A survey on partially dissipative systems: global well-posedness and strong relaxation limit in the critical regularity setting

Timothée Crin-Barat

Friedrich-Alexander-Universität (FAU) Erlangen-Nuremberg

Nanjing University of Aeronautics and Astronautics (NUAA)
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1. **First part:** Stability of partially dissipative hyperbolic systems

2. **Second part:** Hyperbolisation via partial dissipation

Stability of hyperbolic systems

We consider n-component hyperbolic systems of the form:

$$\begin{cases} \partial_t U + \sum_{j=1}^{d} A^j(U)\partial_{x_j} U + BU = 0, \\ U_0(x,t) = U_0(x), \end{cases}$$

where

- $U(x,t) \in \mathbb{R}^n$, $x \in \mathbb{R}^d$ or $\mathbb{T}^d$ and $t > 0$,
- The matrices valued maps A^j are symmetric,
- The $n \times n$ matrix B is positive and symmetric.

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Three scenarios:

- When $B = 0$, small and smooth initial data lead to local-in-time solutions (Kato, Majda, Serre) that may develop shock waves in finite time (Dafermos, Lax).
- When $\mathrm{rank}(B) = n$, existence of global-in-time solutions (Li) that are exponentially damped.
- Partially dissipative setting: $0 < \mathrm{rank}(B) < n$.

Partially dissipative structure

- For simplicity, we look at one-dimensional hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0, \tag{1}$$

where A is symmetric and B is *partially dissipative*: $\mathrm{rank}(B) = n_2 < n$, $n_1 + n_2 = n$ and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0.$$

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$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0.$$

- Decomposing $U = (U_1, U_2)$, with $U_1 \in \mathbb{R}^{n_1}$ and $U_2 \in \mathbb{R}^{n_2}$, we have

$$\begin{cases} \partial_t U_1 + A_{1,1}\partial_x U_1 + A_{1,2}\partial_x U_2 = 0, \\ \partial_t U_2 + A_{2,1}\partial_x U_1 + A_{2,2}\partial_x U_2 = -DU_2, \end{cases} \quad \text{where } A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

The symmetry of B implies that: there exists $\kappa > 0$ such that

$$\langle DX, X\rangle \geq \kappa \|X\|^2.$$

Applications

Examples of application • The compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) + \rho u = 0, \end{cases}$$

For the pressure law $P(\rho) = A\rho^\gamma$, with $A > 0$ and $\gamma > 1$, we can rewrite System (5) into the symmetric form:

$$\begin{cases} \partial_t c + u\partial_x c + \dfrac{\gamma - 1}{2} c\partial_x u = 0, \\ \partial_t u + u\partial_x u + \dfrac{\gamma - 1}{2} c\partial_x c = -u, \end{cases} \tag{2}$$

where $c = \sqrt{\frac{\partial P(\rho)}{\partial \rho}}$ corresponds to the sound speed.

• *Partial dissipation* occurs in many compressible models including dissipation: Compressible Navier-Stokes equations, Chemotaxis systems, Timoshenko systems, Discrete BGK, Euler-Maxwell equations, Sugimoto model, damped wave equation, Cattaneo's approximation etc.

Large-time stability for partially dissipative systems

Context

Goal: establish time-decay rates for

$$\partial_t U + A\partial_x U + BU = 0.$$

First difficulty: partial dissipation leads to an obvious lack of coercivity:

$$\frac{1}{2}\frac{d}{dt}\|(U_1, U_2)(t)\|_{L^2}^2 + \kappa\|U_2(t)\|_{L^2}^2 \leq 0, \tag{3}$$

$\rightarrow$ no time-decay information on U_1.

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Inspiration to tackle this issue: Theories of hypoellipticity (Hörmander), control (Kalman), and hypocoercivity (Villani):

"There might be regularizing/stabilizing mechanisms *hidden* in the interactions between the hyperbolic part A and the dissipative matrix B."

$\to$ Let's see what how it looks like in the context of ODEs.

ODE toy-model

Consider the ODE

$$\partial_t U + AU + BU = 0 \tag{4}$$

such that A is skew-symmetric and B positive symmetric (rank(B)$< n$).

Lemma

The following statement are equivalent.

- *The pair (A, B) satisfies the Kalman rank condition:*

$$\mathit{rank}(B, BA, BA^2, \ldots, BA^{n-1}) = n \tag{K}$$

- *The solution of (4) with the initial data $U_0 \in L^2$ satisfies*

$$\|U(t)\|_{L^2} \leq Ce^{-\lambda t}\|U_0\|_{L^2}.$$

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Sketch of proof: Since A is skew-symmetric, we have

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{L^2} + \kappa\|U_2(t)\|^2_{L^2} \leq 0. \tag{5}$$

Using the interactions between A and B,

$$\frac{d}{dt}\left(\sum_{k=1}^{n-1} < BA^{k-1}U, BA^kU >\right) + \sum_{k=1}^{n-1}\|BA^kU(t)\|^2_{L^2} \leq C\|U_2(t)\|^2_{L^2} + \dots$$

