

Hyperbolic approximation: Hypocoercivity and hybrid Besov spaces

Timothée Crin-Barat

Chair of Dynamics, Control and Numerics, Department of Data Science, FAU, Erlangen, Germany

Séminaire A3
Université d'Amiens
27 Mars 2023

Paradox of heat conduction

- One of the most successful models in continuum physics is Fourier's law of heat conduction

$$\mathbf{q} = -\kappa \nabla T$$

where \mathbf{q} is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

- One of the most successful models in continuum physics is Fourier's law of heat conduction

$$\mathbf{q} = -\kappa \nabla T$$

where \mathbf{q} is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

- From this law, one can deduce a diffusion equation for the temperature

$$\partial_t T - \kappa \Delta T = 0$$

- One of the most successful models in continuum physics is Fourier's law of heat conduction

$$\mathbf{q} = -\kappa \nabla T$$

where \mathbf{q} is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

- From this law, one can deduce a diffusion equation for the temperature

$$\partial_t T - \kappa \Delta T = 0$$

- And for instance, the full compressible Navier-Stokes system in \mathbb{R}^d reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho \mathbf{u} T + \mathbf{u} p) - \kappa \Delta T - \operatorname{div}(\tau \cdot \mathbf{u}) = 0. \end{cases} \quad (1)$$

- A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

- One of the most successful models in continuum physics is Fourier's law of heat conduction

$$q = -\kappa \nabla T$$

where q is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

- From this law, one can deduce a diffusion equation for the temperature

$$\partial_t T - \kappa \Delta T = 0$$

- And for instance, the full compressible Navier-Stokes system in \mathbb{R}^d reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + \rho \theta) - \kappa \Delta T - \operatorname{div}(\tau \cdot u) = 0. \end{cases} \quad (1)$$

- A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

→ Such behavior contradicts the principle of causality.

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho \mathbf{u} T + \mathbf{u} p) + \operatorname{div} \mathbf{q} - \operatorname{div}(\tau \cdot \mathbf{u}) = 0, \\ \varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} + \kappa \nabla T = 0, \end{cases} \quad (2)$$

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + u p) + \operatorname{div} \mathbf{q} - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} + \kappa \nabla T = 0, \end{cases} \quad (2)$$

- \rightarrow Finite speed of propagation for the temperature.

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + u p) + \operatorname{div} \mathbf{q} - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} + \kappa \nabla T = 0, \end{cases} \quad (2)$$

- \rightarrow Finite speed of propagation for the temperature.
- Question: How to justify rigorously the limit $\varepsilon \rightarrow 0$?

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho \mathbf{u} T + \mathbf{u} p) + \operatorname{div} \mathbf{q} - \operatorname{div}(\tau \cdot \mathbf{u}) = 0, \\ \varepsilon^2 \partial_t \mathbf{q} + \mathbf{q} + \kappa \nabla T = 0, \end{cases} \quad (2)$$

- \rightarrow Finite speed of propagation for the temperature.
 - Question: How to justify rigorously the limit $\varepsilon \rightarrow 0$?
- \rightarrow Element of response to the *paradox of heat conduction*
- \rightarrow Allows to simplify the numerical analysis.

First-order partially dissipative coupling

- During my Ph.D, in collaboration with Raphaël Danchin, we studied the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \rightarrow 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- During my Ph.D, in collaboration with Raphaël Danchin, we studied the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \rightarrow 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02 .

- During my Ph.D, in collaboration with Raphaël Danchin, we studied the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \rightarrow 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02
- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Xu-Fang '09

- During my Ph.D, in collaboration with Raphaël Danchin, we studied the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \rightarrow 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02
- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Xu-Fang '09
- Here: strong convergence in \mathbb{R}^d with $d \geq 1$ for global-in-time strong solutions being small perturbations of $(\bar{\rho}, \bar{u}) = (\bar{\rho}, 0)$ with $\bar{\rho} > 0$.

- During my Ph.D, in collaboration with Raphaël Danchin, we studied the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \rightarrow 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02
- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Xu-Fang '09
- Here: strong convergence in \mathbb{R}^d with $d \geq 1$ for global-in-time strong solutions being small perturbations of $(\bar{\rho}, \bar{u}) = (\bar{\rho}, 0)$ with $\bar{\rho} > 0$.
- Tools: Littlewood-Paley and hypocoercivity theory.

The one-dimensional damped p-system reads

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + \frac{u}{\varepsilon} = 0. \end{cases}$$

- **Goal:** obtain uniform-in- ε a priori estimates.

