}$ and $\widehat{U}^h(\xi,t) := \widehat{U}(\xi,t)\mathbb{1}_{\{|\xi|>1\}}$.

Here, our goal is to adapt the Fourier-based *hyperbolic hypocoercive* approach of [2] to a Fourier-free framework. To that matter, inspired by the works of Beauchard and Zuazua [2], Hérau and Nier [14, 15] and Porretta and Zuazua [27], we will construct new Lyapunov functionals with suitable time weights to recover time-decay estimates.

2.2. **Main results for the linear system.** We are now in position to state our first result concerning time asymptotics of linear hyperbolic partially dissipative system (2.1).

Theorem 2.2. *Assume that (A, B) satisfies the Kalman rank condition and let $U_0 \in H^1(\mathbb{R})$. Then the solution U to System (2.1) satisfies, for all $t > 0$,*

$$\|U(t)\|_{H^1}^2 + \int_0^t (\|U_2(\tau)\|_{H^1}^2 + \|\partial_x U_1(\tau)\|_{L^2}^2)d\tau \leq C\|U_0\|_{H^1}^2, \tag{2.10}$$

and

$$\|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}\|U_0\|_{H^1}, \tag{2.11}$$

where $C > 0$ is a constant independent of time.

Remark 2.3 (On Theorem 2.2). Some remarks are in order.

- The proof of this result does not depend on frequency-based tools. This contrasts widely the results obtained in the literature for these systems (see Section 9.1) and allow us to analyse nonlinear systems, such as the nonlinear compressible Euler system with linear damping (1.4) by assuming the $H^2(\mathbb{R})$-smallness of the initial data, cf. Theorem 7.1 and Section 7.
- We are not able to recover any decay rates for the non-directly damped component U_1 at the energetic level $L^2(\mathbb{R})$ but only in $\dot{H}^1(\mathbb{R})$. This comes from the partially dissipative aspect of the system which forces us to work at a different regularity level for U_1 and U_2. This can be seen directly in the damped equation of U_2: $\partial_t U_2 + DU_2 + A_{2,2}\partial_x U_2 = -A_{2,1}\partial_x U_1$ which essentially implies that $\partial_x U_1$ satisfy similar decay rates to those of U_2.
- Compared to the classical results in the literature (see Proposition 9.2), we recover time-decay estimates without assuming L^1-type assumption on the initial data.
- As discussed in Section 8, we expect our method to provide time-decay estimates at a discrete level, by constructing numerical schemes preserving the hypocoercivity nature of our systems.

The aim of the next two results is to improve the time-decay rates, in particular to recover time-decay for the component U_1 in $L^2(\mathbb{R})$. To that matter, by virtue of a new space-weighted energy method and the Caffarelli-Kohn-Nirenberg inequality, we prove the following algebraic time-decay rates with any orders under suitable weighted Lebesgue integrability properties on the initial data.

Theorem 2.4. *Let $\mu > \frac{1}{2}$, the assumptions of Theorem 2.2 be in force, and U be the global solution to System (2.1). Assume furthermore $A_{1,1} = 0$ and (1.3) with $\kappa \geq \kappa_0$ for some positive constant κ_0. Additionally, the initial data U_0 satisfies*

$$\| |x|^\mu U_0 \|_{L^2} + \| |x|^\mu \partial_x U_0 \|_{L^2} < \infty. \tag{2.12}$$

Then the solution U of System (2.1) satisfies, for all $t > 0$,

$$\begin{cases} \|U_1(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}} X_0, \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}-\frac{1}{2}} X_0, \end{cases} \tag{2.13}$$

where $X_0 := \|U_0\|_{H^1} + \| |x|^\mu U_0 \|_{L^2} + \| |x|^\mu \partial_x U_0 \|_{L^2}$, and $C > 0$ is a constant independent of time.

The next time-decay result is obtained by assuming that the system (2.1) can be formulated as an extended damped wave system. Indeed, the unknown

$$W(x,t) = \int_{-\infty}^{x} U_1(y,t)\, dy,$$

satisfies the damped wave system

$$\partial_{tt}^2 W - A_{1,2}A_{2,1}\partial_x^2 W + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t \partial_x W + A_{1,2} D A_{1,2}^{-1} \partial_t W = 0$$

provided that $A_{1,1} = 0$. In order to take advantage of the techniques for damped wave equations and relax the weighted assumption (2.12), we need to impose additional conditions on the matrix A. Our result is stated as follows.

Theorem 2.5. *Let $\frac{1}{2} < \mu \leq 1$, the assumptions of Theorem 2.2 be in force, and U be the global solution to System (2.1). In addition, assume that $A_{1,1} = 0$, $A_{1,2}$ is invertible, D is symmetric, and $A_{1,2}A_{2,1}$ satisfies the following strong dissipative condition: there exists a constant $\kappa_1 > 0$ such that*

$$(A_{1,2}A_{2,1}X, X) \geq \kappa_1 |X|^2, \qquad X \in \mathbb{R}^{n_2}, \qquad \text{if} \quad \mu = 1, \tag{2.14}$$

$$\kappa_1 |X|^2 \leq (A_{1,2}A_{2,1}X, X) \leq |X|^2, \quad X \in \mathbb{R}^{n_2}, \quad \kappa_1 \leq 1, \quad \text{if} \quad \frac{1}{2} < \mu < 1. \tag{2.15}$$

Assume furthermore that the initial data U_0 satisfies

$$\| |x|^\mu U_{1,0} \|_{L^2} + \| |x|^{\mu - \frac{1}{2}} U_{2,0} \|_{L^2} < \infty, \tag{2.16}$$

and that the following compatibility condition holds:

$$\partial_t U_1|_{t=0} = -A_{1,2}\partial_x U_{2,0}. \tag{2.17}$$

Then the solution U to System (2.1) subject to the initial data U_0 satisfies, for all $t > 0$,

$$\|U_1(t)\|_{L^2} \leq C(1+t)^{-\mu+\frac{1}{2}} Y_0, \tag{2.18}$$

and

$$\begin{cases} \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-1} Y_0, & \text{if} \quad \mu = 1, \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\mu} \log^{\frac{1}{2}}(1+t) Y_0, & \text{if} \quad \frac{1}{2} < \mu < 1, \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\mu'} Y_0, \quad \text{for } \mu' < \mu, & \text{if} \quad \frac{1}{2} < \mu < 1, \end{cases} \tag{2.19}$$

where $Y_0 := \|U_0\|_{H^1} + \| |x|^\mu U_{1,0} \|_{L^2} + \| |x|^{\mu-\frac{1}{2}} U_{2,0} \|_{L^2}$, and $C > 0$ is a constant independent of time.

Remark 2.6 (On Theorems 2.4 and 2.5). Some comments are in order.

- Theorems 2.4 and 2.5 imply that under some space-weighted assumptions on the initial data and structural conditions on the system, the solution can achieve faster time-decay rates compared with the classical decay rates stated in Proposition 9.2.
- In the context of fluid mechanics, the condition $A_{1,1} = 0$ is natural as it is satisfied up to a Galilean change of frame.

In Section 7, using perturbative arguments, we extend the results from Theorem 2.4 and 2.5 to a nonlinear setting and obtain a results for the compressible Euler system (1.4). Such result can be extended to general partially dissipative systems (1.1) under structural conditions.

Adapting the method used to prove these theorems, we are able to study the long time behavior of a purely nonlinearly dissipated system.

2.3. **Asymptotics for the nonlinearly damped p-system.** We are now interested in the asymptotic behavior of the Cauchy problem of the non-linearly damped p-system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + |v|^{r-1}v = 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ (u,v)(x,0) = (u_0, v_0)(x), \qquad x \in \mathbb{R}, \end{cases} \tag{2.20}$$

with some constant $1 < r < 3$. In contrast with the linear damping setting, Fourier analysis tools may be hopeless when trying to capture beneficial properties from the nonlinear damping term $|v|^{r-1}v$ as it would involve a convolution product.

Therefore, we use a method close to the one developed for Theorem 2.5 and employ weighted integrability properties. Formally defining w such that $u = \partial_x w$ and $v = \partial_t w$, one recovers the nonlinearly damped wave equation

$$\partial_{tt}^2 w - \partial_x^2 w + |\partial_t w|^{r-1}\partial_t w = 0. \tag{2.21}$$

Generalizing the approach developed by Mochizuki and Motai in [24] pertaining to the study of (2.21) on the real line, we obtain the following result.

Theorem 2.7. *Let $(u_0, v_0) \in H^1(\mathbb{R})$. System (2.20) supplemented with the initial data (u_0, v_0) admits a unique global solution $(u, v) \in C(\mathbb{R}_+; H^1(\mathbb{R}))$ satisfying, for all $t > 0$,*

$$\|(u,v)(t)\|_{H^1}^2 + \int_0^t (\|v(\tau)\|_{L^{r+1}}^{r+1} + \|(\partial_x v^{\frac{r+1}{2}}, \partial_x u^{\frac{r+1}{2}})(\tau)\|_{L^2}^2) d\tau \leq C\|(u_0, v_0)\|_{H^1}^2, \tag{2.22}$$

where $C > 0$ is a constant independent of time.

Furthermore, suppose that

$$\partial_t u|_{t=0} = -\partial_x v_0, \quad u_0 \in L^1(\mathbb{R}), \quad \log^q(1 + |x|)(u_0, v_0) \in L^2(\mathbb{R}), \tag{2.23}$$

for any $q > 0$. Then the solution (u, v) has the time-decay estimates

$$\|(u,v)(t)\|_{L^2} \leq \frac{C}{\log^q(1+t)} \quad \text{for all } t > 0. \tag{2.24}$$

Remark 2.8 (On Theorem 2.7). Some remarks are in order.

- The restrictions on the index r comes naturally in our method, see inequality (6.26). In the literature, the same condition is used in the analysis of the nonlinearly damped wave equation. It is known that the solutions to the nonlinearly damped wave equation (2.21) decay in time for $1 < r < 1 + \frac{d}{2}$, where d corresponds to the dimension, and do not decay if $r > 1 + \frac{d}{2}$ (cf. [11, 25, 24] and references therein).
- One difficulty here is that we consider (2.20) on the real line $\mathbb{R}$, and therefore Poincaré-type inequalities are not available. To compensate this, we resort to weighted integrability conditions.

3. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by employing pure energy arguments and avoid using the Fourier transform. We introduce Lyapunov functional

$$\mathcal{L}(t) := \|U(t)\|_{H^1}^2 + c_0 t\|\partial_x U(t)\|_{L^2}^2 + \mathcal{I}(t), \tag{3.1}$$

where the corrector term $\mathcal{I}(t)$ is defined by

$$\mathcal{I}(t) := \sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1}U, BA^k\partial_x U)_{L^2}, \tag{3.2}$$

and the constants c_0, ε_i, $i = 1, 2, ..., k-1$, will be determined later.

3.1. Time-derivative of $\mathcal{L}$.

3.1.1. *Energy part.* Standard energy estimates for (2.2) lead to

$$\begin{aligned}
\frac{d}{dt}\|U(t)\|_{L^2}^2 + 2(DU_2, U_2)_{L^2} = 0,\\
\frac{d}{dt}\|\partial_x U(t)\|_{L^2}^2 + 2(D\partial_x U_2, \partial_x U_2)_{L^2} = 0,\\
\frac{d}{dt}(t\|\partial_x U(t)\|_{L^2}^2) + 2t(D\partial_x U_2, \partial_x U_2)_{L^2} = \|\partial_x U(t)\|_{L^2}^2,
\end{aligned}$$

which, together with the strong dissipativity condition (1.3) of D, gives

$$\frac{d}{dt}\big(\mathcal{L}(t) - \mathcal{I}(t)\big) + 2\kappa\|U_2(t)\|_{L^2}^2 + 2\kappa(1 + c_0 t)\|\partial_x U_2(t)\|_{L^2}^2 \le c_0\|\partial_x U(t)\|_{L^2}^2. \tag{3.3}$$

In (3.3), dissipative estimates for $\partial_x U$ in $L^2(\mathbb{R})$ are missing to ensure the negativity of $\frac{d}{dt}\mathcal{L}(t)$. It is recovered in the next section thanks to the corrector term $\mathcal{I}(t)$ in the Lyapunov functional (3.1).

3.1.2. *Estimates of the corrector:* Differentiating $\mathcal{I}(t)$ in time, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) + \sum_{k=1}^{n-1} \varepsilon_k \|BA^k \partial_x U(t)\|_{L^2}^2 = &-\sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1}BU, BA^k\partial_x U)_{L^2}\\
&-\sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1}U, BA^k B\partial_x U)_{L^2}\\
&-\sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1}U, BA^{k+1}\partial_{xx}^2 U)_{L^2}.
\end{aligned} \tag{3.4}$$

Thanks to Lemma 2.1, the second term on the left-hand side of (3.4) leads to time-decay information for $\partial_x U_1$. To deal with the remainder terms, we proceed as in [2, 7, 10] with some adaptations due to the lack of Fourier analysis.

Lemma 3.1 (Time-derivative of $\mathcal{I}$)**.** *For any positive constant ε_0, there exists a sequence $\{\varepsilon_k\}_{k=1,\dots,n-1}$ of small positive constants such that*

$$\frac{d}{dt}\mathcal{I}(t) + \frac{1}{2}\sum_{k=1}^{n-1} \varepsilon_k \|BA^k\partial_x U(t)\|_{L^2}^2 \le \varepsilon_0\|U_2(t)\|_{L^2}^2 + \varepsilon_0\|\partial_x U_2(t)\|_{L^2}^2. \tag{3.5}$$

Proof. To begin with, we fix a positive constant ε_0 and estimate the terms in the right-hand side of (3.4) as follows.

- The terms $\mathcal{I}_k^1 := \varepsilon_k(BA^{k-1}BU, BA^k\partial_x U)$ with $k \in \{1,\cdots,n-1\}$: Due to $BU = DU_2$ and the fact that the matrices A, D are bounded, we obtain

$$\begin{aligned}|\mathcal{I}_k^1| &\leq C\varepsilon_k\|DU_2(t)\|_{L^2}\|BA^k\partial_x U(t)\|_{L^2}\\ &\leq \frac{\varepsilon_0}{4n}\|U_2(t)\|_{L^2}^2 + \frac{C\varepsilon_k^2}{\varepsilon_0}\|BA^k\partial_x U(t)\|_{L^2}^2.\end{aligned}$$

- The term $\mathcal{I}_1^2 := \varepsilon_1(BU, BAB\partial_x U)_{L^2}$: One has

$$\begin{aligned}|\mathcal{I}_1^2| &\leq C\varepsilon_1\|DU_2(t)\|_{L^2}\|D\partial_x U_2(t)\|_{L^2}\\ &\leq \frac{\varepsilon_0}{4n}\|U_2(t)\|_{L^2}^2 + \frac{C\varepsilon_1^2}{\varepsilon_0}\|\partial_x U_2(t)\|_{L^2}^2.\end{aligned}$$

- The terms $\mathcal{I}_k^2 := \varepsilon_k(BA^{k-1}U, BA^kB\partial_x U)_{L^2}$ with $k \in \{2,\cdots,n-1\}$ if $n \geq 3$: We deduce after integrating by part that

$$\begin{aligned}|\mathcal{I}_k^2| &= \varepsilon_k|(BA^{k-1}\partial_x U, BA^kBU)_{L^2}|\\ &\leq C\varepsilon_k\|BA^{k-1}\partial_x U(t)\|_{L^2}\|BU(t)\|_{L^2}\\ &\leq \frac{\varepsilon_0}{4n}\|U_2(t)\|_{L^2}^2 + \frac{C\varepsilon_{k-1}^2}{\varepsilon_0}\|BA^{k-1}\partial_x U(t)\|_{L^2}^2.\end{aligned}$$

- The terms $\mathcal{I}_k^3 := \varepsilon_k(BA^{k-1}U, BA^{k+1}\partial_{xx}^2 U)_{L^2}$ with $k \in \{1,\cdots,n-2\}$ if $n \geq 3$: A similar argument yields

$$\begin{aligned}|\mathcal{I}_k^3| &= \varepsilon_k|(BA^{k-1}\partial_x U, BA^{k+1}\partial_x U)_{L^2}|\\ &\leq \frac{\varepsilon_{k-1}}{8}\|BA^{k-1}\partial_x U(t)\|_{L^2}^2 + \frac{C\varepsilon_k^2}{\varepsilon_{k-1}}\|BA^{k+1}\partial_x U(t)\|_{L^2}^2.\end{aligned}$$