Under the Kalman rank condition, we have

$$\sum_{k=0}^{n-1} \|BA^k U(t)\|_{L^2}^2 \sim \|U(t)\|_{L^2}^2.$$

Therefore, the following functional is a Lyapunov functional

$$\mathcal{L}(t) = \|U(t)\|_{L^2}^2 + \eta \left(\sum_{k=1}^{n-1} < BA^{k-1}U, BA^k U >_{L^2} \right)$$

verifying

$$\frac{d}{dt}\mathcal{L}(t) + \|U_2(t)\|_{L^2}^2 + \eta \|U(t)\|_{L^2}^2 \leq \eta \|U_2(t)\|_{L^2}^2.$$

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$$\frac{d}{dt}\mathcal{L}(t) + \|U_2(t)\|_{L^2}^2 + \eta \|U(t)\|_{L^2}^2 \leq \eta \|U_2(t)\|_{L^2}^2.$$

For η small enough, we have

$$\mathcal{L}(t) \sim \|U(t)\|_{L^2}^2$$

and thus

$$\frac{d}{dt}\mathcal{L}(t) + \eta \mathcal{L}(t) \leq 0. \quad \square$$

Morale: The conservative part A of the system helped to *propagate/rotate* the partial dissipation of B.

Partially dissipative hyperbolic systems

- In the hyperbolic setting, the idea is essentially the same.

Main difficulty: The operators $A\partial_x$ and B are of a different order.

$\to$ Need to find a way to make them communicate as in the ODE setting.

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$\rightarrow$ Need to find a way to make them communicate as in the ODE setting.

Two approaches:

- Fourier-based approach. (Shizuta-Kawashima, Yong, Beauchard-Zuazua, CB-Danchin)

Roughly, one can proceed as in the ODE setting by adding frequency weights to the Lyapunov functional.

- Time-weighted Fourier-free approach. (CB-Shou-Zuazua)

$\rightarrow$ Not optimal results but a broader range of applications e.g. numerics, bounded domains, nonlinear dissipation.

Partially dissipative hyperbolic systems

• In the hyperbolic setting, we follow the same idea.

Main difficulty: The operator $A\partial_x$ and B are of a different order $\rightarrow$ how to make them communicate as in the ODE setting.

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Essentially, one can proceed as in the ODE setting by adding frequency-weights to the Lyapunov functional.

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Beauchard and Zuazua's Result

We have the following result for

$$\partial_t U + A\partial_x U + BU = 0. \tag{6}$$

Lemma (Beauchard-Zuazua '11)

The following statements are equivalent.

- *The pair (A, B) satisfies the Kalman rank condition:*

$$\mathit{rank}(B, BA, BA^2, \ldots, BA^{n-1}) = n \tag{K}$$

- *The solution of (6) with the initial data $U_0 \in L^1 \cap L^2$ satisfies*

$$\|U(t)\|_{L^2} \leq Ce^{-\min(1,\xi^2)t}\|U_0\|_{L^2}$$

and, for $U^\ell = \widehat{U}(t,\xi)\mathbf{1}_{|\xi|\leq 1}$ *and* $U^h = \widehat{U}(t,\xi)\mathbf{1}_{|\xi|\geq 1}$,

$$\|U^\ell(t)\|_{L^\infty} \leq Ct^{-1/2}\|U_0\|_{L^1}, \tag{7}$$

$$\|U^h(t)\|_{L^2} \leq Ce^{-\gamma_* t}\|U_0\|_{L^2}, \tag{8}$$

In the multi-dimensional setting: The Kalman rank condition leads to similar decay estimates but is not necessary to justify the stability.

Toy-model analysis

Toy-model analysis

Let us look at the damped p-system:

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + u = 0. \end{cases}$$

Standard H^1 estimates:

$$\frac{d}{dt}\|(\rho, u, \partial_x \rho \partial_x u)\|_{L^2}^2 + \|(u, \partial_x u)\|_{L^2}^2 = 0$$

Cross estimates:

$$\frac{d}{dt}\int_{\mathbb{R}} u\partial_x \rho \, dx + \|\partial_x \rho\|_{L^2}^2 = \|\partial_x u\|_{L^2}^2 + \int_{\mathbb{R}} u\partial_x \rho.$$

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Using Young inequality and gathering the estimates, we get

$$\frac{d}{dt}\mathcal{L}_1(t) + \|(u, \partial_x u)(t)\|_{L^2}^2 + \|\partial_x \rho(t)\|_{L^2}^2 \leq 0, \tag{9}$$

where

$$\mathcal{L}_1(t) = \|(\rho, u, \partial_x \rho, \partial_x u)\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} u\partial_x \rho \, dx \;\sim \|(\rho, u, \partial_x \rho, \partial_x u)\|_{L^2}^2$$

How to get decay estimates from here?