The one-dimensional damped p-system reads

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + \frac{u}{\varepsilon} = 0. \end{cases}$$

- **Goal:** obtain uniform-in- ε a priori estimates.
- **First difficulty:** how to handle the *partially dissipative* structure? Indeed, standard energy estimates leads to:

$$\frac{d}{dt} \|(\rho, u)\|_{L^2}^2 + \frac{1}{\varepsilon} \|u\|_{L^2}^2 \leq 0$$

→ **lack of coercivity: no time-decay information for ρ .**

The one-dimensional damped p-system reads

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + \frac{u}{\varepsilon} = 0. \end{cases}$$

- **Goal:** obtain uniform-in- ε a priori estimates.
- **First difficulty:** how to handle the *partially dissipative* structure? Indeed, standard energy estimates leads to:

$$\frac{d}{dt} \|(\rho, u)\|_{L^2}^2 + \frac{1}{\varepsilon} \|u\|_{L^2}^2 \leq 0$$

→ **lack of coercivity: no time-decay information for ρ .**

- **Idea:** Inspired by the hypocoercivity theory, consider the following perturbed functional

$$\mathcal{L}^2 = \|(\rho, u, \partial_x \rho, \partial_x u)\|_{L^2}^2 + \varepsilon \int_{\mathbb{R}} u \partial_x \rho.$$

Differentiating in time this functional, one obtains

$$\frac{d}{dt} \mathcal{L}^2 + \frac{1}{\varepsilon} \|(u, \partial_x u)\|_{L^2}^2 + \varepsilon \|\partial_x \rho\|_{L^2}^2 \leq 0.$$

- **Second difficulty:** the decay rates depend on the frequencies and the relaxation parameter ε .

From the previous estimate, one obtains formally

$$\ll \frac{d}{dt} \|(\rho, u)\|_{L^2} + \min\left(\frac{1}{\varepsilon}, \varepsilon|\xi|^2\right) \|(\rho, u)\|_{L^2} \leq 0. \gg$$

- **Second difficulty:** the decay rates depend on the frequencies and the relaxation parameter ε .

From the previous estimate, one obtains formally

$$\ll \frac{d}{dt} \|(\rho, u)\|_{L^2} + \min\left(\frac{1}{\varepsilon}, \varepsilon|\xi|^2\right) \|(\rho, u)\|_{L^2} \leq 0. \gg$$

- Therefore, in low frequencies $|\xi| < \frac{1}{\varepsilon}$, the solution behaves as the solution of the heat equation.
- And, in high frequencies $|\xi| > \frac{1}{\varepsilon}$, the solution is exponentially damped.

- **Second difficulty:** the decay rates depend on the frequencies and the relaxation parameter ε .

From the previous estimate, one obtains formally

$$\ll \frac{d}{dt} \|(\rho, u)\|_{L^2} + \min\left(\frac{1}{\varepsilon}, \varepsilon|\xi|^2\right) \|(\rho, u)\|_{L^2} \leq 0. \gg$$

- Therefore, in low frequencies $|\xi| < \frac{1}{\varepsilon}$, the solution behaves as the solution of the heat equation.
- And, in high frequencies $|\xi| > \frac{1}{\varepsilon}$, the solution is exponentially damped.
- One has

$$\|(\rho, u)^h(t)\|_{L^2(\mathbb{R}^d)} \leq C e^{-\lambda \frac{t}{\varepsilon}} \|(\rho_0, u_0)\|_{L^2(\mathbb{R}^d)},$$

$$\|(\rho, u)^\ell(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(\varepsilon t)^{-\frac{d}{2}} \|(\rho_0, u_0)\|_{L^1(\mathbb{R}^d)}$$

where $(\rho, u)^h$ and $(\rho, u)^\ell$ correspond, respectively, to the high and low frequencies of the solution.

Hyperbolic hypocoercivity

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A \partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \quad (\text{SK})$$

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A \partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \quad (\text{SK})$$

Such condition is actually equivalent to the Kalman rank condition for the couple (A, B) .

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A \partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \quad (\text{SK})$$

Such condition is actually equivalent to the Kalman rank condition for the couple (A, B) .

Inspired by this fact and the theories of hypo-coercivity and hypo-ellipticity, Beauchard and Zuazua constructed the following Lyapunov functional

$$\mathcal{L}^2 \triangleq \|U\|_{H^1}^2 + \int_{\mathbb{R}^d} \mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1} \widehat{U} \cdot BA^k \widehat{U}).$$

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A \partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \quad (\text{SK})$$

Such condition is actually equivalent to the Kalman rank condition for the couple (A, B) .