- The term $\mathcal{I}_{n-1}^3 := \varepsilon_{n-1}(BA^{n-2}U, BA^n\partial_{xx}^2 U)_{L^2}$: Owing to the Cayley-Hamilton theorem, there exists coefficients c_ω^j $(j = 1, 2, ..., n-1)$ (that are uniformly bounded on $\mathbb{S}^{d-1}$) such that

$$A^n = \sum_{j=0}^{n-1} c^j A^j. \tag{3.6}$$

Consequently, one gets

$$\begin{aligned}|\mathcal{I}_{n-1}^3| &\leq \varepsilon_{n-1}\sum_{j=0}^{n-1}\|BA^{n-2}\partial_x U(t)\|_{L^2}\|BA^j\partial_x U(t)\|_{L^2}\\ &\leq \frac{\varepsilon_{n-2}}{8}\|BA^{n-2}\partial_x U(t)\|_{L^2}^2 + \sum_{j=1}^{n-1}\frac{C\varepsilon_{n-1}^2}{\varepsilon_{n-2}}\|BA^j\partial_x U(t)\|_{L^2}^2 + \frac{C\varepsilon_{n-1}^2}{\varepsilon_{n-2}}\|\partial_x U_2(t)\|_{L^2}^2.\end{aligned}$$

In order to absorb the right-hand side terms of $\mathcal{I}_k^1$ and $\mathcal{I}_k^2$ by the left-hand side of (3.4), we take the constant ε_k small enough so that

$$C\varepsilon_1^2 \leq \frac{\varepsilon_0^2}{8}, \qquad C\varepsilon_k^2 \leq \frac{\varepsilon_k\varepsilon_0}{8}, \qquad k = 1, 2, ..., n-1. \tag{3.7}$$

To handle the above estimates of $\mathcal{I}_k^3$ with $k = 1, 2, ..., n-2$, one may let

$$C\varepsilon_k^2 \leq \frac{1}{8}\varepsilon_{k-1}\varepsilon_{k+1}, \qquad k = 1, 2, ..., n-2 \quad \text{if } n \geq 3. \tag{3.8}$$

In addition, to handle the term $\mathcal{I}_{n-1}^3$, we assume

$$C\varepsilon_{n-1}^2 \leq \frac{1}{8}\varepsilon_j\varepsilon_{n-2}, \qquad j = 0, \cdots, n-1. \tag{3.9}$$

Clearly, the inequality (3.5) holds if we find $\varepsilon_1, \cdots, \varepsilon_{n-1}$ fulfilling (3.7) and (3.9). As in [2], one can take $\varepsilon_k = \varepsilon^{m_k}$ with some suitably small constant $\varepsilon \leq \varepsilon_0$ and $m_1, \cdots, m_{n-1}$ satisfying for some $\delta > 0$ (that can be taken arbitrarily small):

$$m_k > 1, \quad m_k \geq \frac{m_{k-1} + m_{k+1}}{2} + \delta \quad \text{and} \quad m_{n-1} \geq \frac{m_k + m_{n-2}}{2} + \delta, \quad k = 1, \cdots, n-2.$$

This concludes the proof of Lemma 3.1. □

3.2. **Decay for $\partial_x U$.** First, we fix suitably small $\varepsilon_k, k = 1, 2, ..., n-1$, such that (3.5) holds and

$$\mathcal{L}(t) \sim \|U(t)\|_{H^1}^2 + c_0 t \|\partial_x U(t)\|_{L^2}^2. \tag{3.10}$$

Combining the Lyapunov inequality (3.3), the estimate (3.5) of the corrector term and Lemma 2.1 together, we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{L}(t) + 2\kappa \|U_2(t)\|_{L^2}^2 + \kappa(1 + 2c_0 t)\|\partial_x U_2(t)\|_{L^2}^2 + \frac{\varepsilon_*}{C_K}\|\partial_x U(t)\|_{L^2}^2 \\ &\leq c_0 \|\partial_x U(t)\|_{L^2}^2 + \varepsilon_0 \|U_2(t)\|_{L^2}^2 + \varepsilon_0 \|\partial_x U_2(t)\|_{L^2}^2, \end{aligned} \tag{3.11}$$

with $\varepsilon_* := \min\{\kappa, \varepsilon_1, \varepsilon_1, ..., \varepsilon_{n-1}\}$ and $C_K > 0$ a constant depending only on (A, B) and n. In order to ensure the coercivity of (3.11), we adjust the coefficients appropriately as

$$0 < c_0 < \frac{\varepsilon_*}{2C_K}, \qquad 0 < \varepsilon_0 < \frac{\kappa}{2}$$

such that

$$\frac{d}{dt}\mathcal{L}(t) + \frac{3\kappa}{2}\|U_2(t)\|_{L^2}^2 + \kappa(\frac{1}{2} + c_0 t)\|\partial_x U_2(t)\|_{L^2}^2 + \frac{\varepsilon_*}{2C_K}\|\partial_x U(t)\|_{L^2}^2 \leq 0, \tag{3.12}$$

Therefore, by (3.10) and (3.12), we have

$$\|U(t)\|_{L^2} + (1+t)^{\frac{1}{2}}\|\partial_x U(t)\|_{L^2} \leq C\|U_0\|_{H^1}. \tag{3.13}$$

3.3. **Decay for U_2.** Taking the inner product of $(2.2)_2$ with U_2 and using the property (1.3), we get

$$\frac{d}{dt}\|U_2(t)\|_{L^2}^2 + 2\kappa\|U_2(t)\|_{L^2}^2 \lesssim \|\partial_x U(t)\|_{L^2}\|U_2(t)\|_{L^2}. \tag{3.14}$$

Dividing the above inequality (3.14) by $\sqrt{\|U_2(t)\|_{L^2}^2 + \varepsilon}$, employing Grönwall's inequality and then letting $\varepsilon \to 0$, we have

$$\|U_2(t)\|_{L^2} \lesssim e^{-\kappa t}\|U_{2,0}\|_{L^2} + \int_0^t e^{-\kappa(t-\tau)}\|\partial_x U(\tau)\|_{L^2} dx. \tag{3.15}$$

Together with the time-decay estimates (3.13) of $\partial_x U$, this leads to

$$\|U_2(t)\|_{L^2} \leq e^{-\kappa t}\|U_{2,0}\|_{L^2} + \int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-\frac{1}{2}}\, d\tau \|U_0\|_{H^1} \lesssim (1+t)^{-\frac{1}{2}}\|U_0\|_{H^1},$$

which concludes the proof of Theorem 2.2.

4. Faster time-decay rates: space-weighted energy method

The purpose of this section is to capture the time-decay rates of U_1 in $L^2(\mathbb{R})$. Our method allows to recover strong decay rates compared to the $(1+t)^{-\frac{1}{4}}$ decay rates obtained in the works [3, 7]. In these references, inspired by the work of Matsumura and Nishida [23], the authors assume L^1-in-space regularity on the initial data so as to recover decay in low frequencies. Here, to avoid the use of the Fourier transform, we rely on weighted Lebesgue integrability of the initial data and perform weighted estimates by taking advantage of the Caffarelli-Kohn-Nirenberg inequality.

Lemma 4.1. *Let U be the solution to the Cauchy problem (2.1) from Theorem 2.2. There exists a constant $\widetilde{X}_0$ such that if*

$$\widetilde{X}_0 := \sup_{t>0} \||x|^\mu U_1(t)\|_{L^2} < \infty, \qquad \mu > 0, \tag{4.1}$$

then for all $t > 0$, the following time-decay estimates hold:

$$\begin{cases} \|U_1(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}), \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}-\frac{1}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}). \end{cases} \tag{4.2}$$

Proof. Our starting point is the inequality (3.12), we prove that thanks to the weighted integrability condition (4.1), it can be rewritten so that we can apply Lemma 9.5. From the Caffarelli-Kohn-Nirenberg inequality (9.3), we have

$$\|U_1(t)\|_{L^2} \lesssim \|\partial_x U_1(t)\|_{L^2}^{\frac{\mu}{1+\mu}} \||x|^\mu U_1(t)\|_{L^2}^{\frac{1}{1+\mu}}, \tag{4.3}$$

which together with (4.1) yields

$$\|\partial_x U_1(t)\|_{L^2}^2 \gtrsim (\widetilde{X}_0 + \varepsilon)^{-\frac{2}{\mu}} \|U_1(t)\|_{L^2}^{2+\frac{2}{\mu}}. \tag{4.4}$$

for any $\varepsilon > 0$. On the other hand, due to (2.11), one has

$$\|U_2(t)\|_{L^2}^2 \gtrsim (\varepsilon + \|U_0\|_{H^1})^{-\frac{2}{\mu}} \|U_2(t)\|_{L^2}^{2+\frac{2}{\mu}}, \qquad \|\partial_x U(t)\|_{L^2}^2 \gtrsim (\varepsilon + \|U_0\|_{H^1})^{-\frac{2}{\mu}} \|\partial_x U(t)\|_{L^2}^{2+\frac{2}{\mu}}. \tag{4.5}$$

Thus, it follows from (3.12) and (4.4)-(4.5) that

$$\frac{d}{dt}(\mathcal{L}_*(t) + c_0 t\|\partial_x U(t)\|_{L^2}^2) + (\varepsilon + \widetilde{X}_0 + \|U_0\|_{H^1})^{-\frac{2}{\mu}} \mathcal{L}_*(t)^{2+\frac{2}{\mu}} + \|\partial_x U(t)\|_{L^2}^2 \lesssim 0, \tag{4.6}$$

with

$$\mathcal{L}_*(t) := \mathcal{L}(t) - c_0 t\|\partial_x U(t)\|_{L^2}^2 \sim \|U(t)\|_{H^1}^2.$$

Choosing the constant c sufficiently small, applying Lemma 9.5 to the differential inequality (4.6) and letting $\varepsilon \to 0$, we conclude that

$$\|U(t)\|_{H^1} \lesssim t^{-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}), \qquad \|\partial_x U(t)\|_{L^2} \lesssim t^{-\frac{1}{2}-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}).$$

Combing these two estimates and (2.10), we obtain the time-decay estimates for U_1 and $\partial_x U$ in (4.2).

Finally, (3.15) together with (4.2) leads to

$$\begin{aligned} \|U_2(t)\|_{L^2} &\leq e^{-\kappa t}\|U_{2,0}\|_{L^2} + (\widetilde{X}_0 + \|U_0\|_{H^1}) \int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-\frac{1}{2}-\frac{\mu}{2}}\, d\tau \\ &\lesssim (1+t)^{-\frac{1}{2}-\frac{\mu}{2}}(\widetilde{X}_0 + \|U_0\|_{H^1}) \end{aligned}$$

which concludes the proof of Lemma 4.1. □

In the next lemma we justify that the assumption (4.1) from Lemma 4.1 holds under weighted integrability condition on the initial data.

Lemma 4.2. *Let U be the solution to the Cauchy problem (2.1) supplemented with the initial data U_0. Then, under the assumptions of Theorem 2.4 and $\mu > \frac{1}{2}$, for all $t > 0$, it holds that*

$$\||x|^\mu U(t)\|_{L^2}^2 + \||x|^\mu \partial_x U(t)\|_{L^2}^2 + \int_0^t (\||x|^\mu U_2(\tau)\|_{L^2}^2 + \||x|^\mu \partial_x U(\tau)\|_{L^2}^2)\, d\tau \leq C X_0^2, \tag{4.7}$$

with $X_0 := \|U_0\|_{H^1} + \||x|^\mu U_0\|_{L^2} + \||x|^\mu \partial_x U_0\|_{L^2}$.

Proof. To recover weighted energy estimates, we adapt the hypocoercive approach from the Section 3 to a weighted framework by taking advantage of the Caffarelli–Kohn–Nirenberg inequality (9.4). The proof is split into three steps.

- *Step 1: Space-weighted estimates of U.*

We first derive the weighted estimates for U_2. In what follows, $\widetilde{C}>0$ denotes a suitable large constant independent of time and the matrix D. Taking the $L^2(\mathbb{R})$-inner product of $(2.2)_2$ with $|x|^{2\mu}U_2$ and using Young's inequality yields

$$\begin{aligned}&\frac{d}{dt}\||x|^{\mu}U_2(t)\|_{L^2}^2+2\kappa\||x|^{\mu}U_2(t)\|_{L^2}^2\\ &=2\int_{\mathbb{R}}(-|x|^{2\mu}A_{2,2}\partial_xU_2U_2-|x|^{2\mu}A_{2,1}\partial_xU_1U_2)\,dx\\ &\leq\frac{\kappa}{2}\||x|^{\mu}U_2(t)\|_{L^2}^2+\frac{\widetilde{C}}{\kappa}\||x|^{\mu}\partial_xU(t)\|_{L^2}^2.\end{aligned}\tag{4.8}$$

where κ is given by the strong dissiativity of D in (1.3). Before estimating U_1, we notice that the Caffarelli-Kohn-Nirenberg inequality (9.4) gives

$$\||x|^{\mu-1}U_1(t)\|_{L^2}\leq\frac{2\mu-1}{2}\||x|^{\mu}\partial_xU_1(t)\|_{L^2}.\tag{4.9}$$

Thence, it follows from $(2.2)_2$ with $A_{1,1}=0$ and (4.9) that

$$\begin{aligned}\frac{d}{dt}\||x|^{\mu}U_1(t)\|_{L^2}^2&=2\int_{\mathbb{R}}|x|^{2\mu}A_{1,2}\partial_xU_2U_1\,dx\\ &=2\int_{\mathbb{R}}(\partial_x|x|^{2\mu}A_{1,2}U_2U_1+|x|^{2\mu}A_{1,2}U_2\partial_xU_1)\,dx\\ &\leq\widetilde{C}\||x|^{\mu}U_2(t)\|_{L^2}(\||x|^{\mu-1}U_1(t)\|_{L^2}+\||x|^{\mu}\partial_xU_1(t)\|_{L^2})\\ &\leq\frac{\kappa}{2}\||x|^{\mu}U_2(t)\|_{L^2}^2+\frac{\widetilde{C}}{\kappa}\||x|^{\mu}\partial_xU_1(t)\|_{L^2}^2.\end{aligned}$$

This together with (4.8) yields

$$\begin{aligned}&\||x|^{\mu}U(t)\|_{L^2}^2+\kappa\int_0^t\||x|^{\mu}U_2(\tau)\|_{L^2}^2\,d\tau\\ &\leq\||x|^{\mu}U_0\|_{L^2}^2+\frac{\widetilde{C}}{\kappa}\int_0^t\||x|^{\mu}\partial_xU_1(\tau)\|_{L^2}^2\,d\tau.\end{aligned}\tag{4.10}$$

In addition, it is necessary to derive weighted estimate for BU which is useful for the cross estimates in Step 3. From (2.2), we have

$$\frac{d}{dt}\||x|^{\mu}BU(t)\|_{L^2}^2+2(|x|^{\mu}BBU,|x|^{\mu}BU)_{L^2}=-2(|x|^{\mu}BA\partial_xU,|x|^{\mu}BU)_{L^2}.\tag{4.11}$$

Using the partially dissipative structure of B (1.2)-(1.3), we obtain

$$2(|x|^{\mu}BBU,|x|^{\mu}BU)_{L^2}=2(|x|^{\mu}DDU_2,|x|^{\mu}DU_2)_{L^2}\geq2\kappa\||x|^{\mu}DU_2\|_{L^2}^2.$$

And Young inequality leads to

$$2(|x|^{\mu}BA\partial_xU,|x|^{\mu}BU)_{L^2}\leq\frac{\widetilde{C}}{\kappa}\||x|^{\mu}BA\partial_xU(t)\|_{L^2}^2+\kappa\||x|^{\mu}DU_2\|_{L^2}^2.$$

Thus, integration of (4.11) in time gives

$$\||x|^{\mu}BU(t)\|_{L^2}^2+\kappa\||x|^{\mu}DU_2(t)\|_{L^2}^2\leq\||x|^{\mu}BU_0\|_{L^2}^2+\frac{\widetilde{C}}{\kappa}\||x|^{\mu}BA\partial_xU(t)\|_{L^2}^2.\tag{4.12}$$

- *Step 2: Space-weighted estimates of ∂_xU.*

Differentiating (2.1) with respect to x and taking the $L^2(\mathbb{R})$-inner product of the resulting system with $|x|^{2\mu}\partial_x U$, we get

$$\frac{d}{dt}\||x|^\mu\partial_x U(t)\|_{L^2}^2 + 2\kappa\||x|^\mu\partial_x U_2(t)\|_{L^2}^2 = 4\mu\int_{\mathbb{R}}|x|^{2\mu-1}A\partial_x U\partial_x U\,dx. \tag{4.13}$$

For a constant $\eta_1\in(0,1)$ to be chosen later, since $\mu>\frac{1}{2}$, a use of Young's inequality yields

$$4\mu\int_{\mathbb{R}}|x|^{2\mu-1}A\partial_x U\partial_x U\,dx \le \eta_1\||x|^\mu\partial_x U(t)\|_{L^2}^2 + \frac{\widetilde{C}}{\eta_1}\|\partial_x U(t)\|_{L^2}^2.$$

Thus, integrating in time (4.13), we obtain

$$\begin{aligned}
&\||x|^\mu\partial_x U(t)\|_{L^2}^2 + 2\kappa\int_0^t\||x|^\mu\partial_x U_2(\tau)\|_{L^2}^2\,d\tau\\
&\le \||x|^\mu\partial_x U_0\|_{L^2}^2 + \eta_1\int_0^t\||x|^\mu\partial_x U(\tau)\|_{L^2}^2\,d\tau + \frac{\widetilde{C}}{\eta_1}\int_0^t\|\partial_x U(\tau)\|_{L^2}^2\,d\tau\\
&\le \||x|^\mu\partial_x U_0\|_{L^2}^2 + \eta_1\int_0^t\||x|^\mu\partial_x U(\tau)\|_{L^2}^2\,d\tau + \frac{\widetilde{C}}{\eta_1}\|U_0\|_{H^1}^2,
\end{aligned}\tag{4.14}$$

where we used (2.11).