Fourier heuristics

We have

$$\frac{d}{dt}\mathcal{L}_1(t) + \|(u, \partial_x u)(t)\|_{L^2}^2 + \|\partial_x \rho(t)\|_{L^2}^2 \leq 0. \tag{10}$$

Heuristically, applying the Fourier transform, it reads

$$\frac{d}{dt}\mathcal{L}_1(t) + \|\min(1, \xi)(\widehat{u}, \widehat{\rho})\|_{L^2}^2 \leq 0. \tag{11}$$

From which it is easy to obtain

- A heat behavior for low frequencies,
- Exponential decay for high frequencies:

$$\|(\rho, u)^{\ell}(t)\|_{L^\infty} \leq Ct^{-1/2}\|(\rho_0, u_0)\|_{L^1}, \tag{12}$$

$$\|(\rho, u)^{h}(t)\|_{L^2} \leq Ce^{-\gamma_* t}\|(\rho_0, u_0)\|_{L^2}. \tag{13}$$

How to obtain (11) rigorously?

First approach: Beauchard-Zuazua's method

Consider

$$\mathcal{L}_\xi(t) = |(\widehat{\rho}, \widehat{u})(\xi, t)|^2 + \frac{1}{2} \min\left(\frac{1}{|\xi|}, |\xi|\right) < \widehat{u} \cdot \widehat{\rho} >_{\mathbb{C}^n} . \tag{14}$$

Second approach:

Homogeneous Littlewood-Paley decomposition

→ Allows to obtain precise decay rates, critical GWP results and to justify the strong relaxation limit.

Littlewood-Paley decomposition

Littlewood-Paley decomposition

- We define $\dot{\Delta}_j$ as dyadic blocks such that $f \in \mathcal{S}'_h(\mathbb{R}^d)$

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4} 2^j \leq |\xi| \leq \frac{8}{3} 2^j\}.$$

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- The main motivation behind this decomposition is the following Bernstein inequality: $\forall k \in \mathbb{N}$, $p \in [1,\infty]$,

$$c2^{jk}\|\dot{\Delta}_j f\|_{L^p} \leq \|D^k\dot{\Delta}_j f\|_{L^p} \leq C2^{jk}\|\dot{\Delta}_j f\|_{L^p}.$$

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- The homogeneous Besov semi-norms are defined as follows:

$$\|f\|_{\dot{B}^s_{p,1}} \triangleq \sum_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}.$$

- We have $\dot{B}^0_{p,1} \hookrightarrow L^p$, $\dot{B}^1_{2,1} \hookrightarrow \dot{H}^1$, $\dot{B}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$ and $\dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{W}^{1,\infty}$

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- For a threshold $J_0 \in \mathbb{Z}$ and $s, s' \in \mathbb{R}$, we define the high and low norms:

$$\|f\|^h_{\dot B^s_{2,1}} \triangleq \sum_{j\geq J_0} 2^{js}\|\dot\Delta_j f\|_{L^2} \quad \text{and} \quad \|f\|^\ell_{\dot B^{s'}_{p,1}} \triangleq \sum_{j\leq J_0} 2^{js'}\|\dot\Delta_j f\|_{L^p}.$$

Toy-model analysis

Back to the damped p-system:

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + u = 0.. \end{cases} \tag{15}$$

Applying the localisation operator $\dot{\Delta}_j$ to (15) and denoting $\dot{\Delta}_j f = f_j$, we have

$$\begin{cases} \partial_t \rho_j + \partial_x u_j = 0, \\ \partial_t u_j + \partial_x \rho_j + u_j = 0. \end{cases} \tag{16}$$

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Differentiating in time $\mathcal{L}_j(t) = \|(\rho_j, u_j, \partial_x \rho_j, \partial_x u_j)(t)\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} u_j \partial_x \rho_j \, dx$, we get

$$\frac{d}{dt}\mathcal{L}_j(t) + \|(u_j, \partial_x u_j)\|_{L^2}^2 + \|\partial_x \rho_j\|_{L^2}^2 \leq 0. \tag{17}$$

Using Bernstein inequality, we have

$$\frac{d}{dt}\mathcal{L}_j(t) + \min(1, 2^{2j})\|(u_j, \rho_j)\|_{L^2}^2 \leq 0, \tag{18}$$

where $2^{2j} \sim |\xi|^2$.

We are going to use the following lemma.

Lemma

Let $p \geq 1$ and $X : [0, T] \to \mathbb{R}^+$ be a continuous function such that X^p is a.e. differentiable. If

$$\frac{1}{p}\frac{d}{dt}X^p + bX^p \leq AX^{p-1} \quad a.e.\ on \quad [0, T].$$

Then, for all $t \in [0, T]$, we have

$$X(t) + b\int_0^t X \leq X_0 + \int_0^t A.$$

Applying this lemma to

$$\frac{d}{dt}\mathcal{L}_j(t) + \min(1, 2^{2j})\|(u_j, \rho_j)\|_{L^2}^2 \leq 0, \tag{19}$$

since $\mathcal{L}_j \sim \|(u_j, \rho_j)\|_{L^2}^2$, we obtain

$$\sqrt{\mathcal{L}_j(t)} + \min(1, 2^{2j})\int_0^t \|(u_j, \rho_j)\|_{L^2} \leq 0. \tag{20}$$

Using that $\sqrt{\mathcal{L}_j(t)} \sim \|(u_j, \rho_j)\|_{L^2}$, we get

$$\|(u_j, \rho_j)(t)\|_{L^2} + \min(1, 2^{2j}) \int_0^t \|(u_j, \rho_j)\|_{L^2} \leq 0. \tag{21}$$

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- For high frequencies: $j \geq 0 \implies \min(1, 2^{2j}) = 1$.