Inspired by this fact and the theories of hypo-coercivity and hypo-ellipticity, Beauchard and Zuazua constructed the following Lyapunov functional

$$\mathcal{L}^2 \triangleq \|U\|_{H^1}^2 + \int_{\mathbb{R}^d} \mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \sum_{k=1}^{n-1} \varepsilon_k (BA^{k-1} \widehat{U} \cdot BA^k \widehat{U}).$$

If the (SK) condition is satisfied, differentiating in time this functional leads to

$$\frac{d}{dt} \mathcal{L} + \kappa \min(1, |\xi|^2) \mathcal{L} \leq 0 \quad \text{and} \quad \mathcal{L} \sim \|U\|_{H^1}$$

- However, this *hyperbolic hypocoercivity* approach does not depict the full story for these systems.
- It is suitable to study the high frequencies of the solution but not the low frequencies.
- As we shall see, the distinction between these two regime is crucial: if you do not distinguish two frequency regimes then you can only recover the worst behavior of the two regimes for all the frequencies.

- Back to the damped p -system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0. \end{cases} \quad (3)$$

A spectral analysis of the matrix associated to the system:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \frac{1}{\varepsilon} \end{pmatrix}$$

shows that:

- Back to the damped p -system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0. \end{cases} \quad (3)$$

A spectral analysis of the matrix associated to the system:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \frac{1}{\varepsilon} \end{pmatrix}$$

shows that:

- In low frequencies ($|\xi| \ll \frac{1}{\varepsilon}$), there are two **real** eigenvalues $\frac{1}{\varepsilon}$ and $\varepsilon\xi^2$.

- Back to the damped p -system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0. \end{cases} \quad (3)$$

A spectral analysis of the matrix associated to the system:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \frac{1}{\varepsilon} \end{pmatrix}$$

shows that:

- In low frequencies ($|\xi| \ll \frac{1}{\varepsilon}$), there are two **real** eigenvalues $\frac{1}{\varepsilon}$ and $\varepsilon\xi^2$.
- In high frequencies ($|\xi| \gg \frac{1}{\varepsilon}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\frac{1}{2\varepsilon}$.

- Back to the damped p -system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0. \end{cases} \quad (3)$$

A spectral analysis of the matrix associated to the system:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \frac{1}{\varepsilon} \end{pmatrix}$$

shows that:

- In low frequencies ($|\xi| \ll \frac{1}{\varepsilon}$), there are two **real** eigenvalues $\frac{1}{\varepsilon}$ and $\varepsilon\xi^2$.
- In high frequencies ($|\xi| \gg \frac{1}{\varepsilon}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\frac{1}{2\varepsilon}$.
- The threshold between low and high frequencies is at $\frac{1}{\varepsilon}$.

- Back to the damped p -system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0. \end{cases} \quad (3)$$

A spectral analysis of the matrix associated to the system:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \frac{1}{\varepsilon} \end{pmatrix}$$

shows that:

- In low frequencies ($|\xi| \ll \frac{1}{\varepsilon}$), there are two **real** eigenvalues $\frac{1}{\varepsilon}$ and $\varepsilon\xi^2$.
- In high frequencies ($|\xi| \gg \frac{1}{\varepsilon}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\frac{1}{2\varepsilon}$.
- The threshold between low and high frequencies is at $\frac{1}{\varepsilon}$.
- **→ The behavior of solution depend on the relation between ξ and ε and there is an extra property to use in low frequencies.**

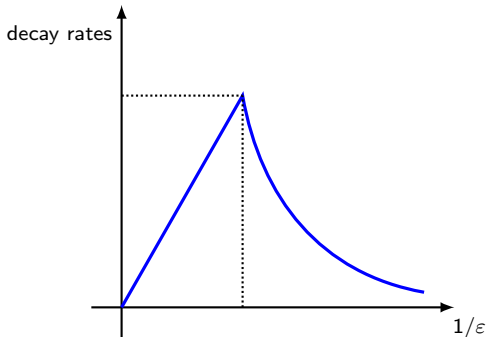
- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon}$ → **Leads to crucial uniform estimates**

Insights from the spectral analysis

- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon}$ → **Leads to crucial uniform estimates**
- The asymptotic behaviour of the solution when $\varepsilon \rightarrow 0$ is not so intuitive.

- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon}$ → **Leads to crucial uniform estimates**
- The asymptotic behaviour of the solution when $\varepsilon \rightarrow 0$ is not so intuitive.
 - Naively, we expect that as the damping coefficient becomes larger the dissipation becomes more dominant.

- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon} \rightarrow$ **Leads to crucial uniform estimates**
- The asymptotic behaviour of the solution when $\varepsilon \rightarrow 0$ is not so intuitive.
 - Naively, we expect that as the damping coefficient becomes larger the dissipation becomes more dominant.
 - However, the so-called *overdamping* effect occurs: **the decay rates are related to $\min(\varepsilon, 1/\varepsilon)$.**



- To handle this phenomenon, not mixing the frequencies and isolating the bad mode is crucial. (Not possible with the hypocoercivity approach)

New goal: replicate what the spectral analysis tells us at the level of a priori estimates.

- We define $\dot{\Delta}_j$ the dyadic blocks such that for $f \in \mathcal{S}'(\mathbb{R}^d)$

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}.$$

- We define $\dot{\Delta}_j$ the dyadic blocks such that for $f \in \mathcal{S}'(\mathbb{R}^d)$

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}.$$

- One of the main motivation behind this decomposition is the following Bernstein inequality: for all $j \in \mathbb{Z}$

$$\|D^k \dot{\Delta}_j f\|_{L^p} \sim 2^{jk} \|\dot{\Delta}_j f\|_{L^p}.$$

- We define $\dot{\Delta}_j$ the dyadic blocks such that for $f \in \mathcal{S}'(\mathbb{R}^d)$

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}.$$

- One of the main motivation behind this decomposition is the following Bernstein inequality: for all $j \in \mathbb{Z}$

$$\|D^k \dot{\Delta}_j f\|_{L^p} \sim 2^{jk} \|\dot{\Delta}_j f\|_{L^p}.$$

- The homogeneous Besov semi-norms are defined as follows:

$$\|f\|_{\dot{\mathbb{B}}_{p,1}^s} \triangleq \sum_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}.$$

- We define $\dot{\Delta}_j$ the dyadic blocks such that for $f \in \mathcal{S}'(\mathbb{R}^d)$

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}.$$

- One of the main motivation behind this decomposition is the following Bernstein inequality: for all $j \in \mathbb{Z}$

$$\|D^k \dot{\Delta}_j f\|_{L^p} \sim 2^{jk} \|\dot{\Delta}_j f\|_{L^p}.$$

- The homogeneous Besov semi-norms are defined as follows:

$$\|f\|_{\dot{\mathbb{B}}_{p,1}^s} \triangleq \sum_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}.$$

- The high-frequency restricted norms: $\|f\|_{\dot{\mathbb{B}}_{2,1}^s}^h \triangleq \sum_{j \geq \frac{1}{\varepsilon}} 2^{js} \|\dot{\Delta}_j f\|_{L^2},$
- The low-frequency restricted norms: $\|f\|_{\dot{\mathbb{B}}_{p,1}^{s'}}^\ell \triangleq \sum_{j \leq \frac{1}{\varepsilon}} 2^{js'} \|\dot{\Delta}_j f\|_{L^p}.$

To recover the "missing" property in low-frequency, let us look again at the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0, \end{cases}$$

To recover the "missing" property in low-frequency, let us look again at the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0, \end{cases}$$

Defining the damped mode $w = v + \varepsilon \partial_x u$, the system can be rewritten

$$\begin{cases} \partial_t u - \varepsilon \partial_{xx}^2 u = -\partial_x w \\ \partial_t w + \frac{w}{\varepsilon} = -\varepsilon \partial_{xx}^2 v. \end{cases}$$

To recover the "missing" property in low-frequency, let us look again at the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u + \frac{v}{\varepsilon} = 0, \end{cases}$$

Defining the damped mode $w = v + \varepsilon \partial_x u$, the system can be rewritten

$$\begin{cases} \partial_t u - \varepsilon \partial_{xx}^2 u = -\partial_x w \\ \partial_t w + \frac{w}{\varepsilon} = -\varepsilon \partial_{xx}^2 v. \end{cases}$$

→ **Highlights the behavior observed in the spectral analysis, not just the heat effect as depicted in the previous references.**

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

$$\begin{cases} \partial_t \dot{\Delta}_j u - \varepsilon \partial_{xx}^2 \dot{\Delta}_j u = -\partial_x \dot{\Delta}_j w, \\ \partial_t \dot{\Delta}_j w + \frac{\dot{\Delta}_j w}{\varepsilon} = -\varepsilon \partial_{xx}^2 \dot{\Delta}_j v. \end{cases}$$