As in Section 3, the main difficulty is now to recover weighted time-decay information on $\partial_x U_1$. Again, we follow the hyperbolic hypocoercive approach.

- *Step 3: Cross estimates.*

To obtain $L^2(\mathbb{R}_+;\dot{H}^1(\mathbb{R}))$-estimate for $\partial_x U$ with the weight $|x|^\mu$, according to Lemma 2.1, it is sufficient to recover the control of

$$\int_0^t\sum_{k=1}^{n-1}\||x|^\mu BA^k\partial_x U(\tau)\|_{L^2}^2 d\tau.$$

To this matter, let constants $\widetilde{\varepsilon}_i$, $i=1,2,...,k-1$ to be chosen later. Direct computations on system (2.1) lead to

$$\frac{d}{dt}\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^k\partial_x U)_{L^2} + \sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2 = \mathcal{R}, \tag{4.15}$$

with

$$\begin{aligned}
\mathcal{R} := &-\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}BU,|x|^\mu BA^k\partial_x U)_{L^2}\\
&-\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^{k+1}\partial_{xx}^2 U)_{L^2}\\
&-\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U,|x|^\mu BA^k B\partial_x U)_{L^2}.
\end{aligned}$$

To control the remainder term $\mathcal{R}$, we use arguments similar to those used for the proof of the lemma 3.1.

- For the terms $\widetilde{\mathcal{I}}_k^1 := \widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}BU,|x|^\mu BA^k\partial_x U)$ with $k\in\{1,\cdots,n-1\}$: We get

$$|\widetilde{\mathcal{I}}_k^1| \le \widetilde{C}\||x|^\mu DU_2(t)\|_{L^2}^2 + \widetilde{\varepsilon}_k^2\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2.$$

- The term $\widetilde{\mathcal{I}}_1^2 := \widetilde{\varepsilon}_1(|x|^\mu BU,|x|^\mu BAB\partial_x U)$: We have

$$\begin{aligned}
|\widetilde{\mathcal{I}}_1^2| &\le \widetilde{C}\widetilde{\varepsilon}_1\||x|^\mu DU_2(t)\|_{L^2}\||x|^\mu D\partial_x U_2(t)\|_{L^2}\\
&\le \widetilde{C}\||x|^\mu DU_2(t)\|_{L^2}^2 + \widetilde{\varepsilon}_1^2\||x|^\mu D\partial_x U_2(t)\|_{L^2}^2.
\end{aligned}$$

- The terms $\widetilde{\mathcal{I}}_k^2 := \widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U, |x|^\mu BA^k B\partial_x U)_{L^2}$ with $k \in \{2, \cdots, n-1\}$ if $n \geq 3$: With the aid of integration by parts and the Caffarelli-Kohn-Nirenberg inequality (9.4), there holds that

$$
\begin{aligned}
|\widetilde{\mathcal{I}}_k^2| &\leq \widetilde{\varepsilon}_k|(|x|^\mu BA^{k-1}\partial_x U, |x|^\mu BA^k BU)_{L^2}| \\
&\quad + 2\mu\widetilde{\varepsilon}_k|(|x|^{\mu-1}BA^{k-1}U, |x|^\mu BA^k BU)_{L^2}| \\
&\leq \widetilde{C}\widetilde{\varepsilon}_k(\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2} + \||x|^{\mu-1}BA^{k-1}U(t)\|_{L^2})\||x|^\mu BU(t)\|_{L^2} \\
&\leq \widetilde{C}\widetilde{\varepsilon}_k\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}\||x|^\mu DU_2(t)\|_{L^2} \\
&\leq \widetilde{\varepsilon}_{k-1}\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}^2 + \frac{\widetilde{C}\widetilde{\varepsilon}_k^2}{\widetilde{\varepsilon}_{k-1}}\||x|^\mu DU_2(t)\|_{L^2}^2.
\end{aligned}
$$

- The terms $\widetilde{\mathcal{I}}_k^3 := \widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U, |x|^\mu BA^{k+1}\partial_{xx}^2 U)_{L^2}$ with $k \in \{1, \cdots, n-2\}$ if $n \geq 3$: Similar arguments to those used for $\widetilde{\mathcal{I}}_k^2$ yields

$$
\begin{aligned}
|\widetilde{\mathcal{I}}_k^3| &\leq \widetilde{\varepsilon}_k|(|x|^\mu BA^{k-1}\partial_x U, |x|^\mu BA^{k+1}\partial_x U)_{L^2}| \\
&\quad + 2\mu\widetilde{\varepsilon}_k|(|x|^{\mu-1}BA^{k-1}U, |x|^\mu BA^{k+1}\partial_x U)_{L^2}| \\
&\leq \frac{\widetilde{\varepsilon}_{k-1}}{8}\||x|^\mu BA^{k-1}\partial_x U(t)\|_{L^2}^2 + \frac{\widetilde{C}\widetilde{\varepsilon}_k^2}{\widetilde{\varepsilon}_{k-1}}\||x|^\mu BA^{k+1}\partial_x U(t)\|_{L^2}^2.
\end{aligned}
$$

- The term $\widetilde{\mathcal{I}}_{n-1}^2 := \widetilde{\varepsilon}_{n-1}(|x|^\mu BA^{n-2}U, |x|^\mu BA^n\partial_{xx}^2 U)_{L^2}$: From the Caffarelli-Kohn-Nirenberg inequality (9.4) and (3.6) due to the Cayley-Hamilton theorem, we have

$$
\begin{aligned}
|\widetilde{\mathcal{I}}_{n-1}^3| &\leq \widetilde{\varepsilon}_{n-1}\sum_{j=0}^{n-1}\||x|^\mu BA^{n-2}\partial_x U(t)\|_{L^2}\||x|^\mu BA^j\partial_x U(t)\|_{L^2} \\
&\quad + 2\mu\widetilde{\varepsilon}_{n-1}\sum_{j=0}^{n-1}\||x|^{\mu-1}BA^{n-2}U(t)\|_{L^2}\||x|^\mu BA^j\partial_x U(t)\|_{L^2} \\
&\leq \frac{\widetilde{\varepsilon}_{n-2}}{8}\||x|^\mu BA^{n-2}\partial_x U(t)\|_{L^2}^2 + \sum_{j=1}^{n-1}\frac{\widetilde{C}\widetilde{\varepsilon}_{n-1}^2}{\widetilde{\varepsilon}_{n-2}}\||x|^\mu BA^j\partial_x U(t)\|_{L^2}^2 \\
&\quad + \frac{\widetilde{C}\widetilde{\varepsilon}_{n-1}^2}{\widetilde{\varepsilon}_{n-2}}\||x|^\mu D\partial_x U_2(t)\|_{L^2}^2.
\end{aligned}
$$

Therefore, we are able to find a small positive constant sequence $\{\widetilde{\varepsilon}_k\}_{k=1,\ldots,n-1}$ satisfying

$$
\begin{cases}
\widetilde{\varepsilon}_k \leq \frac{1}{8}, & k = 1, 2, ..., n-1, \\
8\widetilde{C}\widetilde{\varepsilon}_k^2 \leq \widetilde{\varepsilon}_{k-1}\widetilde{\varepsilon}_{k+1}, & k = 1, 2, ..., n-2, \quad \text{if } n \geq 3, \\
8\widetilde{C}\widetilde{\varepsilon}_{n-1}^2 \leq \widetilde{\varepsilon}_{n-2}\widetilde{\varepsilon}_j, & j = 1, \cdots, n-1.
\end{cases}
\tag{4.16}
$$

such that the term $\mathcal{R}$ can be bounded by

$$
\mathcal{R} \leq \frac{1}{2}\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2 + \widetilde{C}\||x|^\mu DU_2(t)\|_{L^2}^2 + C\||x|^\mu D\partial_x U_2(t)\|_{L^2}^2.
$$

This together with (4.12) and (4.15) leads to

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k\||x|^\mu BA^k\partial_x U(t)\|_{L^2}^2\,d\tau \\
&\leq -\sum_{k=1}^{n-1}\widetilde{\varepsilon}_k(|x|^\mu BA^{k-1}U, |x|^\mu BA^k\partial_x U)_{L^2}\Big|_0^t + \widetilde{C}\||x|^\mu BU_0\|_{L^2}^2 + \widetilde{C}\int_0^t\||x|^\mu D\partial_x U_2(\tau)\|_{L^2}^2\,d\tau \\
&\quad + \frac{\widetilde{C}}{\kappa}\int_0^t\||x|^\mu BA\partial_x U(\tau)\|_{L^2}^2 d\tau.
\end{aligned}
$$

Since $\widetilde{\varepsilon}_k$ and $\widetilde{C}$ are independent of B, letting $\kappa \geq \kappa_0 := \frac{4\widetilde{C}}{\varepsilon_1}$ and using (4.10) and (4.14) we get

$$\begin{aligned}
&\frac{1}{4}\int_0^t \sum_{k=1}^{n-1} \widetilde{\varepsilon}_k \||x|^\mu BA^k \partial_x U(t)\|_{L^2}^2 \, d\tau \\
&\quad \leq \sum_{k=1}^{n-1} \widetilde{\varepsilon}_k (|x|^\mu BA^{k-1}U, |x|^\mu BA^k \partial_x U)_{L^2}\Big|_0^t + C\||x|^\mu BU_0\|_{L^2}^2 + C\int_0^t \||x|^\mu D\partial_x U_2(\tau)\|_{L^2}^2 \, d\tau,
\end{aligned}$$

which, together with Lemma 2.1, (4.10) and (4.14), leads to

$$\begin{aligned}
&\int_0^t \||x|^\mu \partial_x U(\tau)\|_{L^2}^2 \, d\tau \\
&\quad \lesssim \||x|^\mu U_0\|_{L^2}^2 + \||x|^\mu \partial_x U_0\|_{L^2}^2 \\
&\qquad + \eta_1^{\frac{1}{2}} \||x|^\mu U(t)\|_{L^2}^2 + \frac{1}{\eta_1^{\frac{1}{2}}} \||x|^\mu \partial_x U(t)\|_{L^2}^2 + \int_0^t \||x|^\mu \partial_x U_2(\tau)\|_{L^2}^2 d\tau \\
&\quad \lesssim \||x|^\mu U_0\|_{L^2}^2 + (1 + \frac{1}{\eta_1^{\frac{3}{2}}}) \||x|^\mu \partial_x U_0\|_{L^2}^2 + \eta_1^{\frac{1}{2}} \int_0^t \||x|^\mu \partial_x U(\tau)\|_{L^2}^2 \, d\tau.
\end{aligned}$$

Choosing η_1 suitably small, we obtain (4.7). □

Proof of Theorem 2.4: By virtue of Lemma 4.2, the $L^2(\mathbb{R})$-norm of $|x|^\mu U_1$ is uniformly bounded with respect to $t > 0$. Therefore, we are able to employ Lemma 4.1 to show the time-decay estimate (2.13), which completes the proof of Theorem 2.4.

5. Faster time-decay rates: wave formulation method

In this section we prove Theorem 2.5. The key ingredient in the proof of the Theorem 2.5 is the introduction of the unknown

$$W(x,t) = \int_{-\infty}^x U_1(y,t)\, dy, \tag{5.1}$$

which satisfies the following damped wave formulation

$$\partial_{tt}^2 W - A_{1,2}A_{2,1}\partial_x^2 W + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t\partial_x W + A_{1,2}DA_{1,2}^{-1}\partial_t W = 0. \tag{5.2}$$

Indeed, since $A_{1,1} = 0$ in this section, integrating $(2.2)_1$ over $(-\infty, x)$, we obtain

$$\partial_t W + A_{1,2}U_2 = 0, \tag{5.3}$$

and differentiating in time the above system and making use of $(2.2)_2$, we get

$$\partial_{tt}^2 W - A_{1,2}A_{2,1}\partial_x U_1 - A_{1,2}A_{2,2}\partial_x U_2 - A_{1,2}DU_2 = 0. \tag{5.4}$$

Combining (5.1), (5.3) and (5.4) together, we have (5.2).

Note that $\partial_x W = U_1$ and $|\partial_t W| \sim |U_2|$, and therefore we can exhibit $L^2(\mathbb{R})$-decay rates for U if we establish decay estimates of the wave energy $\|(\partial_t W, \partial_x W)(t)\|_{L^2}^2$. In the following lemma we establish time-space weighted energy estimates for W.

Lemma 5.1. *Let W be defined by (5.1). Then under the assumptions of Theorem 2.5, for all $t > 0$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}} (1+t+|x|)^{2\mu-1}(|\partial_t W|^2 + |\partial_x W|^2)\, dx \\
&\quad + \int_0^t \int_{\mathbb{R}} \left((1+t+|x|)^{2\mu-1}|\partial_t W|^2 + (1+t+|x|)^{2\mu-2}|\partial_x W|^2\right) dx d\tau \leq CY_0^2,
\end{aligned} \tag{5.5}$$

with $Y_0 := \|U_0\|_{H^1} + \||x|^\mu U_{1,0}\|_{L^2} + \||x|^{\mu-\frac{1}{2}} U_{2,0}\|_{L^2}$.

Proof. The proof is split into the cases $\mu = 1$ and $\frac{1}{2} < \mu < 1$ separately.

- **Case 1:** $\mu = 1$.

Taking the $L^2(\mathbb{R})$-inner product of (5.2) with $(1+t+|x|)\partial_t W$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(1+t+|x|)(|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W \partial_x W)\,dx \\
&\quad + \int_{\mathbb{R}} \big((1+t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W\partial_t W - \frac{1}{2}A_{1,2}A_{2,1}\partial_x W\partial_x W\big)\,dx \\
&= \int_{\mathbb{R}} \big(\frac{1}{2}|\partial_t W|^2 - \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W + \frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)\,dx,
\end{aligned} \tag{5.6}
$$

where we used

$$
\begin{aligned}
&\int_{\mathbb{R}}(1+t+|x|)(W\partial_t W - A_{1,2}A_{2,1}\partial_x^2 W\partial_t W)\,dx \\
&\quad = \frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}(1+t+|x|)(|W|^2 + A_{1,2}A_{2,1}\partial_x W\partial_x W)\,dx \\
&\quad\quad - \frac{1}{2}\int_{\mathbb{R}}(|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W\partial_x W)dx + \int_{\mathbb{R}}\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W\,dx,
\end{aligned}
$$

and

$$
\int_{\mathbb{R}}(1+t+|x|)A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t\partial_x W\partial_t W\,dx = -\frac{1}{2}\int_{\mathbb{R}}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\,dx.
$$

In addition, taking the inner product of (5.2) with W, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\big(W\partial_t W + \frac{1}{2}A_{1,2}DA_{1,2}^{-1}|W|^2\big)\,dx + \int_{\mathbb{R}}A_{1,2}A_{2,1}\partial_x W\partial_x W\,dx \\
&= \int_{\mathbb{R}}\big(|\partial_t W|^2 + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_x W\big)\,dx.
\end{aligned} \tag{5.7}
$$

It follows from (5.6)-(5.7) that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{W}(t) + \mathcal{H}(t) \\
&= \int_{\mathbb{R}}\big(\frac{3}{2}|\partial_t W|^2 - \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W \\
&\quad + \frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W + A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_x W\big)\,dx.
\end{aligned} \tag{5.8}
$$

where $\mathcal{W}(t)$ and $\mathcal{H}(t)$ are defined by

$$
\begin{aligned}
\mathcal{W}(t) :&= \int_{\mathbb{R}}\frac{1}{2}(1+t+|x|)(|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W\partial_x W)\,dx \\
&\quad + \int_{\mathbb{R}}\big(W\partial_t W + \frac{1}{2}A_{1,2}DA_{1,2}^{-1}|W|^2\big)\,dx, \\
\mathcal{H}(t) :&= \int_{\mathbb{R}}(1+t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W\partial_t W\,dx + \int_{\mathbb{R}}\frac{1}{2}A_{1,2}A_{2,1}\partial_x W\partial_x W\,dx.
\end{aligned}
$$