Multiplying (21) by 2^{js} for $s \in \mathbb{R}$ and summing on $j \geq 0$, we obtain

$$\|(u, \rho)(t)\|^h_{\dot{B}^s_{2,1}} + \|(u, \rho)\|^h_{L^1_T(\dot{B}^s_{2,1})} \leq 0.$$

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$$\|(u, \rho)(t)\|^\ell_{\dot{B}^s_{2,1}} + \|(u, \rho)\|^\ell_{L^1_T(\dot{B}^{s+2}_{2,1})} \leq 0.$$

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- Heat effect in low frequencies and exponential decay in high frequencies.
- From here: optimal decay rates using time-weights and interpolations.
- Notice the $L^1_T(B^{s+2}_{2,1})$ norm compared to the usual $L^2_T(H^{s+1})$ norm.

General hyperbolic hypocoercivity

Back to

$$\partial_t U + A\partial_x U + BU = 0.$$

Under the Kalman rank condition (or the Shizuta-Kawashima) condition for (A, B), differentiating in time the following functional

$$\mathcal{L}_j(t) = \|U_j(t)\|_{H^1}^2 + \eta \int_{\mathbb{R}} \left(\sum_{k=1}^{n-1} < BA^{k-1}U_j, BA^k \partial_x U_j > \right)$$

leads to

$$\frac{d}{dt}\mathcal{L}_j + \min(1, 2^{2j})\mathcal{L}_j \leq 0$$

and thus similar estimates.

- What we have just seen allows us to recover the classical existence results for nonlinear systems in a slightly better framework:

$$\dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \quad \textbf{vs} \quad H^s \quad \text{for } s > \frac{d}{2} + 1.$$

- Recalling that

$$H^s(s > \frac{d}{2} + 1) \hookrightarrow B^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p},\frac{d}{2}+1}_{p,2}(p > 2) \hookrightarrow \mathcal{C}^1_b.$$

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- **However, that is not the full story for these systems.** The low-frequency behaviour is more complex than what we just saw.
- A sharper understanding allow us to establish new results.

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- Recalling that

$$H^s(s > \frac{d}{2} + 1) \hookrightarrow B^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}, \frac{d}{2}+1}_{p,2}(p > 2) \hookrightarrow \mathcal{C}^1_b.$$

- **However, that is not the full story for these systems.** The low-frequency behaviour is more complex than what we just saw.
- A sharper understanding allow us to establish new results.

Essentially:

- We have to go beyond "standard hypocoercivity" in the low frequencies.
- The eigenvalues in low-frequency are purely real $\rightarrow$ It is possible to decouple the system, up to linear high-order terms (good in LF).
- For that matter we introduce a purely damped mode, in contrast with the heat behavior, in the low-frequency regime,

Low-frequency analysis.

Low frequencies in a simple case

Back to the localized damped p-system:

$$\begin{cases} \partial_t u_j + \partial_x v_j = 0 \\ \partial_t v_j + \partial_x u_j + v_j = 0, \end{cases}$$

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$$\begin{cases} \partial_t u_j - \partial^2_{xx} u_j = -\partial_x w_j \\ \partial_t w_j + w_j = -\partial^2_{xx} w_j - \partial^3_{xxx} \rho_j. \end{cases}$$

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• This diagonalisation exhibits the low-frequency behaviour observed in the spectral analysis: $\lambda_1(\xi) = \xi^2$ and $\lambda_2(\xi) = 1$ for $\xi \ll 1$.

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• To deal with the linear source terms, we use the Bernstein inequality

$$\|\partial_x f\|^{\ell}_{B^s_{p,1}} = \|f\|^{\ell}_{B^{s+1}_{p,1}} = \sum_{j \leq J_0} 2^{j(s+1)} \|f_j\|_{L^p} \leq \sum_{j \leq J_0} 2^{js} 2^j \|f_j\|_{L^p} \leq J_0 \|f\|^{\ell}_{B^s_{p,1}}.$$

where J_0 is the threshold between low and high frequencies that has to be chosen small enough.

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where J_0 is the threshold between low and high frequencies that has to be chosen small enough.

• A priori estimates in a L^p framework for $2 \leq p \leq 4$ is available in the low-frequency regime.

General case

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In the general case, the system can be rewritten as follows:

$$\begin{cases} \partial_t U_1 + A_{1,1}\partial_x U_1 + A_{1,2}\partial_x U_2 = 0, \\ \partial_t U_2 + A_{2,1}\partial_x U_1 + A_{2,2}\partial_x U_2 + DU_2 = 0. \end{cases} \tag{22}$$

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$$W \triangleq U_2 + D^{-1}A_{2,1}\partial_x U_1 + D^{-1}A_{2,2}\partial_x U_2 = D^{-1}\partial_t U_2.$$

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The system can be rewritten

$$\begin{cases} \partial_t U_1 - A_{1,2}D^{-1}A_{2,1}\partial_x\partial_x U_1 = f \\ \partial_t W + DW = g \end{cases} \tag{24}$$

where f and g are controllable in the low-frequency regime with Bernstein-type inequalities.