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

$$\begin{cases} \partial_t \dot{\Delta}_j u - \varepsilon \partial_{xx}^2 \dot{\Delta}_j u = -\partial_x \dot{\Delta}_j w, \\ \partial_t \dot{\Delta}_j w + \frac{\dot{\Delta}_j w}{\varepsilon} = -\varepsilon \partial_{xx}^2 \dot{\Delta}_j v. \end{cases}$$

- We can now study these two equations in a decoupled fashion thanks to the Bernstein inequality:

$$\|\partial_x f\|_{B_{p,1}^s}^\ell \lesssim 2^j \|f\|_{B_{p,1}^s}^\ell \lesssim \frac{1}{\varepsilon} \|f\|_{B_{p,1}^s}^\ell.$$

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

$$\begin{cases} \partial_t \dot{\Delta}_j u - \varepsilon \partial_{xx}^2 \dot{\Delta}_j u = -\partial_x \dot{\Delta}_j w, \\ \partial_t \dot{\Delta}_j w + \frac{\dot{\Delta}_j w}{\varepsilon} = -\varepsilon \partial_{xx}^2 \dot{\Delta}_j v. \end{cases}$$

- We can now study these two equations in a decoupled fashion thanks to the Bernstein inequality:

$$\|\partial_x f\|_{B_{p,1}^s}^\ell \lesssim 2^j \|f\|_{B_{p,1}^s}^\ell \lesssim \frac{1}{\varepsilon} \|f\|_{B_{p,1}^s}^\ell.$$

- And thus **in L^p spaces!**

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

$$\begin{cases} \partial_t \dot{\Delta}_j u - \varepsilon \partial_{xx}^2 \dot{\Delta}_j u = -\partial_x \dot{\Delta}_j w, \\ \partial_t \dot{\Delta}_j w + \frac{\dot{\Delta}_j w}{\varepsilon} = -\varepsilon \partial_{xx}^2 \dot{\Delta}_j v. \end{cases}$$

- We can now study these two equations in a decoupled fashion thanks to the Bernstein inequality:

$$\|\partial_x f\|_{B_{p,1}^s}^\ell \lesssim 2^j \|f\|_{B_{p,1}^s}^\ell \lesssim \frac{1}{\varepsilon} \|f\|_{B_{p,1}^s}^\ell.$$

- And thus **in L^p spaces!**
- One obtains

$$\|(u, w)\|_{B_{p,1}^s}^\ell + \varepsilon \|u\|_{L_T^1(B_{p,1}^{s+2})}^\ell + \frac{1}{\varepsilon} \|w\|_{L_T^1(B_{p,1}^s)}^\ell \leq \|(u_0, w_0)\|_{B_{p,1}^s}^\ell$$

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

$$\begin{cases} \partial_t \dot{\Delta}_j u - \varepsilon \partial_{xx}^2 \dot{\Delta}_j u = -\partial_x \dot{\Delta}_j w, \\ \partial_t \dot{\Delta}_j w + \frac{\dot{\Delta}_j w}{\varepsilon} = -\varepsilon \partial_{xx}^2 \dot{\Delta}_j v. \end{cases}$$

- We can now study these two equations in a decoupled fashion thanks to the Bernstein inequality:

$$\|\partial_x f\|_{B_{p,1}^s}^\ell \lesssim 2^j \|f\|_{B_{p,1}^s}^\ell \lesssim \frac{1}{\varepsilon} \|f\|_{B_{p,1}^s}^\ell.$$

- And thus **in L^p spaces!**
- One obtains

$$\|(u, w)\|_{B_{p,1}^s}^\ell + \varepsilon \|u\|_{L_T^1(B_{p,1}^{s+2})}^\ell + \frac{1}{\varepsilon} \|w\|_{L_T^1(B_{p,1}^s)}^\ell \leq \|(u_0, w_0)\|_{B_{p,1}^s}^\ell$$

- Due to the non-zero imaginary part of the eigenvalues in high frequencies, such L^p procedure is not available in this region.
- Nevertheless, we can still perform our analysis in such hybrid $L^2 - L^p$ framework.

The system reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

Here the damped mode verifying better properties in low frequencies is $w = u + \frac{\nabla P(\rho)}{\rho}$ (note that $w = 0$ corresponds to the Darcy law for porous media).