Notice that $A_{1,2}DA_{1,2}^{-1}$ satisfies (1.3) since D is positive definite and the eigenvalues of $A_{1,2}DA_{1,2}^{-1}$ are the same as that of D. By the strong dissipation conditions (1.3) and (2.14), we have

$$
\mathcal{W}(t) \geq \int_{\mathbb{R}}\big((1+t+|x|)(\frac{1}{2}|\partial_t W|^2 + \kappa_1|\partial_x W|^2) + \kappa|W|^2 - C|\partial_t W|^2\big)\,dx,
$$

and

$$
\mathcal{H}(t) \geq \int_0^t\int_{\mathbb{R}}\big(\kappa(1+t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{2}|\partial_x W|^2\big)\,dxd\tau.
$$

Since $|\partial_t W|\sim |U_2|$ due to (5.3) and the fact that $A_{1,2}$ is invertible, one can estimate the right-hand side of (5.8) as follows:

$$\begin{aligned}&\int_{\mathbb{R}}\big(\frac{3}{2}|\partial_t W|^2-\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W+\frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W+A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_x W\big)\,dx\\&\leq\frac{\kappa_1}{4}\int_{\mathbb{R}}|\partial_x W|^2\,dx+C\int_{\mathbb{R}}|U_2|^2\,dx.\end{aligned}$$

The above estimates give rise to

$$\begin{aligned}&\int_{\mathbb{R}}\big((1+t+|x|)(\frac{1}{2}|\partial_t W|^2+\kappa_1|\partial_x W|^2)+\kappa|W|^2\big)\,dx\\&\quad+\int_0^t\int_{\mathbb{R}}\big(\kappa(1+\tau+|x|)|\partial_t W|^2+\frac{\kappa_1}{4}|\partial_x W|^2\big)\,dxd\tau\\&\leq\mathcal{W}(0)+C\int_{\mathbb{R}}|U_2|^2\,dx+C\int_0^t\int_{\mathbb{R}}|U_2|^2\,dxd\tau.\end{aligned}\tag{5.9}$$

Under the assumptions (2.16) and (2.17), one deduces from the Caffarelli-Kohn-Nirenberg inequality (9.4) that

$$\mathcal{W}(0)\lesssim\int_{\mathbb{R}}\big((1+|x|)|(u_0,v_0)|^2+|\int_{-\infty}^x u_0(y)\,dy|^2\big)\,dx\lesssim Y_0^2.\tag{5.10}$$

By $|\partial_t W|\sim|U_2|$ and (2.11), it holds that

$$C\int_{\mathbb{R}}|\partial_t W|^2\,dx+C\int_0^t\int_{\mathbb{R}}|\partial_t W|^2\,dxd\tau\lesssim\int_{\mathbb{R}}|U_2|^2\,dx+C\int_0^t\int_{\mathbb{R}}|U_2|^2\,dxd\tau\lesssim Y_0^2.\tag{5.11}$$

Inserting (5.10) and (5.11) into (5.9), we get (5.5) with $\mu=1$.

- **Case 2:** $\frac{1}{2}<\mu<1$.

In this case, let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a weight function to be determined later. Similarly to the case $\mu=1$, one gets

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}\varphi(t+|x|)(|\partial_t W|^2+A_{1,2}A_{2,1}\partial_x W\partial_x W)\,dx\\&\quad+\int_{\mathbb{R}}\big(\varphi(t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W\partial_t W-\frac{1}{2}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W\big)\,dx\\&=\int_{\mathbb{R}}\varphi'(t+|x|)\big(\frac{1}{2}|\partial_t W|^2-\frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W+\frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)\,dx.\end{aligned}$$

After taking the inner product of (5.2) with $\varphi'(t+|x|)W$, we verify that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\big(\varphi'(t+|x|)\partial_t WW-\varphi''(t+|x|)|W|^2+\frac{1}{2}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}WW\big)\,dx\\&\quad+\int_{\mathbb{R}}\big(\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W+\frac{1}{2}\varphi'''(t+|x|)|W|^2\big)\,dx\\&\quad-\int_{\mathbb{R}}\big(\frac{1}{2}\varphi'''(t+|x|)+\varphi''(t+|x|)\delta_0(x)\big)A_{1,2}A_{2,1}WW\,dx\\&=\int_{\mathbb{R}}\varphi'(t+|x|)|\partial_t W|^2\,dx.\end{aligned}$$

Here we have used

$$
\begin{aligned}
&\int_{\mathbb{R}} \varphi'(t+|x|)\partial_{tt}^2 W W\,dx\\
&\quad= \frac{d}{dt}\int_{\mathbb{R}} \big(\varphi'(t+|x|)\partial_t W W - \frac{1}{2}\varphi''(t+|x|)|W|^2\big)\,dx + \int_{\mathbb{R}} \big(\frac{1}{2}\varphi'''(t+|x|)|W|^2 - \varphi'(t+|x|)|\partial_t W|^2\big)\,dx,\\
&\int_{\mathbb{R}} \varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W W\,dx\\
&\quad= \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}WW\,dx - \int_{\mathbb{R}} \varphi''(t+|x|)A_{1,2}DA_{1,2}^{-1}WW\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}} \varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x^2 W W\,dx\\
&\quad= -\int_{\mathbb{R}} \Big(\big(\frac{1}{2}\varphi'''(t+|x|) + \varphi''(t+|x|)\delta_0(x)\big)A_{1,2}A_{2,1}WW + \varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W\Big)\,dx,
\end{aligned}
$$

where $\delta_0(x)$ denotes the Dirac function at 0. Gathering the previous estimates we get the following inequality

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{W}_\mu(t) + \mathcal{H}_\mu(t)\\
&= \int_{\mathbb{R}} \varphi'(t+|x|)\big(\frac{3}{2}|\partial_t W|^2 - \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W + \frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)\,dx,
\end{aligned}
\tag{5.12}
$$

with

$$
\begin{aligned}
\mathcal{W}_\mu(t) :=& \int_{\mathbb{R}} \frac{1}{2}\varphi(t+|x|)(|\partial_t W|^2 + A_{1,2}A_{2,1}\partial_x W\partial_x W)\,dx\\
&+ \int_{\mathbb{R}} \big(\varphi'(t+|x|)\partial_t W W - \frac{1}{2}\varphi''(t+|x|)|W|^2 + \frac{1}{2}\varphi'(t+|x|)A_{1,2}DA_{1,2}^{-1}WW\big)\,dx\\
\mathcal{H}_\mu(t) :=& \int_{\mathbb{R}} \big(\varphi(t+|x|)A_{1,2}DA_{1,2}^{-1}\partial_t W\partial_t W + \frac{1}{2}\varphi'(t+|x|)A_{1,2}A_{2,1}\partial_x W\partial_x W\big)\,dx,\\
&+ \int_{\mathbb{R}} \Big(\frac{1}{2}\varphi'''(t+|x|)|W|^2 - \big(\frac{1}{2}\varphi'''(t+|x|) + \varphi''(t+|x|)\delta_0(x)\big)A_{1,2}A_{2,1}WW\Big)\,dx.
\end{aligned}
$$

In order to recover the coercive estimates of $\mathcal{W}_\mu(t)$ and $\mathcal{H}_\mu(t)$, one requires that $\varphi(s)$ satisfies

$$
\varphi' > 0, \qquad \varphi'' < 0, \qquad \varphi''' > 0, \qquad \frac{1}{4}\varphi(t+|x|) \geq \frac{1}{\kappa_1}\varphi'(t+|x|). \tag{5.13}
$$

Indeed, due to the strong dissipation conditions (1.3) and (2.14) and the properties (5.13), there holds

$$
\mathcal{W}_\mu(t) \geq \int_{\mathbb{R}} \big(\frac{1}{4}\varphi(t+|x|)|\partial_t W|^2 + \kappa\varphi(t+|x|)|\partial_x W|^2 + (\frac{\kappa_1}{4}\varphi'(t+|x|) - \frac{1}{2}\varphi''(t+|x|))|W|^2\big)\,dx,
$$

and

$$
\begin{aligned}
\mathcal{H}_\mu(t) \geq& \int_{\mathbb{R}} \big(\kappa\varphi(t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{2}\varphi'(t+|x|)|\partial_x W|^2\big)\,dx\\
&+ \int_{\mathbb{R}} \frac{1}{2}\varphi'''(t+|x|)(|W|^2 - A_{1,2}A_{2,1}WW)\,dx - \kappa_1\varphi''(t)|W(0,t)|^2\\
\geq& \int_{\mathbb{R}} \big(\kappa\varphi(t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{2}\varphi'(t+|x|)|\partial_x W|^2\big)\,dx,
\end{aligned}
$$

where we have used the condition (2.15). In addition, one has

$$
\begin{aligned}
&\int_{\mathbb{R}} \varphi'(t+|x|)\big(\frac{3}{2}|\partial_t W|^2 - \frac{x}{|x|}A_{1,2}A_{2,1}\partial_x W\partial_t W + \frac{1}{2}\frac{x}{|x|}A_{1,2}A_{2,2}A_{1,2}^{-1}\partial_t W\partial_t W\big)\,dx\\
&\leq \int_{\mathbb{R}} \big(C\varphi'(t+|x|)|\partial_t W|^2 + \frac{\kappa_1}{4}\varphi'(t+|x|)|\partial_x W|^2\big)\,dx.
\end{aligned}
$$

which can be controlled by the side of (5.12) provided that

$$C\varphi'(t+|x|) \le \frac{\kappa}{2}\varphi(t+|x|). \tag{5.14}$$

In addition, under the assumptions (2.16) and (2.17), one needs $\mu > \frac{1}{2}$ and

$$\varphi(s) \sim (1+s)^{2\mu-1}, \qquad \varphi'(s) \sim (1+s)^{2\mu-2}, \tag{5.15}$$

so as to bound the initial energy $\mathcal{W}(0)$ by Y_0^2 in terms of (2.16) and the Caffarelli-Kohn-Nirenberg inequality (9.4). One can show that the function

$$\varphi(s) = (a+s)^{2\mu-1} \quad \text{with } \frac{1}{2} < \mu < 1 \text{ and some constant } a > \max\{\frac{4}{\kappa_1}, \frac{2C}{\kappa}\} \tag{5.16}$$

fulfills the conditions (5.13), (5.14) and (5.15). Therefore, integrating (5.12) over $[0,t]$, we obtain the desired inequality (5.5).

□

Proof of Theorem 2.5: In view of the estimate (5.5) obtained in Lemma 5.1 and the facts that $U_1 = \partial_x W$, $U_2 = -A_{1,2}^{-1}\partial_t W$ and that, for $x \in \mathbb{R}, t > 0$ and $\mu \ge \frac{1}{2}$,

$$(1+t)^{2\mu-1} \le (1+t+|x|)^{2\mu-1},$$

we get the $L^2(\mathbb{R})$-rate $(1+t)^{-\mu+\frac{1}{2}}$ of U in (2.18).

We now focus on recovering faster time-decay rates for $\partial_x U$ and the damped component U_2. Multiplying the Lyapunov inequality (3.12) by $t^{2\mu-1}$ for $\frac{1}{2} < \mu \le 1$, we obtain

$$\begin{aligned}&\frac{d}{dt}(t^{2\mu-1}\mathcal{L}(t)) + \frac{3\kappa}{2}t^{2\mu-1}\|U_2(t)\|_{L^2}^2 + \kappa t^{2\mu-1}(\frac{1}{2}+c_0 t)\|\partial_x U_2(t)\|_{L^2}^2 + \frac{\varepsilon_*}{2C_K}t^{2\mu-1}\|\partial_x U(t)\|_{L^2}^2\\ &\le (2\mu-1)t^{2\mu-2}\mathcal{L}(t).\end{aligned}$$

Since that $t^{2\mu-1}\mathcal{L}(t)|_{t=0} = 0$ for $\mu > \frac{1}{2}$ and choosing c small enough, we arrive at

$$\begin{aligned}&t^{2\mu-1}\|U(t)\|_{L^2}^2 + (t^{2\mu-1}+c_0 t^{2\mu})\|\partial_x U(t)\|_{L^2}^2\\ &\quad + \int_0^t \big(\tau^{2\mu-1}\|U_2(\tau)\|_{L^2}^2 + (\tau^{2\mu-1}+c_0\tau^{2\mu})\|\partial_x U_2(\tau)\|_{L^2}^2 + \tau^{2\mu-1}\|\partial_x U(\tau)\|_{L^2}^2\big)\,d\tau\\ &\lesssim \int_0^t \big(\tau^{2\mu-2}\|U(\tau)\|_{L^2}^2 + \tau^{2\mu-2}\|\partial_x U(\tau)\|_{L^2}^2\big)\,d\tau.\end{aligned} \tag{5.17}$$

We split the proof of the time-decay estimates (2.19) into two cases:

- Case 1: $\mu = 1$.

The weighted estimate (5.5) lead to

$$\int_0^t \tau^{2\mu-2}\|U(\tau)\|_{L^2}^2\,d\tau \le \int_0^t \|\partial_x W(\tau)\|_{L^2}^2\,d\tau \lesssim Y_0^2.$$

On the other hand, by (2.10), one has

$$\int_0^t \tau^{2\mu-2}\|\partial_x U(\tau)\|_{L^2}^2\,d\tau = \int_0^t \|\partial_x U(\tau)\|_{L^2}^2\,d\tau \lesssim Y_0^2.$$

Putting the above two estimates into (5.17) and using (2.10), we have $(2.19)_1$.

- Case 2: $\frac{1}{2} < \mu < 1$.

In this case, we rely on the properties of Theorem 2.2 to control the right-hand side terms. However, (2.11) does not contain any time-decay information on the component U_1 which weakens the result we obtained in this setting.

Thanks to (2.11), for $t \geq 1$, it holds that

$$\begin{aligned}\int_0^t \tau^{2\mu-2}(\|U_2(\tau)\|_{L^2}^2 + \|\partial_x U(\tau)\|_{L^2}^2)\,d\tau &\lesssim Y_0^2 \int_0^t \tau^{2\mu-2}(1+\tau)^{-1}\,d\tau \\ &\lesssim Y_0^2 \int_0^1 \tau^{2\mu-2}\,d\tau + Y_0^2 \int_1^t (1+\tau)^{2\mu-3}\,d\tau \\ &\lesssim Y_0^2.\end{aligned}$$

And for $t \geq 1$, the time-decay estimate (2.18) at hand implies

$$\begin{aligned}\int_0^t \tau^{2\mu-2}\|U_1(\tau)\|_{L^2}^2\,d\tau &\lesssim Y_0^2 \int_0^1 \tau^{2\mu-2}(1+\tau)^{-2\mu+1}\,d\tau + Y_0^2 \int_1^t \tau^{2\mu-2}(1+\tau)^{-2\mu+1}\,d\tau \\ &\lesssim Y_0^2 \int_0^1 \tau^{2\mu-2}\,d\tau + Y_0^2 \int_1^t (1+\tau)^{-1}\,d\tau \\ &\lesssim Y_0^2 \log(1+t).\end{aligned}$$

In addition, for $\mu' < \mu$ and $t \geq 1$, we have

$$\begin{aligned}\int_0^t \tau^{2\mu-2}\|U_1(\tau)\|_{L^2}^2\,d\tau &\lesssim Y_0^2 \int_0^1 \tau^{2\mu-2}\,d\tau + Y_0^2 \int_1^t (1+\tau)^{2\mu-2\mu'-1}\,d\tau \\ &\lesssim Y_0^2(1+t)^{2\mu-2\mu'}.\end{aligned}$$

Combining the above estimates, (2.10), (5.17) together, we get $(2.19)_2$ and $(2.19)_3$ for $\partial_x U$. The faster decay rates for U_2 follows from the decay rates obtained for $\partial_x U$ and (3.15). The proof of Theorem 2.5 is now complete.

6. Time decay for the p-system with nonlinear damping

In this section, we establish the logarithmic time-decay rates of the nonlinearly damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + |v|^{r-1}v = 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ (u,v)(x,0) = (u_0,v_0)(x), \qquad x \in \mathbb{R}, \end{cases} \tag{6.1}$$

where r is a constant verifying $1 < r < 3$.

6.1. Global existence and dissipation estimates.

For brevity, we omit the details concerning the local well-posedness of solutions to (6.1) subject to the initial data in $H^1(\mathbb{R})$ since it can be proved by standard iteration arguments, see e.g. [1, 3, 28, 22, 33]. To extend the local solution to a global one, we establish the uniform a-priori estimates as follows.

- L^2-estimates:

Standard energy estimates lead to

$$\|(u,v)(t)\|_{L^2}^2 + 2\int_0^t \|v(\tau)\|_{L^{r+1}}^{r+1}\,d\tau = \|(u_0,v_0)\|_{L^2}^2. \tag{6.2}$$

- H^1-estimates

From direct energy estimates in (6.1), we get

$$\frac{d}{dt}\|(\partial_x u, \partial_x v)(t)\|_{L^2}^2 + r\int_{\mathbb{R}} |v|(\partial_x v)^2\,dx = 0. \tag{6.3}$$

Here one has used

$$\int_{\mathbb{R}} \partial_x(|v|^{r-1}v)\partial_x v\,dx = \int_{\mathbb{R}} \partial_x|v|^{r-1}v\partial_x v\,dx + \int_{\mathbb{R}} |v|^{r-1}(\partial_x v)^2\,dx = r\int_{\mathbb{R}} |v|^{r-1}(\partial_x v)^2\,dx.$$

- Dissipation of $\partial_x u$ with weights.