Question: What can we say about the second order operator $A_{1,2}D^{-1}A_{2,1}\partial_x\partial_x$ in the equation of U_1?

General case

To study the equation of U_1, we have the following property

Lemma

For $D > 0$, the following assertions are equivalent:

- *(A,B) satisfy the Kalman rank condition,*
- *the operator $\mathcal{A} := A_{1,2}D^{-1}A_{2,1}\partial_{xx}^2$ is strongly elliptic.*

$\rightarrow$ We may study the equations of W and U_1 separately, the former as a damped equation and the latter as a heat equation.

- This approach can be applied to general systems of the form:

$$\begin{cases} \partial_t U + \sum_{j=1}^{d} A^j(U)\partial_{x_j} U + G(U) = 0, \\ U_0(x,t) = U_0(x), \end{cases} \tag{25}$$

for solutions close to a constant equilibrium $\bar{U}$ such that $G(\bar{U}) = 0$.

Important assumptions:

- $A_{1,1}(\bar{U}) = 0$ which means that $\bar{u} = 0$ for fluid-type systems (Galilean transformation).
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Tools to deal with the nonlinear terms:

• Embeddings for the type:

$$\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty, \quad \dot{B}^{\frac{d}{p}+1}_{p,1} \hookrightarrow \dot{W}^{1,\infty} \quad \text{and} \quad B^s_{2,1} \hookrightarrow B^s_{p,1}$$

• Advanced product laws, commutators estimate and composition estimates to deal with the $(L^2)^h \cap (L^p)^\ell$ setting:

$$\|ab\|^h_{\dot{B}^s_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^h_{\dot{B}^s_{2,1}} + \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^h_{\dot{B}^s_{2,1}} + \|a\|^\ell_{\dot{B}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|b\|^\ell_{\dot{B}^\sigma_{p,1}} + \|b\|^\ell_{\dot{B}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|a\|^\ell_{\dot{B}^\sigma_{p,1}}.$$

Well-posedness result for nonlinear systems.

We set $Z = U - \bar{U}$.

Theorem (Danchin, C-B '22 Math. Ann.)

Let $d \geq 1$, $p \in [2,4]$. There exists $c_0 = c_0(p) > 0$ and J_0 such that if

$$\|Z_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|Z_0\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

then the system admits a unique solution Z satisfying

$$X_p(t) \lesssim \|Z_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|Z_0\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \quad \text{for all } t \geq 0,$$

where

$$X_p(t) \triangleq \|Z\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|Z\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|Z_2\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}$$
$$+ \|Z\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|Z_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|W\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Proof: Previous linear analysis + Perturbation and Bootstrap arguments.

Decay estimates

Theorem (Danchin, C-B '22)

Assuming additionally that $Z_0 \in \dot{B}^{-\sigma_1}_{2,\infty}$ *for* $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ *then there exists* $C > 0$ *such that*

$$\|Z(t)\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \leq C\,\|Z_0\|_{\dot{B}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.$$

Moreover, if $\sigma_1 > 1 - d/2$,

$$\langle t\rangle \triangleq \sqrt{1+t^2}, \quad \alpha_1 \triangleq \frac{\sigma_1 + \frac{d}{2} - 1}{2} \quad \textit{and} \quad C_0 \triangleq \|Z_0\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}} + \|Z_0\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}},$$

then Z *satisfies the following decay estimates:*

$$\sup_{t\geq 0}\left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}} Z(t)\right\|^{\ell}_{\dot{B}^{\sigma}_{2,1}} \leq CC_0 \quad \textit{if} \quad -\sigma_1 < \sigma \leq d/2 - 1,$$

$$\sup_{t\geq 0}\left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}+\frac{1}{2}} Z_2(t)\right\|^{\ell}_{\dot{B}^{\sigma}_{2,1}} \leq CC_0 \quad \textit{if} \quad -\sigma_1 < \sigma \leq d/2 - 2,$$

$$\textit{and} \quad \sup_{t\geq 0}\left\|\langle t\rangle^{2\alpha_1} Z(t)\right\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq CC_0.$$

Extensions

- The hypocoercive-type analysis can be extended to general system of any order
$$\partial_t V + A(D)V + L(D)V = 0, \quad \text{where}$$
  - $A(D)$ is a skew-symmetric homogeneous Fourier multiplier of order α,
  - $L(D)$ is a partially elliptic homogeneous Fourier multiplier of order β.
- **What dictates the decay rates is difference of order between A and L.**

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- Anisotropic case (cf. Bianchini-CB-Paicu) concerning stably stratified solutions of the 2D-Boussinesq system.
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- **Open question:** What kind of nonlinearities can we include depending on the partial effect occurring? Relation between partial dissipation, hyperbolicity and anisotropy.

- Another interesting case
$$\partial_t U + A\partial_x U + BU = 0$$
for A symmetric and B non-symmetric e.g. Euler-Maxwell system or Timoshenko system
- One must consider Kalman rank condition for (B^s, B^a) where B^s is the symmetric part of B and B^a the skew-symmetric part.