The system reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

Here the damped mode verifying better properties in low frequencies is $w = u + \frac{\nabla P(\rho)}{\rho}$ (note that $w = 0$ corresponds to the Darcy law for porous media).

Inserting it in the above equation one recovers:

$$\partial_t \rho - \Delta P(\rho) = \operatorname{div} w.$$

The system reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \frac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \quad (\text{E})$$

Here the damped mode verifying better properties in low frequencies is $w = u + \frac{\nabla P(\rho)}{\rho}$ (note that $w = 0$ corresponds to the Darcy law for porous media).

Inserting it in the above equation one recovers:

$$\partial_t \rho - \Delta P(\rho) = \operatorname{div} w.$$

Then, using that $\|w\|_{B_{p,1}^s} = \mathcal{O}(\varepsilon)$ (as it solves a purely damped equation), in the error estimates we can deduce that ρ converge strongly, at the rate ε , toward the solution of the porous media equation:

$$\partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0.$$

Theorem (Danchin, C-B, Math. Ann. 2022)

Let $d \geq 1$, $p \in [2, 4]$ and $\varepsilon > 0$.

- Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution of the compressible Euler system with damping (constructed with the previous arguments)
- Let $\mathcal{N} \in C_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})$ be the unique solution associated to the Cauchy problem:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0, x) = \mathcal{N}_0 \in \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}} \end{cases}$$

If we assume that

$$\|\tilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-1}} \leq C\varepsilon,$$

then

$$\|\tilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-1})} + \|\tilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} + \left\| \frac{\nabla P(\tilde{\rho}^\varepsilon)}{\tilde{\rho}^\varepsilon} + \tilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} \leq C\varepsilon.$$

Sobolev spaces

- Performing a similar analysis with Sobolev spaces does not allow (to the best of my knowledge) to exhibit an explicit convergence rate.
- It only leads to $\|w\|_{L_T^2(H^s)} = \mathcal{O}(1)$ (instead of $\|w\|_{L_T^1(B^s)} = \mathcal{O}(\varepsilon)$)

Sobolev spaces

- Performing a similar analysis with Sobolev spaces does not allow (to the best of my knowledge) to exhibit an explicit convergence rate.
- It only leads to $\|w\|_{L_T^2(H^s)} = \mathcal{O}(1)$ (instead of $\|w\|_{L_T^1(B^s)} = \mathcal{O}(\varepsilon)$)

Relaxation limit and threshold

- One has to be careful when justifying the limit $\varepsilon \rightarrow 0$. Indeed, recall that the threshold between low and high frequencies is situated at $\frac{1}{\varepsilon}$.
- Therefore, when $\varepsilon \rightarrow 0$, the low frequencies recovers the whole space of frequency and the high frequencies disappear.
- This is coherent as the behavior of the solution in low frequencies is similar to the one of the limit system.

Application to the hyperbolic-parabolic chemotaxis system

In collaboration with Q. He and L-Y Shou, we showed that the solutions of the hyperbolic-parabolic chemotaxis system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon \partial_t(\rho u) + \varepsilon \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \rho u - \mu \rho \nabla \phi = 0, \\ \varepsilon \partial_t \phi - \Delta \phi - a \rho + b \phi = 0, \end{cases} \quad x \in \mathbb{R}^d, \quad t > 0, \quad (\text{HPC})$$

converges as $\varepsilon \rightarrow 0$ to the solutions of the Keller-Segel system:

$$\begin{cases} \partial_t \rho^* - \operatorname{div}(\nabla P(\rho^*)) - \mu \rho^* \nabla \phi^* = 0, \\ \rho^* u^* = -\nabla P(\rho^*) + \mu \rho^* \nabla \phi^*, \\ -\Delta \phi^* - a \rho^* + b \phi^* = 0. \end{cases} \quad (\text{KS})$$

Such models are widely used in biology to describe the movement of an organism or entity in response to a chemical stimulus.

In collaboration with Q. He and L-Y Shou, we showed that the solutions of the hyperbolic-parabolic chemotaxis system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon \partial_t(\rho u) + \varepsilon \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \rho u - \mu \rho \nabla \phi = 0, \\ \varepsilon \partial_t \phi - \Delta \phi - a \rho + b \phi = 0, \end{cases} \quad x \in \mathbb{R}^d, \quad t > 0, \quad (\text{HPC})$$

converges as $\varepsilon \rightarrow 0$ to the solutions of the Keller-Segel system:

$$\begin{cases} \partial_t \rho^* - \operatorname{div}(\nabla P(\rho^*)) - \mu \rho^* \nabla \phi^* = 0, \\ \rho^* u^* = -\nabla P(\rho^*) + \mu \rho^* \nabla \phi^*, \\ -\Delta \phi^* - a \rho^* + b \phi^* = 0. \end{cases} \quad (\text{KS})$$

Such models are widely used in biology to describe the movement of an organism or entity in response to a chemical stimulus.