Multiplying $(6.1)_2$ with $|v|^{r-1}\partial_x u$, we infer

$$\int_{\mathbb{R}} \partial_t v|v|^{r-1}\partial_x u\,dx + \int_{\mathbb{R}} |v|^{r-1}|\partial_x u|^2\,dx - \int_{\mathbb{R}} |v|^{2r-2}v\partial_x u\,dx = 0. \tag{6.4}$$

Similarly, from $(6.1)_1$, we get

$$\int_{\mathbb{R}} \frac{1}{r}|v|^{r-1}v\partial_x\partial_t u\,dx + \int_{\mathbb{R}} \frac{1}{r}|v|^{r-1}v\partial_x^2 v\,dx = 0. \tag{6.5}$$

By (6.4)-(6.5),the fact that $\partial_t v|v|^{r-1} = \frac{1}{r}\partial_t(|v|^{r-1}v)$ and integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{r}|v|^{r-1}v\partial_x u\,dx + \int_{\mathbb{R}} |v|^{r-1}|\partial_x u|^2\,dx - \int_{\mathbb{R}} |v|^{2r-2}v\partial_x u\,dx - \int_{\mathbb{R}} |v|^{r-1}|\partial_x v|^2\,dx = 0. \tag{6.6}$$

Defining

$$\begin{aligned}
\mathcal{W}_*(t) &:= \|(u,v,\partial_x u,\partial_x v)(t)\|_{L^2}^2 + \eta_2\int \frac{1}{r}|v|^{r-1}v\partial_x u\,dx,\\
\mathcal{H}_*(t) &:= 2\int_{\mathbb{R}} |v|^{r-1}(|v|^2 + |\partial_x v|^2)\,dx\\
&\quad + \eta_2\Big(\int_{\mathbb{R}} |v|^{r-1}|\partial_x u|^2 dx - \int_{\mathbb{R}} |v|^{2r-2}v\partial_x u\,dx - \int_{\mathbb{R}} |v|^{r-1}|\partial_x v|^2\,dx\Big),
\end{aligned}$$

we obtain from (6.2), (6.3) and (6.6) that

$$\frac{d}{dt}\mathcal{W}_*(t) + \mathcal{H}_*(t) = 0. \tag{6.7}$$

Since due to (6.2) and (6.3),

$$\|v\|_{L_t^\infty(L_x^\infty)} \lesssim \|v\|_{L_t^\infty(L_x^2)}^{\frac{1}{2}}\|\partial_x v\|_{L_t^\infty(L_x^\infty)}^{\frac{1}{2}} \lesssim 1,$$

we are able to choose a suitably small constant $\eta_2 > 0$ such that

$$\mathcal{W}_*(t) \sim \|(u,v,\partial_x u,\partial_x v)(t)\|_{L^2}^2, \qquad \mathcal{H}_*(t) \gtrsim \int_{\mathbb{R}} |v|^{r-1}(|v|^2 + |\partial_x v|^2 + |\partial_x u|^2)\,dx. \tag{6.8}$$

Integrating (6.7) over $[0,t]$ and making use of (6.8), we have

$$\|(u,v)(t)\|_{H^1}^2 + \int_0^t \big(\|v(\tau)\|_{L^{r+1}}^{r+1} + \|(\partial_x v^{\frac{r+1}{2}}, \partial_x u^{\frac{r+1}{2}})(\tau)\|_{L^2}^2\big)\,d\tau \lesssim \|(u_0,v_0)\|_{H^1}^2.$$

The above estimates enable us to prove the global existence of the solution to (6.1) with a standard bootstrap argument.

6.2. **Wave formulation.** Differentiating $(6.1)_1$ and $(6.1)_2$ with respect to t, we rewrite (6.1) into two damped wave-like equations

$$\begin{cases} \partial_{tt}^2 u - \partial_x^2 u + r|v|^{r-1}\partial_t u = 0,\\ \partial_{tt}^2 v - \partial_x^2 v + r|v|^{r-1}\partial_t v = 0. \end{cases} \tag{6.9}$$

From the equation $(6.1)_1$ it follows that

$$v = -\partial_t\int_{-\infty}^x u(y,t)\,dy, \tag{6.10}$$

from which we infer that

$$\begin{aligned} r|v|^{r-1}\partial_t u &= r\Big|\partial_t \int_{-\infty}^x u(y,t)\,dy\Big|^{r-1} \partial_x\partial_t \int_{-\infty}^x u(y,t)\,dy \\ &= \partial_x\Big(\Big|\partial_t \int_{-\infty}^x u(y,t)\,dy\Big|^{r-1} \partial_t \int_{-\infty}^x u(y,t)\,dy\Big), \end{aligned}$$

Thus, defining the new unknown w by

$$w := \int_{-\infty}^x u(y,t)\,dy$$

and integrating the equation $(6.9)_1$ over $(-\infty, x)$, we obtain the nonlinearly damped wave equation:

$$\partial_{tt}^2 w - \partial_x^2 w + |\partial_t w|^r \partial_t w = 0. \tag{6.11}$$

From $(6.1)_1$, we see that

$$\|\partial_t w(t)\|_{L^2}^2 = \|v(t)\|_{L^2}^2, \qquad \|\partial_x w(t)\|_{L^2}^2 = \|u(t)\|_{L^2}^2. \tag{6.12}$$

Thus, once we get the decay rate of $\|(\partial_t w, \partial_x w)(t)\|_{L^2}^2$, the $L^2(\mathbb{R})$-decay of (u,v) follows.

6.3. **Time-decay rate.** In this subsection, we prove Theorem 2.7 pertaining to the logarithmic time-decay rates of solutions to System (6.1). To that matter, we capture the nonlinearly dissipative structures in (6.11) by adapting the method developed by Mochizuki and Motai in [24]. The key ingredient is the $L^2(\mathbb{R})$-coercive estimates for $\partial_t w$ that we derive from the damped term $|\partial_t w|^{r-1}\partial_t w$ with suitable weights (see (6.21) below).

Let two weight functions $\varphi_1(s), \varphi_2(s)$ for $s \geq 0$ to be determined later. Taking the $L^2(\mathbb{R})$-inner product of (6.11) with $\varphi_1(t+|x|)\partial_t w$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)\,dx \\ &\quad + \int_{\mathbb{R}} \big(\varphi_1(t+|x|)|\partial_t w|^r - \frac{1}{2}\varphi_1'(t+|x|)|\partial_x w|^2\big)dx \\ &= \int_{\mathbb{R}} \frac{1}{2}\varphi_1'(t+|x|)|\partial_t w|^2\,dx. \end{aligned} \tag{6.13}$$

In addition, multiplying (6.11) by $\varphi_1'(t+|x|)w$ and integrating by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \big(\varphi_1'(t+|x|)w\partial_t w - \frac{1}{2}\varphi_1''(t+|x|)|w|^2\big)\,dx \\ &\quad + \int_{\mathbb{R}} \big(\varphi_1'(t+|x|)|\partial_x w|^2 - \varphi_1''(t+|x|)\delta_0(x)|w|^2\big)\,dx \\ &= \int_{\mathbb{R}} \big(\varphi_1'(t+|x|)|\partial_t w|^2 - \varphi_1'(t+|x|)|\partial_t w|^{r-1}\partial_t w w\big)\,dx, \end{aligned} \tag{6.14}$$

where $\delta_0(x)$ denotes the Dirac function at 0 and we have used that

$$\begin{aligned} &\int_{\mathbb{R}} \varphi_1'(t+|x|)w\partial_t^2 w\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}} \big(\varphi_1'(t+|x|)w\partial_t w - \frac{1}{2}\varphi_1''(t+|x|)|w|^2\big)\,dx \\ &\quad + \int_{\mathbb{R}} \big(\frac{1}{2}\varphi_1'''(t+|x|)|w|^2 - \int_{\mathbb{R}} \varphi_1'(t+|x|)|\partial_t w|^2\big)\,dx, \end{aligned}$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}} \varphi_1'(t+|x|) w\partial_x^2 w\, dx \\
&= \int_{\mathbb{R}} \Big(\varphi_1'(t+|x|)|\partial_x w|^2 - \big(\frac{1}{2}\varphi_1'''(t+|x|) + \varphi_1''(t+|x|)\delta_0(x)\big)|w|^2\Big)\, dx.
\end{aligned}
$$

To control the second term on the right-hand side of (6.14), one needs to capture the dissipation of $|w|^{r+1}$ with a suitable weight. To this matter, a direct calculation yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \varphi_2(t+|x|)|w|^{r+1}\, dx - \int_{\mathbb{R}} \varphi_2'(t+|x|)|w|^{r+1}\, dx \\
&= -\int_{\mathbb{R}} (r+1)\varphi_2(t+|x|)|w|^{r-1} w\partial_t w\, dx.
\end{aligned} \tag{6.15}
$$

For some small constant $\eta_3 > 0$ to be chosen later, we define

$$
\begin{aligned}
\mathcal{W}_\varphi(t) :=& \int_{\mathbb{R}} \frac{1}{2}\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)\, dx \\
&+ \eta_3 \int_{\mathbb{R}} \big(\varphi_1'(t+|x|) w\partial_t w - \frac{1}{2}\varphi_1''(t+|x|)|w|^2 + \varphi_2(t+|x|)|w|^{r+1}\big)\, dx.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}_\varphi(t) :=& \int_{\mathbb{R}} \varphi_1(\tau+|x|)|\partial_t w|^r\, dx \\
&+ \eta_3 \int_{\mathbb{R}} \big(\varphi_1'(t+|x|)|\partial_x w|^2 - \varphi_1''(t+|x|)\delta_0(x)|w|^2 - \varphi_2'(t+|x|)|w|^{r+1}\big)\, dx.
\end{aligned}
$$

Thus, from (6.13), (6.14) and (6.15), we get

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{W}_\varphi(t) + \mathcal{H}_\varphi(t) \\
&= \int_{\mathbb{R}} \big(\frac{1+\eta_3}{2}\varphi_1'(t+|x|)|\partial_t w|^2 \\
&\quad - \eta_3\varphi_1'(t+|x|)|\partial_t w|^{r-1}\partial_t w w - \eta_3(r+1)\varphi_2(\tau+|x|)|w|^{r-1} w\partial_t w\big)\, dx.
\end{aligned} \tag{6.16}
$$

In order to control $\mathcal{W}_\varphi(t), \mathcal{H}_\varphi(t)$ and derive the desired dissipation estimates, we require

$$
\varphi_1 > 0, \quad \varphi_1' > 0, \quad \varphi_1'' < 0, \quad |\varphi_1'|^2 \le C\varphi_1|\varphi_1''|, \quad \varphi_2 > 0, \quad \varphi_2' < 0. \tag{6.17}
$$

Indeed, under the condition (6.17), one has

$$
\int_{\mathbb{R}} \varphi_1'(t+|x|)|w\partial_t w|\, dx \le C\int_{\mathbb{R}} \varphi_1(t+|x|)|\partial_t w|^2\, dx - \frac{1}{4}\int_{\mathbb{R}} \varphi_1''(t+|x|)|w|^2\, dx, \tag{6.18}
$$

which implies

$$
\begin{aligned}
\mathcal{W}_\varphi(t) \ge& \int_{\mathbb{R}} \big(\frac{1}{2} - C\eta_3)\varphi_1(t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)\, dx \\
&- \frac{\eta_3}{4}\varphi_1''(t+|x|)|w|^2 + \eta_3\varphi_2(t+|x|)|w|^{r+1}\big)\, dx.
\end{aligned} \tag{6.19}
$$

And (6.17) also leads to

$$
\mathcal{H}_\varphi(t) \ge \int_{\mathbb{R}} \big(\varphi_1(t+|x|)|\partial_t w|^r + \eta_3\big(\varphi_1'(t+|x|)|\partial_x w|^2 - \eta_3\varphi_2'(t+|x|)|w|^{r+1}\big)\, dx. \tag{6.20}
$$

We now focus on the estimation of the right-hand side terms (6.16). First, a use of Young's inequality gives

$$
\begin{aligned}
&\frac{3}{2}\int_0^t\int_{\mathbb{R}}\varphi_1'(\tau+|x|)|\partial_t w|^2\,dxd\tau\\
&\leq \frac{3}{2}\Big(\int_0^t\int_{\mathbb{R}}\varphi_1(\tau+|x|)|\partial_t w|^{r+1}\,dxd\tau\Big)^{\frac{2}{r+1}}\Big(\int_0^t\int_{\mathbb{R}}\frac{|\varphi_1'(\tau+|x|)|^{\frac{r+1}{r-1}}}{\varphi_1(\tau+|x|)^{\frac{2}{r-1}}}\,dxd\tau\Big)^{\frac{r-1}{r+1}}\\
&\leq \frac{1}{4}\int_0^t\int_{\mathbb{R}}\varphi_1(\tau+|x|)|\partial_t w|^{r+1}\,dxd\tau+C\int_0^t\int_{\mathbb{R}}\frac{|\varphi_1'(\tau+|x|)|^{\frac{r+1}{r-1}}}{\varphi_1(\tau+|x|)^{\frac{2}{r-1}}}\,dxd\tau.
\end{aligned}
\tag{6.21}
$$

Similarly, we infer

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\varphi_1'(\tau+|x|)|\partial_t w|^r|w|\,dxd\tau\\
&\leq \frac{1}{4}\int_0^t\int_{\mathbb{R}}\varphi_1(\tau+|x|)|\partial_t w|^{r+1}\,dxd\tau+C\int_0^t\int_{\mathbb{R}}\frac{|\varphi_1'(\tau+|x|)|^{r+1}}{\varphi_1(\tau+|x|)^r}|w|^{r+1}\,dxd\tau.
\end{aligned}
\tag{6.22}
$$

and

$$
\begin{aligned}
&(r+1)\int_0^t\int_R\varphi_2(\tau+|x|)|w|^r|\partial_t w|\,dxd\tau\\
&\leq \frac{1}{4}\int_0^t\int_{\mathbb{R}}\varphi_1(\tau+|x|)|\partial_t w|^{r+1}\,dxd\tau+C\int_0^t\int_{\mathbb{R}}\frac{|\varphi_2(\tau+|x|)|^{r+1}}{\varphi_1(\tau+|x|)^r}|w|^{r+1}\,dxd\tau.
\end{aligned}
\tag{6.23}
$$

Let $\eta_3=\frac{1}{4C}$. Then it follows from (6.18)-(6.23) that

$$
\begin{aligned}
&\int_{\mathbb{R}}\Big(\frac{1}{4}\varphi_1(t+|x|)(|\partial_t w|^2+|\partial_x w|^2)-\frac{1}{4}\varphi_1''(t+|x|)|w|^2+\varphi_2(t+|x|)|w|^{r+1}dx\Big)\,dx\\
&\qquad+\int_0^t\int_{\mathbb{R}}\big(\frac{1}{2}\varphi_1(\tau+|x|)|\partial_t w|^r+\frac{1}{2}\varphi_1'(\tau+|x|)|\partial_x w|^2+\mathcal{C}_1(\tau+|x|)|w|^{r+1}\big)\,dxd\tau\\
&\leq \mathcal{W}_\varphi(0)+C\int_0^t\int_{\mathbb{R}}\mathcal{C}_2(\tau+|x|)\,dxd\tau.
\end{aligned}
\tag{6.24}
$$

with

$$
\mathcal{C}_1(s):=-\varphi_2'(s)-\frac{C(|\varphi_1'(s)|^{r+1}+|\varphi_2(s)|^{r+1})}{\varphi_1(s)^r},\qquad \mathcal{C}_2(s):=\frac{|\varphi_1'(s)|^{\frac{r+1}{r-1}}}{\varphi_1(s)^{\frac{2}{r-1}}}.
$$

Therefore, one needs to choose φ',φ'' such that

$$
\mathcal{C}_1(s)>0,\qquad \int_0^\infty\int_{\mathbb{R}}\mathcal{C}_2(\tau+|x|)\,dxd\tau<\infty.
\tag{6.25}
$$

For all $q>0$ and a suitable large constant a, we choose the functions

$$
\varphi_1(s)=\log^{2q}(a+s),\qquad \varphi_2(s)=\frac{\log^{2q-r+1}(a+s)}{|a+s|^r},
$$

which fulfills the conditions (6.17) and (6.25). Indeed, for suitable large $a>0$, it easy to verify that

$$
\begin{aligned}
&\varphi_1'(s)=\frac{2q\log^{2q-1}(a+s)}{a+s}>0, &&\varphi_1''(s)=-\frac{2q\log^{2q-2}(a+s)(\log(a+s)-1+2q)}{|a+s|^2}<0,\\
&|\varphi_1'|^2\leq\frac{1}{4}\varphi_1|\varphi_1''|, &&\varphi_2'(s)=-\frac{\log(a+s)^{2q-r}(r\log(a+s)-2q+r-1)}{|a+s|^{r+1}}<0,
\end{aligned}
$$

and

$$
\mathcal{C}_1(s)\geq\frac{\log(a+s)^{2q-r}\Big(r\log(a)-2q+r-1-C\big((2q)^{r+1}\log^{-1}(a)-C\log^{r-r^2+1}(a)\big)a^{-r^2+1}\Big)}{|a+s|^{r+1}}>0.
$$