Second part: Relaxation procedure and hyperbolisation

Cattaneo approximation of the heat equation

Let us consider the heat equation on $\mathbb{R}^d$

$$\partial_t \rho - \Delta \rho = 0.$$

Its hyperbolic **Cattaneo approximation** reads

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x u_\varepsilon = 0, \\ \varepsilon^2 \partial_t u_\varepsilon + \partial_x \rho_\varepsilon + u_\varepsilon = 0. \end{cases} \tag{26}$$

When $\varepsilon \to 0$, we recover a heat equation for ρ and a Darcy-type law $u = \partial_x \rho$.

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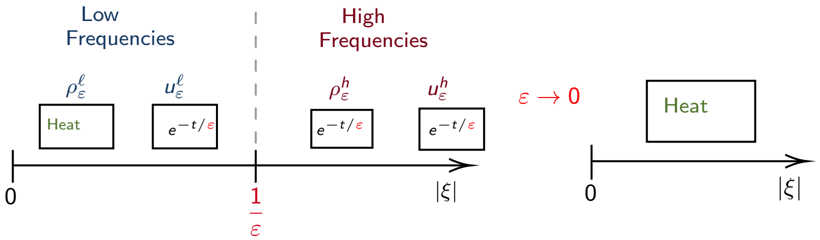
When $\varepsilon \to 0$, we recover a heat equation for ρ and a Darcy-type law $u = \partial_x \rho$.

- System (26) has a partially dissipative and hyperbolic structure.
- $\rightarrow$ *Dissipative hyperbolisation*.
- **How to justify the limit $\varepsilon \to 0$ rigorously?**

Solution first! Spectral analysis

Cattaneo approximation:

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x u_\varepsilon = 0 \\ \varepsilon^2 \partial_t u_\varepsilon + \partial_x \rho_\varepsilon + u_\varepsilon = 0 \end{cases} \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \partial_t \rho - \partial_{xx}^2 \rho = 0$$



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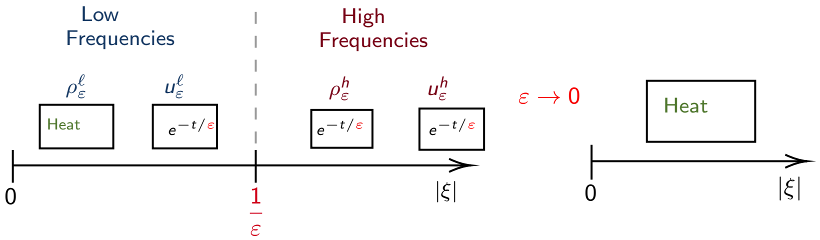

- The Cattaneo approximation creates a high-frequency regime where the solution is exponentially damped.
- The high-frequency regime vanishes in the relaxation limit.
- **Goal:** Justify this process for nonlinear systems.

Spaces

- We work with the following hybrid homogeneous Besov norms:

$$\|f\|^{h}_{\dot{B}^{s}_{2,1}} \triangleq \sum_{j \geq \frac{\eta}{\varepsilon}} 2^{js} \|\dot{\Delta}_j f\|_{L^2} \quad \text{and} \quad \|f\|^{\ell}_{\dot{B}^{s'}_{p,1}} \triangleq \sum_{j \leq \frac{\eta}{\varepsilon}} 2^{js'} \|\dot{\Delta}_j f\|_{L^p}.$$

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- For low-frequencies: $j \leq \dfrac{\eta}{\varepsilon}$,

$$\begin{cases} \partial_t \rho_j + \partial_x u_j = 0 \\ \varepsilon^2 \partial_t u_j + \partial_x \rho_j + u_j = 0, \end{cases}$$

defining the damped mode $w = v + \partial_x u$, the system can be rewritten as

$$\begin{cases} \partial_t \rho_j - \partial^2_{xx} \rho_j = -\partial_x w, \\ \varepsilon \partial_t w_j + \dfrac{w_j}{\varepsilon} = -\varepsilon \partial^3_{xxx} \rho_j - \varepsilon \partial^2_{xx} w. \end{cases}$$

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Due to the different threshold, the Bernstein inequality becomes:

$$\|\partial_x f\|^{\ell}_{B^{s}_{p,1}} \leq \frac{\eta}{\varepsilon}\|f\|^{\ell}_{B^{s}_{p,1}}.$$

Details of computations

For $s \in \mathbb{R}$, we have

$$\|(u,\varepsilon w)(t)\|^{\ell}_{B^{s}_{p,1}} + \|\rho\|^{\ell}_{L^1_T(B^{s+2}_{p,1})} + \frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_T(B^{s}_{p,1})} \leq \|(u_0,w_0)\|^{\ell}_{B^{s}_{p,1}} + \varepsilon\|w\|^{\ell}_{L^1_T(B^{s+2}_{p,1})} + \varepsilon\|\rho\|^{\ell}_{L^1_T(B^{s+3}_{p,1})}$$

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With the Berstein inequality, we have

$$\varepsilon\|\rho\|^{\ell}_{L^1_T(B^{s+3}_{p,1})} \leq \eta\|\rho\|^{\ell}_{L^1_T(B^{s+2}_{p,1})} \quad \text{and} \quad \varepsilon\|w\|^{\ell}_{L^1_T(B^{s+2}_{p,1})} \leq \frac{\eta^2}{\varepsilon}\|w\|^{\ell}_{L^1_T(B^{s}_{p,1})}.$$

Thus, choosing η small enough, these terms can be absorbed by the l.h.s.