Tools:

- Low-high frequency decomposition
- Two damped modes related to physical quantities.

Application to a (partially) hyperbolic Navier-Stokes system

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε , by the following hyperbolic system

$$\begin{cases} \partial_t u + \operatorname{div} v = 0 \\ \varepsilon \partial_t v + \nabla u + v = 0. \end{cases}$$

- Our aim is now to understand to what extent this approximation can be used to approximate systems modelling physical phenomena.

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε , by the following hyperbolic system

$$\begin{cases} \partial_t u + \operatorname{div} v = 0 \\ \varepsilon \partial_t v + \nabla u + v = 0. \end{cases}$$

- Our aim is now to understand to what extent this approximation can be used to approximate systems modelling physical phenomena.

Performing such approximation for the compressible Navier-Stokes system, one has

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + u p) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \partial_t q + q + \kappa \nabla T = 0, \end{cases} \quad (\text{NSCC})$$

Let us now see how to justify that the solution of this system converge to the solution of the classical Navier-Stokes equations.

Frequency splitting

- First of all, knowledge about the limit system is necessary. Danchin showed the existence of solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.

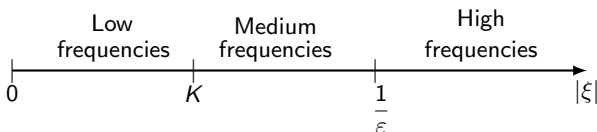
Frequency splitting

- First of all, knowledge about the limit system is necessary. Danchin showed the existence of solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.
- On the other hand, the hyperbolic approximation suggests to distinguish two distinct frequency regimes with a threshold located at $\frac{1}{\varepsilon}$.

Frequency splitting

- First of all, knowledge about the limit system is necessary. Danchin showed the existence of solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.
- On the other hand, the hyperbolic approximation suggests to distinguish two distinct frequency regimes with a threshold located at $\frac{1}{\varepsilon}$.

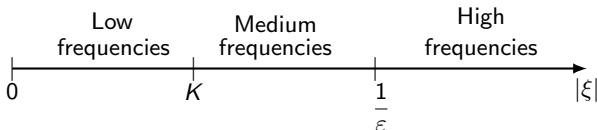
Therefore, in order to obtain the complete picture, it appears natural to divide the frequency space as follows



Frequency splitting

- First of all, knowledge about the limit system is necessary. Danchin showed the existence of solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.
- On the other hand, the hyperbolic approximation suggests to distinguish two distinct frequency regimes with a threshold located at $\frac{1}{\varepsilon}$.

Therefore, in order to obtain the complete picture, it appears natural to divide the frequency space as follows



Formally, when $\varepsilon \rightarrow 0$, it means that:

- The low frequency regime is not modified.
- The mid-frequency regime becomes larger and larger and recovers the high-frequency regime.
- The high frequency regime disappears.

And, in the limit, we retrieve the behavior of the compressible Navier-Stokes system.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|_{\dot{B}_{p,1}^s}^\ell := \sum_{j \leq J_0} 2^{js} \|f_j\|_{L^2}, \quad \|f\|_{\dot{B}_{p,1}^s}^{m,\varepsilon} := \sum_{J_0 \leq j \leq J_\varepsilon} 2^{js} \|f_j\|_{L^2}$$

and $\|f\|_{\dot{B}_{p,1}^s}^{h,\varepsilon} := \sum_{j \geq J_\varepsilon - 1} 2^{js} \|f_j\|_{L^2}$

where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\log_2(\varepsilon)$.

- Then, in each regime, different methods have to be developed to derive a priori estimates. Hypocoercivity + efficient unknowns.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|_{\dot{B}_{p,1}^s}^\ell := \sum_{j \leq J_0} 2^{js} \|f_j\|_{L^2}, \quad \|f\|_{\dot{B}_{p,1}^s}^{m,\varepsilon} := \sum_{J_0 \leq j \leq J_\varepsilon} 2^{js} \|f_j\|_{L^2}$$

and

$$\|f\|_{\dot{B}_{p,1}^s}^{h,\varepsilon} := \sum_{j \geq J_\varepsilon - 1} 2^{js} \|f_j\|_{L^2}$$

where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\log_2(\varepsilon)$.