For the second term, that is where the condition $1 < r < 3$ comes into play. Indeed, it implies that $\frac{r+1}{r-1} > 2$, and therefore

$$\int_0^\infty \int_{\mathbb{R}} \mathcal{C}_2(\tau+|x|)dxd\tau = \int_0^\infty \int_{\mathbb{R}} \frac{\log^{2q-\frac{r+1}{r-1}}(a+\tau+|x|)}{|a+\tau+|x||^{\frac{r+1}{r-1}}} dxd\tau < \infty. \tag{6.26}$$

Therefore, by (6.17), (6.25) and the facts that $\partial_x w = u$, $\partial_t w = -v$, we substitute $\varphi_1(s) = \log(a+s)^{2q}$ into (6.24) to obtain

$$\begin{aligned}
&\log^{2q}(1+t)\int_{\mathbb{R}} (|u|^2+|v|^2)\, dx \\
&\leq \int_{\mathbb{R}} \log^{2q}(1+t+|x|)(|\partial_t w|^2 + |\partial_x w|^2)\, dx \leq C\mathcal{W}_\varphi(0) + C.
\end{aligned}$$

To bound the initial energy $\mathcal{W}_\varphi(0)$, we use (2.23) and find that

$$\begin{aligned}
\mathcal{W}_\varphi(0) &\leq C\int_{\mathbb{R}} \log^{2q}(1+|x|)(|v_0|^2+|u_0|^2)\, dx \\
&\quad + C\int_{\mathbb{R}} \frac{\log^{2q-2}(1+|x|)}{(1+|x|)^2} |\int_{-\infty}^x u_0(y)dy|^2\, dx \\
&\quad + C\int_{\mathbb{R}} \frac{\log^{2q-r-1}(1+|x|)}{(1+|x|)^r} |\int_{-\infty}^x u_0(y)dy|^{r+1}\, dx \\
&\leq C + C\int_{\mathbb{R}} \frac{1}{(1+|x|)^{2-\varepsilon}}\, dx \|u_0\|_{L^1}^2 + C\int_{\mathbb{R}} \frac{1}{(1+|x|)^{r-\varepsilon}}\, dx \|u_0\|_{L^1}^{r+1} < \infty,
\end{aligned}$$

for some sufficiently small $\varepsilon \in (0, \min\{1, r-1\})$. Gathering the last two estimates, we obtain (2.24) which concludes the proof of Theorem 2.7. □

7. Application to the nonlinear compressible Euler system with linear damping

7.1. **Asymptotics for the compressible Euler equations with damping.** In this subsection, we apply the methods developed in Theorems 2.4 and 2.5 to a concrete nonlinear partially dissipative hyperbolic system: the damped compressible Euler equations (1.4) with a general pressure function $P(\rho)$ satisfying

$$P(\rho) \in C^\infty(\mathbb{R}), \qquad P'(\rho) > 0. \tag{7.1}$$

For example, one can choose the γ-law $P(\rho) = \rho^\gamma$ where the adiabatic exponent $\gamma > 1$ corresponds to the isentropic flow and $\gamma = 1$ corresponds to the isothermal flow.

We now establish the long time behavior of System (1.4) as follows.

Theorem 7.1. *Let (7.1) hold and $\bar\rho > 0$ be a given constant. Assume that the initial data (ρ_0, v_0) fulfills $(\rho_0 - \bar\rho, v_0) \in H^2(\mathbb{R})$ and*

$$\|(\rho_0 - \bar\rho, v_0)\|_{H^2} \leq \delta_0, \tag{7.2}$$

where $\delta_0 > 0$ is a suitably small constant. Then, System (1.4) admits a unique global-in-time solution (ρ, v) which satisfies $(\rho - \bar\rho, v) \in C(\mathbb{R}_+; H^2(\mathbb{R}))$,

$$\|(\rho-\bar\rho, v)(t)\|_{H^2}^2 + \int_0^t (\|\partial_x(\rho-\bar\rho)(\tau)\|_{H^1}^2 + \|v(\tau)\|_{H^2}^2)d\tau \lesssim \|(\rho_0-\bar\rho, v_0)\|_{H^2}^2, \tag{7.3}$$

and

$$\|v(t)\|_{L^2} + \|\partial_x(\rho-\bar\rho, v)(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}, \tag{7.4}$$

for all $t > 0$.

Furthermore, the following statements hold:

- 1) *For* $\mu > \frac{1}{2}$, *if in addition to* (7.2) *we assume* $|x|^{\mu}(\rho_0 - \bar{\rho}, u_0) \in L^2(\mathbb{R})$ *and* $|x|^{\mu}\partial_x(\rho_0 - \bar{\rho}, u_0) \in L^2(\mathbb{R})$ *and that there exists a constant* $\lambda_0 > 0$ *such that the friction coefficient* λ *satisfies* $\lambda \geq \lambda_0$, *then for all* $t > 0$, *the solution* (ρ, v) *satisfies*

$$\begin{cases} \|(\rho - \bar{\rho})(t)\|_{L^2} \lesssim (1+t)^{-\frac{\mu}{2}}, \\ \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2} \lesssim (1+t)^{-\frac{\mu}{2}-\frac{1}{2}}. \end{cases} \tag{7.5}$$

- 2) *For* $\frac{1}{2} < \mu \leq 1$, *if in addition to* (7.2) *we assume that the initial data* (ρ_0, v_0) *verifies* $|x|^{\mu}(\rho_0 - \bar{\rho}) \in L^2(\mathbb{R})$, $|x|^{\frac{\mu}{2}} v_0 \in L^2(\mathbb{R})$, $\partial_t \rho|_{t=0} = -\partial_x(\rho_0 v_0)$ *and* $P'(\bar{\rho}) \leq 1$, *then for all* $t > 0$, *there holds*

$$\|(\rho - \bar{\rho})(t)\|_{L^2} \lesssim (1+t)^{-\mu+\frac{1}{2}}, \tag{7.6}$$

and

$$\begin{cases} \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2} \lesssim (1+t)^{-1}, & \text{if} \quad \mu = 1, \\ \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2} \lesssim (1+t)^{-\mu} \log^{\frac{1}{2}}(1+t), & \text{if} \quad \dfrac{1}{2} < \mu < 1, \\ \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2} \lesssim (1+t)^{-\mu'}, \qquad \mu' < \mu, & \text{if} \quad \dfrac{1}{2} < \mu < 1. \end{cases} \tag{7.7}$$

7.2. **Proof of global existence and time-decay.** To prove the global existence of System (1.4), we establish the a-priori estimates as follows.

Lemma 7.2. (*A priori estimates*) *Let* (ρ, v) *be the solution to System* (1.4) *on* $[0, T]$ *for any given time* $T > 0$. *Define*

$$\begin{aligned} X(t) :&= \|(\rho - \bar{\rho}, v)(t)\|_{H^2}^2 + t\|v(t)\|_{L^2}^2 + t\|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2}^2 \\ &\quad + \int_0^t (\|\partial_x(\rho - \bar{\rho})(t)\|_{H^1}^2 + \|v(\tau)\|_{H^2}^2 + \tau\|\partial_x v(t)\|_{L^2}^2)\, d\tau. \end{aligned} \tag{7.8}$$

If (ρ, u) *satisfies*

$$\|(\rho - \bar{\rho}, v)(t)\|_{H^2} \ll 1, \qquad 0 < t < T, \tag{7.9}$$

then there exists a generic constant $C_0 > 0$ *such that*

$$X(t) \leq C_0 \|(\rho_0 - \bar{\rho}, v_0)\|_{H^2}^2, \qquad 0 < t < T. \tag{7.10}$$

Proof. We use similar arguments to those used in the subsection 3.1. Denote

$$n := \rho - \bar{\rho}.$$

Then, (n, v) solves

$$\begin{cases} \partial_t n + \bar{\rho}\partial_x v = F_1, \\ \partial_t v + \dfrac{P'(\bar{\rho})}{\bar{\rho}}\partial_x n + \lambda v = F_2, \\ (n, v)|_{t=0} = (\rho_0 - \bar{\rho}, v_0) \end{cases} \tag{7.11}$$

where

$$F_1 := -\partial_x(nv) \quad \text{and} \quad F_2 := -v\partial_x v - \Big(\frac{P'(\bar{\rho}+n)}{\bar{\rho}+n} - \frac{P'(\bar{\rho})}{\bar{\rho}}\Big)\partial_x n.$$

By direct computations on (7.11), we obtain the $H^1(\mathbb{R})$-estimate

$$\begin{aligned} &\frac{d}{dt}\big(\|n(t)\|_{H^1}^2 + \frac{\bar{\rho}^2}{P'(\bar{\rho})}\|v(t)\|_{H^1}^2\big) + \frac{2\bar{\rho}^2\lambda}{P'(\bar{\rho})}\|v(t)\|_{H^1}^2 \\ &\leq 2\|F_1(t)\|_{H^1}\|n(t)\|_{H^1} + \frac{2\bar{\rho}^2}{P'(\bar{\rho})}\|F_2(t)\|_{H^1}\|v(t)\|_{H^1}. \end{aligned} \tag{7.12}$$

In addition, we have the following time-weighted estimates

$$
\begin{aligned}
&\frac{d}{dt}\big(t\|\partial_x n(t)\|_{L^2}^2+\frac{\bar\rho^2}{P'(\bar\rho)}t\|\partial_x v(t)\|_{L^2}^2\big)+\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}t\|\partial_x v(t)\|_{L^2}^2\\
&\le \|\partial_x n(t)\|_{L^2}^2+\frac{\bar\rho^2}{P'(\bar\rho)}\|\partial_x v(t)\|_{L^2}^2\\
&\quad+2t\|\partial_x F_1(t)\|_{L^2}\|\partial_x n(t)\|_{L^2}+\frac{2\bar\rho^2}{P'(\bar\rho)}t\|\partial_x F_2(t)\|_{L^2}\|\partial_x v(t)\|_{L^2}.
\end{aligned}\tag{7.13}
$$

Unlike linear analysis, $H^2(\mathbb{R})$-estimates are needed to control the nonlinear terms. The system satisfied by $(\partial_x^2 n,\partial_x^2 v)$ reads

$$
\begin{cases}
\partial_t\partial_x^2 n+v\cdot\partial_x^3 n+\bar\rho\partial_x^3 v+n\partial_x^3 v=R_1,\\
\partial_t\partial_x^2 v+v\cdot\partial_x^3 v+\dfrac{P'(\bar\rho+n)}{\bar\rho+n}\partial_x^2 n+\lambda\partial_x^2 u=R_2,
\end{cases}\tag{7.14}
$$

where

$$
R_1:=[v,\partial_x^3]n+[n,\partial_x^2]\partial_x v\quad\text{and}\quad R_2:=[v,\partial_x^2]\partial_x n+[\frac{P'(\bar\rho+n)}{\bar\rho+n},\partial_x^2]\partial_x n.
$$

It holds that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\big(|\partial_x^2 n|^2+\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2\big)dx+2\lambda\int_{\mathbb{R}}\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2dx\\
&\le\|\partial_t\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L_t^\infty(L_x^\infty)}\|\partial_x^2 v\|_{L^2}^2+2\|R_1\|_{L^2}\|\partial_x^2 n\|_{L^2}+2\|\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L^\infty}\|R_2\|_{L^2}\|\partial_x^2 v\|_{L^2}
\end{aligned}\tag{7.15}
$$

where we added weight function in the energy estimates to overcome the loss of derivatives due to the lack of symmetry of (7.14).

To capture time-decay information for n, from (7.11) we follow the hyperbolic hypocoercive approach and perform cross estimates

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(v\partial_x n+\partial_x v\partial_x^2 n)\,dx+\frac{P'(\bar\rho)}{\bar\rho}\|\partial_x n(t)\|_{H^1}^2-\|\partial_x v(t)\|_{H^1}^2+\lambda\int_{\mathbb{R}}(v\partial_x n+\partial_x v\partial_x^2 n)\,dx\\
&\le\|F_1(t)\|_{H^1}\|\partial_x v(t)\|_{H^1}+\|F_2(t)\|_{H^1}\|\partial_x n(t)\|_{H^1}.
\end{aligned}\tag{7.16}
$$

Let $c_1,c_2\in(0,1)$ be two constants to be chosen later. Define

$$
\begin{aligned}
\mathcal{L}_{euler}(t):=&\|n(t)\|_{H^1}^2+\frac{\bar\rho^2}{P'(\bar\rho)}\|v(t)\|_{H^1}^2+c_1t\|\partial_x n(t)\|_{L^2}^2+\frac{\bar\rho^2}{P'(\bar\rho)}t\|\partial_x v(t)\|_{L^2}^2\\
&+\int_{\mathbb{R}}\big(|\partial_x^2 n|^2+\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2\big)\,dx+c_2\int_{\mathbb{R}}(v\partial_x n+\partial_x v\partial_x^2 n)\,dx.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_{euler}(t)=&\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}\|v(t)\|_{H^1}^2+\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}c_1t\|\partial_x v(t)\|_{L^2}^2-c_1\|\partial_x n(t)\|_{L^2}^2-c_1\frac{\bar\rho^2}{P'(\bar\rho)}\|\partial_x v(t)\|_{L^2}^2\\
&+\big(2\lambda-\|\partial_t\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L_t^\infty(L_x^\infty)}\big)\int_{\mathbb{R}}\frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}|\partial_x^2 v|^2\,dx\\
&+c_2\big(\frac{P'(\bar\rho)}{\bar\rho}\|\partial_x^2 n\|_{L^2}^2-t\|\partial_x^2 v\|_{L^2}^2+\lambda\int_{\mathbb{R}}\partial_x v\partial_x^2 n\,dx\big).
\end{aligned}
$$

Due to the smallness condition (7.9), there holds that

$$
0<\frac{\bar\rho}{2P'(\bar\rho)}\le\frac{\bar\rho+n}{P'(\bar\rho+n)}(x,t)\le\frac{2\bar\rho}{P'(\bar\rho)},\qquad(x,t)\in\mathbb{R}\times(0,T),\tag{7.17}
$$

and

$$\begin{aligned}\|\partial_t \frac{(\bar\rho+n)\bar\rho}{P'(\bar\rho+n)}\|_{L^\infty_t(L^\infty_x)} &\le C\|\partial_t n\|_{L^\infty_t(L^\infty_x)}\\ &\le C\|v\|_{L^\infty}\|\partial_x n\|_{L^\infty_t(L^\infty_x)}+C\|\partial_x v\|_{L^\infty_t(L^\infty_x)}(1+\|n\|_{L^\infty_t(L^\infty_x)})\\ &\le \lambda.\end{aligned} \tag{7.18}$$

Adjusting the coefficients c_1, c_2 suitably and making use of (7.17)-(7.18), we obtain

$$\mathcal{L}_{euler}(t) \sim \|(n,v)(t)\|_{H^2}^2 + c_1 t\|\partial_x(n,v)(t)\|_{L^2}^2. \tag{7.19}$$

and

$$\mathcal{D}_{euler}(t) \gtrsim \|v(t)\|_{H^2}^2 + \|\partial_x n(t)\|_{H^1}^2 + t\|\partial_x v(t)\|_{L^2}^2. \tag{7.20}$$

Then, it follows from (7.12)-(7.13) and (7.15)-(7.17) that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_{euler}(t)+\mathcal{D}_{euler}(t) \lesssim{}& \|F_1(t)\|_{H^1}\|(n,\partial_x v)(t)\|_{H^1} + \|F_2(t)\|_{H^1}\|(\partial_x n, v)(t)\|_{H^1}\\ &+ t\|\partial_x F_1(t)\|_{L^2}\|\partial_x n(t)\|_{L^2} + t\|\partial_x F_2(t)\|_{L^2}\|\partial_x v(t)\|_{L^2}\\ &+ \|R_1\|_{L^2}\|\partial_x^2 n(t)\|_{L^2} + \|R_2\|_{L^2}\|\partial_x^2 v(t)\|_{L^2}.\end{aligned} \tag{7.21}$$