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• This estimate provides $\mathcal{O}(\varepsilon)$ bounds on $w = u + \partial_x \rho$ which is crucial to justify the relaxation.

• **High frequencies $j \geq \frac{\eta}{\varepsilon}$:** Hypocoercivity-type approach but there is no damped mode!

High frequencies trick

To be able to recover $\mathcal{O}(\varepsilon)$ bounds on w in high frequencies, we use the Bernstein inequality

$$\|f\|^h_{B^s_{2,1}} \leq \frac{\varepsilon}{\eta}\|\partial_x f\|^h_{B^s_{2,1}}.$$

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Say you want to obtain uniform bounds for w in $B^{\frac{d}{2}}_{2,1}$, then you should assume that the initial data are in $B^{\frac{d}{2}+1}_{2,1}$ and use that

$$\|w\|^h_{B^{\frac{d}{2}}_{2,1}} \leq \frac{\varepsilon}{\eta}\|w\|^h_{B^{\frac{d}{2}+1}_{2,1}}.$$

$\Longrightarrow$ We must study the low and high frequencies at different regularities.

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In the general case, the system can be rewritten as follows:

$$\begin{cases} \partial_t Z_1 + \sum_{k=1}^{d} \Big(A^k_{1,1}(V)\partial_k Z_1 + A^k_{1,2}(V)\partial_k Z_2 \Big) = 0, \\ \partial_t Z_2 + \sum_{k=1}^{d} \Big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2 \Big) + \dfrac{L_2 Z_2}{\varepsilon} = 0. \end{cases}$$

We define the damped mode:

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$$W \triangleq Z_2 + \varepsilon \sum_{k=1}^{d} L_2^{-1}(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2) = -{L_2}^{-1}\partial_t Z_2.$$

The system can be rewritten

$$\begin{cases} \partial_t W + \dfrac{L_2 W}{\varepsilon} = g \\ \partial_t Z_1 - \varepsilon \sum_{k=1}^{d} \sum_{\ell=1}^{d} \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell Z_1 = f \end{cases} \tag{27}$$

where f and g are controllable in the low-frequency regime.

General case

To study the equation of Z_1, we have the following property

Lemma

Assume that $\forall\ k \in \{1, \cdots, d\}$, $\bar{A}^k_{1,1} = 0$. *The following assertions are equivalent:*

- *the system satisfy the (SK) condition at* $\bar{V}$*;*
- *the operator* $\mathcal{A} := \sum_{k=1}^{d} \sum_{\ell=1}^{d} \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell$ *is strongly elliptic.*

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$\rightarrow$ We may study the equations of W and Z_1 separately, the former as a damped equation and the latter as a heat equation.

Back to the compressible Euler equations

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The system reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

The damped mode associated to the relaxation is $w = u + \dfrac{\nabla P(\rho)}{\rho}$.

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• Let $\mathcal{N}$ be the solution of the porous media equation:

$$\partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0.$$

Then, using that $\|w\|_{L^1_T(B^s_{p,1})} = \mathcal{O}(\varepsilon)$, in the error estimates for $\widetilde{\rho} = \rho - \mathcal{N}$, we can justify that ρ converges strongly toward $\mathcal{N}$ in $B^{s-1}_{p,1}$.

Relaxation result

Theorem (Danchin, C-B, Math. Ann. 2022)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$.

- *Let $\bar{\rho}$ be a strictly positive constant and $(\rho^\varepsilon - \bar{\rho}, u^\varepsilon)$ be the solution of the compressible Euler system with damping (constructed with the previous arguments)*
- *Let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:*

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \in \dot{B}^{\frac{d}{p}}_{p,1} \end{cases}$$

If we assume that

$$\|\rho_0^\varepsilon - \mathcal{N}_0\|_{B^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\rho^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\rho^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}+1}_{p,1})} + \left\|\frac{\nabla P(\rho^\varepsilon)}{\rho^\varepsilon} + u^\varepsilon\right\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Remarks

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- Performing a similar analysis with Sobolev spaces does not allow (to the best of my knowledge) to exhibit an explicit convergence rate.
- It only leads to $\|w\|_{L^2_T(H^s)} = \mathcal{O}(1)$ **vs** $\|w\|_{L^1_T(B^s_{2,1})} = \mathcal{O}(\varepsilon)$

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- It only leads to $\|w\|_{L^2_T(H^s)} = \mathcal{O}(1)$ **vs** $\|w\|_{L^1_T(B^s_{2,1})} = \mathcal{O}(\varepsilon)$
- First result to establish the strong relaxation limit in the multi-dimensional setting.
- It can be employed in many other contexts.

The Jin-Xin Approximation.