- Then, in each regime, different methods have to be developed to derive a priori estimates. Hypocoercivity + efficient unknowns.
- Difficulty: finding the appropriate regularity indexes to work with.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|_{\dot{B}_{p,1}^s}^\ell := \sum_{j \leq J_0} 2^{js} \|f_j\|_{L^2}, \quad \|f\|_{\dot{B}_{p,1}^s}^{m,\varepsilon} := \sum_{J_0 \leq j \leq J_\varepsilon} 2^{js} \|f_j\|_{L^2}$$

and $\|f\|_{\dot{B}_{p,1}^s}^{h,\varepsilon} := \sum_{j \geq J_\varepsilon - 1} 2^{js} \|f_j\|_{L^2}$

where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\log_2(\varepsilon)$.

- Then, in each regime, different methods have to be developed to derive a priori estimates. Hypocoercivity + efficient unknowns.
- Difficulty: finding the appropriate regularity indexes to work with.

Morale

- The Cattaneo-hyperbolic approximation *creates* a temporary high frequencies regimes that disappears in the limit.
- The low frequencies corresponds the limit system.
- Difficulty: justify that the nonlinear terms can be handled between each frequency regime.

Extensions

- To what extent this hyperbolic approximation can be used? Numerical schemes, PINNs, non-conservative systems...

- To what extent this hyperbolic approximation can be used? Numerical schemes, PINNs, non-conservative systems...
- What about other operators than the laplacian?

- To what extent this hyperbolic approximation can be used? Numerical schemes, PINNs, non-conservative systems...
- What about other operators than the Laplacian?

In a collaboration with Roberta Bianchini and Marius Paicu, we showed that the stably stratified solutions of the incompressible porous media equation:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

can be approximated by the 0-th order stratified Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \quad (2DB)$$

- To what extent this hyperbolic approximation can be used? Numerical schemes, PINNs, non-conservative systems...
- What about other operators than the Laplacian?

In a collaboration with Roberta Bianchini and Marius Paicu, we showed that the stably stratified solutions of the incompressible porous media equation:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

can be approximated by the 0-th order stratified Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \quad (2DB)$$

Such justification involves anisotropic Besov spaces so as to recover crucial $L_T^1(W^{1,\infty})$ bounds on the solution.

- To what extent this hyperbolic approximation can be used? Numerical schemes, PINNs, non-conservative systems...
- What about other operators than the Laplacian?

In a collaboration with Roberta Bianchini and Marius Paicu, we showed that the stably stratified solutions of the incompressible porous media equation:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

can be approximated by the 0-th order stratified Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \quad (2DB)$$

Such justification involves anisotropic Besov spaces so as to recover crucial $L_T^1(W^{1,\infty})$ bounds on the solution.

- Question: under what conditions can an operator be approximated in this fashion?

- To what extent this hyperbolic approximation can be used? Numerical schemes, PINNs, non-conservative systems...
- What about other operators than the Laplacian?

In a collaboration with Roberta Bianchini and Marius Paicu, we showed that the stably stratified solutions of the incompressible porous media equation:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$






can be approximated by the 0-th order stratified Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \quad (2DB)$$

Such justification involves anisotropic Besov spaces so as to recover crucial $L_T^1(W^{1,\infty})$ bounds on the solution.

- Question: under what conditions can an operator be approximated in this fashion?
- Interplay of partial dissipation, anisotropy and special nonlinear structure on the time evolution of solutions to hyperbolic and dispersive systems.

Thank you for your attention!

-  T. Crin-Barat, R. Danchin, Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit, *Mathematische Annalen*, 2022.
-  T. Crin-Barat, R. Danchin, Partially dissipative hyperbolic systems in the critical regularity setting: The multi-dimensional case. *Journal de Mathématiques Pures et Appliquées*, 2022.
-  T. Crin-Barat, Q. He, L-Y. Shou, The hyperbolic-parabolic chemotaxis system for vasculogenesis: global dynamics and relaxation limit toward a Keller-Segel model. *SIAM Journal on Mathematical Analysis*, 2023.
-  T. Crin-Barat, L-Y. Shou, Diffusive relaxation limit of the multi-dimensional Jin-Xin system. *JDE*, 2023
-  R. Bianchini, T. Crin-Barat, M. Paicu, Relaxation approximation and asymptotic stability of stratified solutions to the IPM equation. [arXiv:2210.02118](https://arxiv.org/abs/2210.02118)