The nonlinear terms on the right-hand side of (7.21) are analyzed as follows. First, standard product laws and (7.9) yield

$$\begin{aligned}&\|F_1(t)\|_{H^1}\|(n,\partial_x v)(t)\|_{H^1} + \|F_2(t)\|_{H^1}\|(\partial_x n, v)(t)\|_{H^1}\\ &\quad\lesssim (\|v(t)\|_{H^2}\|\partial_x n\|_{H^1} + \|v(t)\|_{H^1}\|\partial_x v(t)\|_{H^1})\|(\partial_x n, v)(t)\|_{H^1}\\ &\qquad + \|\big(\frac{P'(\bar\rho+n)}{\bar\rho+n} - \frac{P'(\bar\rho)}{\bar\rho}\big)(t)\|_{H^1}\|\partial_x n(t)\|_{H^1}\|(\partial_x n, v)(t)\|_{H^1}\\ &\quad\lesssim \|(n,v)(t)\|_{H^2}\|(\partial_x n, v)(t)\|_{H^1}.\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}&t\|\partial_x F_1(t)\|_{L^2}\|\partial_x n(t)\|_{L^2} + t\|\partial_x F_2(t)\|_{L^2}\|\partial_x v(t)\|_{L^2}\\ &\quad\lesssim t(\|v(t)\|_{H^1}\|\partial_x n(t)\|_{H^1} + \|n(t)\|_{H^1}\|\partial_x v(t)\|_{H^1})\|\partial_x n(t)\|_{L^2}\\ &\qquad + t\big(\|v(t)\|_{H^1}\|\partial_x v(t)\|_{H^1} + \|\big(\frac{P'(\bar\rho+n)}{\bar\rho+n} - \frac{P'(\bar\rho)}{\bar\rho}\big)(t)\|_{H^1}\|\partial_x n\|_{H^1}\big)\|\partial_x v(t)\|_{L^2}\\ &\quad\lesssim \|(n,v)(t)\|_{H^1}(\|v(t)\|_{H^2}^2 + \|\partial_x n(t)\|_{H^1}^2 + t\|\partial_x v(t)\|_{L^2}^2 + t^2\|\partial_x^2 v(t)\|_{L^2}^2 + t\|\partial_x^2 n(t)\|_{L^2}^2).\end{aligned}$$

From standard commutator estimates, one has

$$\begin{aligned}\|R_1\|_{L^2} &\lesssim \|[v,\partial_x^3]n\|_{L^2} + \|[n,\partial_x^2]\partial_x v\|_{L^2}\\ &\lesssim \|\partial_x^2 v\|_{L^2}\|n\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\|\partial_x^2 n\|_{L^2} + \|\partial_x^2 n\|_{L^2}\|v\|_{L^\infty} + \|\partial_x n\|_{L^\infty}\|\partial_x^2 v\|_{L^2}\\ &\lesssim \|(n,v)(t)\|_{H^2}\|\partial_x^2(n,v)(t)\|_{L^2}\end{aligned}$$

and similarly

$$\|R_2\|_{L^2} \lesssim \|[v,\partial_x^2]\partial_x n\|_{L^2} + \|[\frac{P'(\bar\rho+n)}{\bar\rho+n}, \partial_x^2]\partial_x n\|_{L^2} \lesssim \|(n,v)(t)\|_{H^2}\|\partial_x^2(n,v)(t)\|_{L^2}.$$

Moreover, thanks to (7.9), we have

$$\|R_1\|_{L^2}\|\partial_x^2 n(t)\|_{L^2} + \|R_2\|_{L^2}\|\partial_x^2 v(t)\|_{L^2} \lesssim \|(n,v)(t)\|_{H^2}\|\partial_x^2(n,v)(t)\|_{L^2}^2.$$

Substituting the above estimates into (7.21) and using (7.20) and the fact that $\|(n,v)(t)\|_{H^2}$ is sufficiently small, we derive

$$\frac{d}{dt}\mathcal{L}_{euler}(t) + \|v(t)\|_{H^2}^2 + \|\partial_x n(t)\|_{H^1}^2 + t\|\partial_x v(t)\|_{L^2}^2 \lesssim 0. \tag{7.22}$$

Integrating (7.22) over $[0,t]$ and taking advantage of (7.19), we get

$$\begin{aligned} &\|(n,v)(t)\|_{H^2}^2 + t\|\partial_x(n,v)(t)\|_{L^2}^2 \\ &\quad + \int_0^t (\|\partial_x(\rho-\bar\rho)(\tau)\|_{H^1}^2 + \|v(\tau)\|_{H^2}^2 + \tau\|\partial_x v(\tau)\|_{L^2}^2)\,d\tau \\ &\quad \lesssim \|(\rho_0-\bar\rho, v_0)\|_{H^2}^2. \end{aligned} \tag{7.23}$$

Finally, taking the $L^2(\mathbb{R})$-inner product of $(7.11)_2$ with v, we have

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + 2\lambda\|v(t)\|_{L^2}^2 \lesssim (\|v\|_{L^\infty_t(L^\infty_x)}\|\partial_x v(t)\|_{L^2} + \|\partial_x n(t)\|_{L^2})\|v(t)\|_{L^2}. \tag{7.24}$$

This, together with Grönwall's inequality and (7.23), leads to

$$\begin{aligned} \|v(t)\|_{L^2} &\lesssim e^{-\lambda t}\|v_0(t)\|_{L^2} + \int_0^t e^{-\lambda(t-\tau)}\|\partial_x(n,v)(\tau)\|_{L^2}\,d\tau \\ &\lesssim \|(\bar\rho_0-\bar\rho, v_0)\|_{H^2}^2 (1+t)^{-\frac12} \end{aligned}$$

which concludes the proof of Lemma 7.2. □

Proof of Theorem 7.1: Let (ρ_0, v_0) satisfy (7.2). According to classical local well-posedness results (see e.g. [1, 3, 22, 28, 33]), there exists a time $T_0 > 0$ such that System (1.4) associated to the initial data $(\rho_0, v_0) \in H^2(\mathbb{R})$ admits a unique solution (ρ, v) satisfying $(\rho-\bar\rho, v) \in C([0,T]; H^2(\mathbb{R}))$. Then, according to the a-priori estimates (7.10) established in Lemma 7.10 and a standard bootstrap argument, one can extend the solution (ρ, v) globally in time and recover the properties (7.3)-(7.4).

The time-decay estimates (7.5), (7.6) and (7.7) will be proved in Lemmas 7.3 and 7.4 in the next subsection.

7.3. Faster time-decay rates.

Lemma 7.3. *Let (ρ, u) be the global solution to the Cauchy problem of System (1.4) associated to the initial data (ρ_0, u_0). Assume that there exists a constant $\lambda_0 > 0$ such that the friction coefficient λ satisfies $\lambda \geq \lambda_0$, and in addition to (7.2), (ρ_0, v_0) satisfies $|x|^\mu(\rho_0-\bar\rho, v_0) \in H^1(\mathbb{R})$ with $\mu > \frac12$. Then, (7.5) holds.*

Proof. Following a similar procedure to the one used in the proof of Lemma 4.1, the time-decay estimates (7.5) will follow provided that

$$\sup_{t>0} \||x|^\mu(\rho-\bar\rho)(t)\|_{L^2} < \infty. \tag{7.25}$$

To this end, one needs to perform space-weighted estimates similar to Lemma 4.2. Recall that (n,v) with $n = \rho-\bar\rho$ satisfies (7.11). Due to (9.4), it holds that

$$\||x|^{\mu-1}n(t)\|_{L^2} \leq \frac{2\mu-1}{2}\||x|^\mu\partial_x n(t)\|_{L^2}. \tag{7.26}$$

Taking the $L^2(\mathbb{R})$-inner product of $(7.11)_1$ and $(7.11)_2$ with $|x|^{2\mu}n$ and $|x|^{2\mu}v$ respectively, and applying (7.26), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\big(\||x|^{\mu}n(t)\|_{L^2}^2+\frac{\bar\rho^2}{P'(\bar\rho)}\||x|^{\mu}v(t)\|_{L^2}^2\big)+\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}\||x|^{\mu}v(t)\|_{L^2}^2\\
&=\int_{\mathbb{R}}(-\partial_x|x|^{2\mu}nv+\partial_x(|x|^{2\mu}n)nv+\frac{\bar\rho^2}{P'(\bar\rho)}|x|^{2\mu}F_2v)dx\\
&\leq 2\mu\||x|^{\mu-1}n(t)\|_{L^2}\||x|^{\mu}v(t)\|_{L^2}+\frac{\bar\rho^2}{P'(\bar\rho)}\||x|^{\mu}F_2(t)\|_{L^2}\||x|^{\mu}v(t)\|_{L^2}\\
&\quad+\|n(t)\|_{L^2}(2\mu\||x|^{\mu-1}n(t)\|_{L^2}+\||x|^{\mu}\partial_xn(t)\|_{L^2})\||x|^{\mu}v(t)\|_{L^2}\\
&\leq\frac{\bar\rho^2\lambda}{P'(\bar\rho)}\||x|^{\mu}v(t)\|_{L^2}^2+\frac{\widetilde C}{\lambda}\||x|^{\mu}\partial_xn(t)\|_{L^2}^2\\
&\quad+\widetilde C\|(n,v)(t)\|_{L^2}\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\widetilde C\||x|^{\mu}n(t)\|_{L^2}\|\partial_xn(t)\|_{L^2}\||x|^{\mu}v(t)\|_{L^2}.
\end{aligned}\tag{7.27}
$$

where $\widetilde C>0$ denotes a constant independent of time and λ. Similarly, it also follows that

$$
\begin{aligned}
&\frac{d}{dt}\big(\||x|^{\mu}\partial_xn(t)\|_{L^2}^2+\frac{\bar\rho^2}{P'(\bar\rho)}\||x|^{\mu}\partial_xv(t)\|_{L^2}^2\big)+\frac{2\bar\rho^2\lambda}{P'(\bar\rho)}\||x|^{\mu}\partial_xv(t)\|_{L^2}^2\\
&=\int_{\mathbb{R}}(-\partial_x|x|^{2\mu}\partial_xn\partial_xv+|x|^{2\mu}\partial_xF_1\partial_xn+|x|^{2\mu}\partial_xF_2\partial_xv)dx\\
&\leq\frac{\bar\rho^2\lambda}{P'(\bar\rho)}\||x|^{\mu}\partial_xv(t)\|_{L^2}^2+\varepsilon\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\frac{\widetilde C}{\varepsilon}\|\partial_x(n,v)(t)\|_{L^2}^2\\
&\quad+\widetilde C\|(n,v)(t)\|_{H^2}\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\widetilde C\||x|^{\mu}n(t)\|_{H^1}\|\partial_xn(t)\|_{H^1}\||x|^{\mu}\partial_xv(t)\|_{L^2}.
\end{aligned}\tag{7.28}
$$

where we used that for any $\mu>\frac12$ and $\varepsilon>0$, we have

$$
\Big|\int_{\mathbb{R}}\partial_x|x|^{2\mu}\partial_xn\partial_xv\,dx\Big|\leq\varepsilon\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\frac{\widetilde C}{\varepsilon}\|\partial_x(n,v)(t)\|_{L^2}^2.
$$

Adding (7.27)-(7.28) together and integrating it over $[0,t]$, we have

$$
\begin{aligned}
&\||x|^{\mu}(n,v)(t)\|_{L^2}^2+\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\lambda\int_0^t(\||x|^{\mu}v(\tau)\|_{L^2}^2+\||x|^{\mu}\partial_xv(\tau)\|_{L^2}^2)\,d\tau\\
&\leq\widetilde C\||x|^{\mu}(\rho_0-\bar\rho,v_0)(t)\|_{H^1}+\widetilde C(\frac1\lambda+\varepsilon)\int_0^t\||x|^{\mu}\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau+\frac{\widetilde C}{\varepsilon}\int_0^t\|\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau\\
&+\widetilde C\sup_{t>0}\|(n,v)(t)\|_{H^2}\int_0^t\||x|^{\mu}\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau\\
&+\big(\int_0^t\|\partial_xn(t)\|_{H^1}^2d\tau\big)^{\frac12}\sup_{t>0}\||x|^{\mu}n(t)\|_{H^1}\big(\int_0^t\||x|^{\mu}\partial_xv(\tau)\|_{H^1}^2\,d\tau\big)^{\frac12}.
\end{aligned}\tag{7.29}
$$

The combination of (7.2), (7.3) and (7.29) yields

$$
\begin{aligned}
&\||x|^{\mu}(n,v)(t)\|_{L^2}^2+\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2+\lambda\int_0^t(\||x|^{\mu}v(\tau)\|_{L^2}^2+\||x|^{\mu}\partial_xv(\tau)\|_{L^2}^2)\,d\tau\\
&\leq\||x|^{\mu}(\rho_0-\bar\rho,v_0)(t)\|_{H^1}^2+\frac{\widetilde C}{\varepsilon}\|(\rho_0-\bar\rho,v_0)(t)\|_{H^2}^2\\
&\quad+\widetilde C(\frac1\lambda+\varepsilon)\int_0^t\||x|^{\mu}\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau+\frac{\widetilde C}{\varepsilon}\int_0^t\|\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau.
\end{aligned}\tag{7.30}
$$

In addition, the cross terms estimates are given by

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}|x|^{2\mu}v\partial_x n\,dx+\frac{P'(\bar\rho)}{\bar\rho}\||x|^{\mu}\partial_x n\|_{L^2}^2+\int_{\mathbb{R}}(\lambda|x|^{2\mu}v\partial_x n-(|x|^{2\mu}\partial_x n)_x\partial_x v)\,dx\\
&\leq \||x|^{\mu}F_1(t)\|_{L^2}\||x|^{\mu}\partial_x v(t)\|_{L^2}+\||x|^{\mu}F_2(t)\|_{L^2}\||x|^{\mu}\partial_x n(t)\|_{L^2}\\
&\leq \widetilde{C}\|(n,v)(t)\|_{L^2}\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2.
\end{aligned}
$$

This, together with (7.30) and $\|(n,v)(t)\|_{H^2}\ll 1$, yields

$$
\begin{aligned}
&\||x|^{\mu}(n,v)(t)\|_{L^2}^2+\||x|^{\mu}\partial_x(n,v)(t)\|_{L^2}^2\\
&\qquad+\lambda\int_0^t(\||x|^{\mu}v(\tau)\|_{L^2}^2+\||x|^{\mu}\partial_x v(\tau)\|_{L^2}^2)\,d\tau+\int_0^t\||x|^{\mu}\partial_x n(\tau)\|_{L^2}^2\,d\tau\\
&\leq \widetilde{C}\||x|^{\mu}(\rho_0-\bar\rho,v_0)(t)\|_{L^2}^2+\||x|^{\mu}\partial_x(\rho_0-\bar\rho,u_0)\|_{L^2}+\frac{\widetilde{C}}{\varepsilon}\|(\rho_0-\bar\rho,v_0)(t)\|_{H^2}^2\\
&\qquad+\widetilde{C}(\frac{1}{\lambda}+\varepsilon)\int_0^t\||x|^{\mu}\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau+\frac{\widetilde{C}}{\varepsilon}\int_0^t\|\partial_x(n,v)(\tau)\|_{L^2}^2\,d\tau.
\end{aligned}
\tag{7.31}
$$

Choosing ε sufficiently small and letting $\lambda\geq\lambda_0$ with λ_0 large enough, we obtain (7.25), and therefore the time-decay estimates (7.5) follows. □

Lemma 7.4. *Let (ρ,u) be the global solution to the Cauchy problem of System (1.4) associated to the initial data (ρ_0,u_0). In addition to (7.2), assume $\partial_t\rho|_{t=0}=-\partial_x(\rho_0v_0)$, $|x|^{\mu}(\rho_0-\bar\rho)\in L^2(\mathbb{R})$, $|x|^{\mu-\frac12}v_0\in L^2(\mathbb{R})$ for $\frac12<\mu\leq1$ and $P'(\bar\rho)\leq 1$. Then, (7.6) and (7.7) hold.*

Proof. We mention that in order to apply the method in Section 5, the key idea is to consider the momentum $m=nv$ instead of the velocity v. Then, System (1.4) is rewritten, in terms of $(n,m)=(\rho-\bar\rho,\rho v)$, as

$$
\begin{cases}
\partial_t n+\partial_x m=0,\\
\partial_t m+P'(\bar\rho)\partial_x n+\lambda m=F_3:=-\partial_x\big(\dfrac{m^2}{\bar\rho+n}+P(\bar\rho+n)-P(\bar\rho)-P'(\bar\rho)n\big),\\
(n,m)|_{t=0}=(\rho_0-\bar\rho,\rho_0v_0).
\end{cases}
\tag{7.32}
$$

As in Section 5, we introduce the wave unknown

$$M(x,t):=\int_{-\infty}^{x}n(y,t)\,dy$$

such that

$$\partial_{tt}^2M-P'(\bar\rho)\partial_x^2M+\lambda\partial_tM=F_3. \tag{7.33}$$

Then, following the computation done in the proof of Lemma 5.1, we can show that

$$
\begin{aligned}
&\int_{\mathbb{R}}(1+t+|x|)^{2\mu-1}(|\partial_tM|^2+|\partial_xM|^2)\,dx\\
&\qquad+\int_0^t\int_{\mathbb{R}}\big((1+t+|x|)^{2\mu-1}|\partial_tM|^2+(1+t+|x|)^{2\mu-2}|\partial_xM|^2\big)\,dxd\tau\\
&\leq C+C\int_0^t\int_{\mathbb{R}}(1+t+|x|)^{2\mu-1}|F_3|^2\,dxd\tau.
\end{aligned}
\tag{7.34}
$$

From (7.3) and composition estimates, we obtain

$$|F_3|\lesssim|m||\partial_xm|+|n||\partial_xn|.$$

It thus follows that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} (1+t+|x|)^{2\mu-1}|F_3|^2\, dxd\tau \\
&\lesssim \int_0^t \|\partial_x(m,n)(\tau)\|_{L^\infty}^2 \int_{\mathbb{R}} (1+t+|x|)^{2\mu-1}(|m|^2+|n|^2)\, dxd\tau \\
&\lesssim \int_0^t \|\partial_x(n,v)(\tau)\|_{H^1}^2 \int_{\mathbb{R}} (1+t+|x|)^{2\mu-1}(|\partial_t M|^2+|\partial_x M|^2)\, dxd\tau,
\end{aligned} \tag{7.35}$$

where one has used the facts that $\partial_x M = n$ and $\partial_t M = -m$. Inserting the above estimate into (7.34) and using (7.3) and Grönwall's inequality, we get

$$\int_{\mathbb{R}} (1+t+|x|)^{2\mu-1}(|n|^2+|m|^2)\, dx + \int_0^t \int_{\mathbb{R}} (1+t+|x|)^{2\mu-2}|m|^2\, dxd\tau \lesssim 1, \tag{7.36}$$

which implies (7.6).