Jin-Xin Approximation

We justified the strong convergence of the diffusive Jin-Xin approximation

$$\begin{cases} \dfrac{\partial}{\partial t}u + \displaystyle\sum_{i=1}^{d} \dfrac{\partial}{\partial x_i} v_i = 0, \\ \varepsilon^2 \dfrac{\partial}{\partial t} v_i + A_i \dfrac{\partial}{\partial x_i} u = -\big(v_i - f_i(u)\big), \qquad i = 1, 2, \dots, d, \end{cases} \tag{28}$$

toward viscous conservation laws:

$$\frac{\partial}{\partial t}u^* + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i(u^*) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(A_i \frac{\partial}{\partial x_i} u^*). \tag{29}$$

- In a L^2 framework, collaboration with L-Y. Shou (JDE) '23
- In an hybrid $L^2 - L^p$ framework, collaboration with L-Y Shou and J. Zhang.
- Applications in numerical analysis.

The HPC System

In joint work with Q. He and L-Y. Shou, we studied the following hyperbolic-parabolic system:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) + \dfrac{1}{\varepsilon}\rho u - \mu\rho\nabla\phi = 0, \\ \partial_t\phi - \Delta\phi - a\rho + b\phi = 0, \qquad x \in \mathbb{R}^d, \quad t > 0, \end{cases} \tag{HPC}$$

In this case, when $\varepsilon \to 0$, we show that the diffusive-rescaled solution of (HPC) converges strongly to the solution of the Keller-Segel system:

$$\begin{cases} \partial_t \rho - \mathrm{div}\,(\nabla P(\rho) - \mu\rho\nabla\phi) = 0, \\ \rho u = -\nabla P(\rho) + \mu\rho\nabla\phi, \\ -\Delta\phi - a\rho + b\phi = 0, \end{cases} \tag{KS}$$

Multifluid system

In a joint work with C. Burtea, J. Tan and L.-Y. Shou, we studied the following damped Baer-Nunziato system:

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \dfrac{\alpha_+ \alpha_-}{2\mu + \lambda} (P_+ (\rho_+) - P_- (\rho_-)), \\ \partial_t (\alpha_\pm \rho_\pm) + \mathrm{div}\, (\alpha_\pm \rho_\pm u) = 0, \\ \partial_t (\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = \alpha_+ P_+ (\rho_+) + \alpha_- P_- (\rho_-) \end{cases} \qquad (BN)$$

Limit $\lambda, \mu, \nu \to 0$.

- Difficulties: the entropy that is naturally associated with this system is only positive semi-definite.
- The system (BN) is not a system of conservation laws
- We find an ad-hoc change of variables that enables us to symmetrize the system with a good structure to treat the nonlinear terms.

Other applications:

- Hyperbolic Navier-Stokes system, on-going work with S. Kawashima, J. Xu and E. Zuazua.
- 2D-Boussinesq System (Bianchini-CB-Paicu) ARMA '24.
- Baer-Nunziato System (Burtea-CB-Tan), M3AS '23.
- Chemotaxis/Keller-Segel, (CB-He-Shou) SIAM '23.

Conclusion

- Hypocoercivity tells you that when the dissipation is not strong enough, its interactions with the hyperbolic part can make up for the lack of coercivity.
- When the skew-symmetric operator A and the dissipative B are of different order then the decay rates may not be exponential and the rates depend on the difference of their order.
- In the full space $\mathbb{R}^d$ and the Torus $\mathbb{T}^d$, the classical hypocoercivity techniques need to be extended to treat the low frequencies.
- The hyperbolic relaxation creates a temporary exponentially stable high-frequency regime and the low frequencies correspond to the behavior of the limit system.

Thank you!

Formal link between (IPM) and (2D-B)

The 2-dimensional Boussinesq system read

$$\begin{cases} \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \eta \mathbf{g}, \qquad \mathbf{g} = (0, -g), \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{E}$$

The linearized system around $\overline{\rho}_{\text{eq}}(y) = \rho_0 - y$, reads

$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = 0, \\ \varepsilon^2 \partial_t \Omega - \mathcal{R}_1 b + \Omega = 0. \end{cases} \tag{30}$$

where

$$\mathcal{R}_1 = \frac{\partial_x}{(-\Delta)^{-\frac{1}{2}}}$$

Formally, as $\varepsilon \to 0$, the second equation gives the Darcy's law $\widetilde{\Omega}^\varepsilon = \mathcal{R}_1 \widetilde{b}^\varepsilon$ and inserting it in the first one gives the linear part of the incompressible porous media equation:

$$\partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1^2 \widetilde{b}^\varepsilon = 0.$$

Overdamping

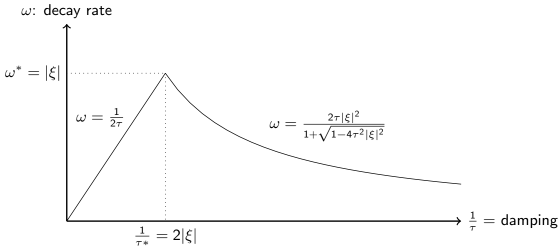

Figure: A graph of overdamping phenomenon for System (??).