Finally, multiplying the inequality (7.22) by $t^{2\mu-1}$, integrating the resulting inequality on $[0,t]$ with $t\geq 1$, using the smallness of c_1 and that $\frac{1}{2}<\mu\leq 1$, we have

$$\begin{aligned}
&t^{2\mu-1}\|(n,v)(t)\|_{L^2}^2 + (t^{2\mu-1}+c_1 t^{2\mu})\|\partial_x(n,v)(t)\|_{L^2}^2 + \int_0^t \tau^{2\mu-1}\|\partial_x(n,v)(\tau)\|_{L^2}^2\, d\tau \\
&\lesssim \int_0^t \big(\tau^{2\mu-2}\|(n,v)(\tau)\|_{L^2}^2 + \tau^{2\mu-2}\|\partial_x(n,v)(\tau)\|_{L^2}^2\big)\, d\tau.
\end{aligned} \tag{7.37}$$

To estimate the right-hand side of (7.37), we distinguish two cases.

- Case 1: $\mu=1$.

Since $\mu=1$, the estimate (7.3) gives

$$\int_0^t \tau^{2\mu-2}\big(\|v(\tau)\|_{L^2}^2 + \|\partial_x(n,v)(\tau)\|_{L^2}^2\big)\, d\tau \lesssim 1,$$

and from (7.36) and $v=\frac{m}{\bar\rho+n}$, one has

$$\int_0^t \tau^{2\mu-2}\|v(\tau)\|_{L^2}^2 d\tau \lesssim \int_0^t \|m(\tau)\|_{L^2}^2\, d\tau \lesssim 1.$$

- Case 2: $\frac{1}{2}<\mu<1$.

We deduce from estimates (7.4) that

$$\int_0^t \tau^{2\mu-2}\big(\|v(\tau)\|_{L^2}^2 + \|\partial_x(n,v)(\tau)\|_{L^2}^2\big)\, d\tau \lesssim \int_0^1 \tau^{2\mu-2}\, d\tau + \int_0^1 (1+\tau)^{2\mu-3}\, d\tau \lesssim 1.$$

Using the time-decay estimates of n in (7.6) at hand, we have

$$\begin{aligned}
\int_0^t \tau^{2\mu-2}\|n(\tau)\|_{L^2}^2\, d\tau &\lesssim \int_0^1 \tau^{2\mu-2}\, d\tau + \int_1^t (1+\tau)^{-1}\, d\tau \\
&\lesssim \log(1+t).
\end{aligned}$$

And for all $\mu'<\mu$, one also has

$$\begin{aligned}
\int_0^t \tau^{2\mu-2}\|n(\tau)\|_{L^2}^2\, d\tau &\lesssim \int_0^1 \tau^{2\mu-2}\, d\tau + \int_1^t (1+\tau)^{2\mu-2\mu'-1}\, d\tau \\
&\lesssim Y_0^2(1+t)^{2\mu-2\mu'}.
\end{aligned}$$

Combining the above two cases together and employing Grönwall's inequality to (7.24), we have (7.7). □

8. Extensions

8.1. **The compressible Euler system with nonlinear damping.** As an extension of Section 7, it is natural to consider the Euler System (1.4) with a nonlinear damping term $\rho u|u|$ instead of ρu. Such modelisation is relevant for gas transports, see e.g.[12]. Actually, following the approach used in Section 7, similar decay rates to the one obtained for the nonlinearly damped p-system (2.20) in Theorem 2.7 (therefore under weighted integrability condition) can be derived if one assumes that we have a solution of the nonlinearly damped Euler system belonging to $L^2(\mathbb{R}_+; H^1(\mathbb{R}))$ (a crucial to control the advection terms in the time-decay analysis). However, different from the linear damping term, the nonlinear damping term $\rho u|u|$ does not seem sufficient, to the best of our knowledge, to provide this bound. Current investigations are devoted to obtaining such bound for the nonlinearly damped compressible Euler system.

8.2. **Numerics.** As an application of the method developed here, inspired by Porretta and Zuazua's paper [27], one can prove that a centered finite-difference approximation of the partially dissipative systems (2.2) in the whole space preserves the asymptotic properties of the continuous solutions as $t \to \infty$. Such result would highlight that the hyperbolic hypocoercive nature of the system can be preserved at the discrete level. Compared to the results obtained by Porretta and Zuazua concerning the Kolmogorov equation, here we employ hypocoercive properties of rank n, meaning that we need information from the whole Kalman matrix $\mathcal{K} = (B, BA, BA^2, \cdots, BA^{n-1})$ to recover dissipation on all the components. In the case of the Kolmogorov equation, only rank-2 hypocoercivity is involved.

8.3. **Multi-dimensional setting:** In the multi-dimensional setting, one can investigate the n-component systems in $\mathbb{R}^d$ $(d \geq 1)$ of the type:

$$\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j \frac{\partial V}{\partial x_j} = BV, \tag{8.1}$$

where the A^j $(j = 1, .., d)$ are symmetric matrices, B is a strongly dissipative matrix and the unknown $V = V(t, x) \in \mathbb{R}^n$ depends on the time and space variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

For these general systems, under the (SK) condition, Beauchard and Zuazua in [2] were able to obtain time-decay rates using hypocoercivity-type arguments. Even if the method developed in the present paper is based on arguments from [2], we are not able to to derive time-decay rates for the multi-dimensional system (8.1). The issue comes from the appearance of mixed derivatives (due the lack of Fourier analysis) when differentiating in time the low-order corrector term in the Lyapunov functional.

Nevertheless, it is possible to obtain results under additional structural conditions on the system: for instance when it has a structure similar to the multi-dimensional compressible Euler system with damping. Indeed, for the multi-dimensional version of (1.4), straightforward computations show that the Lyapunov functional:

$$\mathcal{L}(t) = \|(\rho - \bar{\rho}, u)(t)\|_{H^1}^2 + ct\|(\nabla\rho, \nabla u)(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} u \cdot \nabla\rho dx$$

allows to directly recover time-decay rates in any dimension. For a more precise formulation of the necessary conditions to be able to study (8.1) with the approach developed in the present paper, we refer to the structural conditions used in [7, Theorem 2.3].

9. Appendix

9.1. **A short litterature review on partially dissipative systems.** To capture the dissipative structures for large time and overcome the coercivity issue mentioned in (2.3), Shizuta and Kawashima in [29] developed a condition, the well-known (SK) stability condition

$$\{\text{eigenvectors of } A\xi\} \cap \operatorname{Ker}(B) = \{0\} \quad \forall \xi \in \mathbb{R}^d. \tag{9.1}$$

Under the (SK) assumption, the global well-posedness of classical solutions for multi-dimensional quasi-linear partially dissipative hyperbolic systems near some constant equilibrium state was proved by Yong [37] and Hanouzet-Natalini [13] in Sobolev spaces with high regularity, and then by Xu-Kawashima [33] in critical inhomogenous Besov spaces.

Under the (SK) assumption, the authors in [29] proved the following time-decay theorem for multi-dimensional linear problems by using the Fourier transform.

Theorem 9.1 ([29], (SK) decay estimate)**.** *Let $d \geq 1$. Under the condition* (SK) *and for $U_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, the solution U of* (2.1) *verifies*

$$\|U^\ell(t)\|_{L^\infty} \leq Ct^{-d/2}\|U_0\|_{L^1},$$
$$\|U^h(t)\|_{L^2} \leq Ce^{-\gamma t}\|U_0\|_{L^2},$$

where C, γ are positive constants depending only on A, B and D, $\widehat{U}^h(\xi,t) := \widehat{U}(\xi,t)\mathbb{1}_{\{|\xi|>1\}}$ and $\widehat{U}^\ell(\xi,t) := \widehat{U}(\xi,t)\mathbb{1}_{\{|\xi|<1\}}$.

We also refer to the papers of Bianchini, Hanouzet and Natalini in [3] and Xu and Kawashima [34, 35] for results concerning the asymptotic behavior of these systems in, respectively, the inhomogeneous Sobolev and Besov frameworks. Essentially, under the (SK) condition, one sees that the low-frequency part of the Green function behaves like the heat kernel and the high frequency-part of the solution decays exponentially.

More recently, Beauchard and Zuazua in [2], aiming to handle multi-dimensional systems, retrieved directly the time-decay estimates by employing the following Lyapunov functional

$$\mathcal{L}_\xi(t) \triangleq |\widehat{U}|^2 + \min\big\{\frac{1}{|\xi|}, |\xi|\big\}\mathrm{Re}\sum_{k=1}^{n-1}\varepsilon_k < BA_\omega^{k-1}\widehat{U} \cdot BA_\omega^k\widehat{U} > .$$

Here $\omega \in \mathbb{S}^{d-1}$ satisfying $\xi = |\xi|\omega$, $A_\omega = \sum_{j=1}^d \omega_j A^j$, and $< \cdot >$ designates the Hermitian scalar product in $\mathbb{C}^n$. Under the (SK) assumption, employing the Fourier transform to the linear system and assuming suitable smallness conditions on the coefficients ε_k, one obtains

$$\frac{d}{dt}\mathcal{L}(t) + \min\{1, |\xi|^2\}\mathcal{L}(t) \lesssim 0, \quad \text{and} \quad \mathcal{L}_2(t) \sim |\widehat{U}|^2.$$

This approach, inspired by the hypocoercivity theory of Villani [32], gives the same result as Theorem 9.1 in a direct way and allows to better describe the asymptotic behavior of solutions when the (SK) condition fails (in the multidimensional setting). Inspired by their method, Crin-Barat and Danchin in [7, 8] obtained global well-posedness and sharp time-decay results in the framework of critical homogeneous Besov spaces for general nonlinear hyperbolic partially dissipative systems. We also refer to [6] for a new approach to deal with the relaxation limit associated to these systems by employing a precise frequency decomposition thanks to the Littlewood-Paley theory.

To be able to compare the results obtained in this paper with the classical ones, we now state time-decay estimates that are obtain in e.g. [2, 3].

Proposition 9.2. *Let $d = 1$, $k \geq 1$ and $U \in C(\mathbb{R}_+; H^k(\mathbb{R}))$ be a solution of System (2.1) subject to the initial data $U_0 \in H^k(\mathbb{R}) \cap L^1(\mathbb{R})$. Then it holds that*

$$
\begin{cases}
\|U(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}} \|U_0\|_{H^k \cap L^1}, \\
\|U_2(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}} \|U_0\|_{H^k \cap L^1}, \\
\|\nabla^i U_1(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{i}{2}} \|U_0\|_{H^k \cap L^1}, \quad i = 1, 2, ..., k, \\
\|\nabla^i U_2(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{i}{2}} \|U_0\|_{H^k \cap L^1}, \quad i = 1, 2, ..., k-1.
\end{cases}
\tag{9.2}
$$

Remark 9.3. The L^1-type assumption on the initial data can be replaced by $\dot{B}^{-\frac{1}{2}}_{2,\infty}(\mathbb{R})$, cf. [7, 34, 35]. Moreover, these decay rates are optimal in the sense that they follow the ones of the heat equation, which is expected by the low frequencies (the slowly-decaying part) of the solution.

9.2. **Technical lemmas.**

Lemma 9.4. (*General Caffarelli-Kohn-Nirenberg inequality*)

- ([18]) *For all $h \in \mathcal{C}_c(\mathbb{R})$, it holds that*

$$
\||x|^\kappa h\|_{L^r} \leq C \||x|^\alpha \partial_x h\|_{L^p}^\theta \||x|^\beta h\|_{L^q}^{1-\theta},
\tag{9.3}
$$

  where $1 \leq p, q < \infty$, $0 < r < \infty$, $0 \leq \theta \leq 1$, $\frac{1}{p} + \alpha > 0$, $\frac{1}{q} + \beta > 0$, $\frac{1}{r} + \kappa > 0$ such that

$$
\frac{1}{r} + \kappa = \theta(\frac{1}{p} + \alpha - 1) + (1-\theta)(\frac{1}{q} + \beta), \qquad \kappa = \theta\sigma + (1-\theta)\beta,
$$

  with σ satisfying

$$
\sigma \leq \alpha \quad \text{if } \theta > 0, \qquad \sigma \geq \alpha - 1 \quad \text{if } \theta > 0, \ \text{and } \frac{1}{p} + \alpha - 1 = \frac{1}{r} + \kappa.
$$

- ([4]) *For all $h \in \mathcal{C}_c(\mathbb{R})$, it holds that*

$$
\||x|^{\mu_1} h\|_{L^p} \leq C_{\mu_1,\mu_2} \||x|^{\mu_2} \partial_x h\|_{L^2},
\tag{9.4}
$$

  with $\mu_2 > \frac{1}{2}$, $\mu_2 - 1 \leq \mu_1 \leq \mu_2 - \frac{1}{2}$ and $p = \frac{2}{2(\mu_2-\mu_1)-1}$. If $\mu_1 = \mu_2 - 1$, then $p = 2$ and the best constant in (9.4) is

$$
C_{\mu_1,\mu_2} = C_{\mu_2-1,\mu_2} = \frac{|2\mu_2 - 1|}{2}.
$$

Lemma 9.5. *Let $T > 0$ be given time, and $E_1(t), E_2(t)$ be two nonnegative and absolutely continuous functions on $[0,T)$. Suppose that*

$$
\frac{d}{dt}\big(E_1(t) + a_0 t E_2(t)\big) + a_1 E_1(t)^{1+\frac{1}{\mu}} + a_2 E_2(t) \leq 0, \qquad t \in (0,T),
\tag{9.5}
$$

where a_0, a_1, a_2 and μ are constants satisfying

$$
a_1, a_2, \mu > 0, \qquad 0 < a_0 < \min\{\frac{a_2}{\mu}, a_2\}.
$$

Then it holds that

$$
E_1(t) + a_0 t E_2(t) \leq C a_1^{-\mu} t^{-\mu}, \qquad t \in (0,T),
\tag{9.6}
$$

where $C > 0$ is a constant independent of T and a_1.

Proof. Let the constant p satisfy $\max\{1, \mu\} < p < \frac{a_2}{a_0}$. Multiplying (9.5) with t^p, we get

$$
\frac{d}{dt}\big(t^p E_1(t) + a_0 t^{p+1} E_2(t)\big) + a_1 t^p E_1(t)^{1+\frac{1}{\mu}} + (a_2 - p a_0) t^p E_2(t) \leq p t^{p-1} E_1(t).
\tag{9.7}
$$

Noticing $a_2 - p a_0 > 0$ and

$$
p t^{p-1} E_1(t) = p\big(t^p E_1(t)^{1+\frac{1}{\mu}}\big)^{\frac{\mu}{1+\mu}} \big(t^{p-\mu-1}\big)^{\frac{1}{1+\mu}} \leq a_1 t^p E_1(t)^{1+\frac{1}{\mu}} + \big(\frac{p}{\mu+1}\big)^{\mu+1} \big(\frac{\mu}{a_1}\big)^\mu t^{p-\mu-1},
$$

we prove after integrating (9.7) over $[0,t]$ that

$$t^pE_1(t)+a_0t^{p+1}E_2(t)\leq\big(\frac{p}{\mu+1}\big)^{\mu+1}\big(\frac{\mu}{a_1}\big)^{\mu}\int_0^t\tau^{p-\mu-1}d\tau=\big(\frac{p}{\mu+1}\big)^{\mu+1}\big(\frac{\mu}{a_1}\big)^{\mu}\frac{1}{p-\mu}t^{p-\mu}.$$

□

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