

# The hyperbolic-parabolic chemotaxis system modelling vasculogenesis: global dynamics and relaxation limit

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## Abstract

An Euler-type hyperbolic-parabolic system of chemotactic aggregation describing the vascular network formation is investigated in the critical regularity setting. For small initial data around a constant equilibrium state, the well-posedness of the global classical solution to the Cauchy problem with general pressure laws is established in homogeneous hybrid Besov spaces. Then, the optimal time-decay rates of the global solution are analyzed under an additional regularity assumption on the initial data. Furthermore, the relaxation limit (large friction limit) of the hyperbolic-parabolic system is justified rigorously. It is shown that as the friction coefficient tends to zero, the global solution of the hyperbolic-parabolic chemotaxis system converges to the global solution of the Keller-Segel equations with an explicit convergence rate. To capture the dissipative properties of the nonlinear system, our approach relies on the introduction of new effective unknowns in low frequencies and the construction of a Lyapunov functional in the spirit of Beauchard and Zuazua's in [5] to treat the high frequencies.

**Keywords:** Hyperbolic-parabolic chemotaxis, vascular network, hybrid Besov spaces, global well-posedness, optimal time-decay rate, relaxation limit

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# 1 Introduction

Chemotaxis is a phenomenon describing the influence of environmental chemical substances on the motion of various cells. It plays an important role in many biological process, such as cells aggregations (amoebae, bacteria, etc), embryonic development, cancer growth, vascular networks and so on, cf. [28, 34, 43, 44]. Vasculogenesis, the process of new blood vessels formation led by chemotaxis, has complex and varied mechanisms as shown recently in numerous experimental investigations. Understanding the formation of blood vessels in organisms is a long-standing problem in the life sciences, cf. [1, 2, 8, 25, 28].

In this paper, we consider the following  $d$ -dimensional ( $d \geq 1$ ) hyperbolic-parabolic chemotaxis (HPC) system modelling vasculogenesis:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \frac{1}{\varepsilon} \rho u - \mu \rho \nabla \phi = 0, \\ \partial_t \phi - \Delta \phi - a \rho + b \phi = 0, \end{cases} \quad x \in \mathbb{R}^d, \quad t > 0, \quad (\text{HPC})$$

where  $\rho = \rho(x, t) \in \mathbb{R}_+$  and  $u = u(x, t) \in \mathbb{R}^d$  are, respectively, the cell density and the cell velocity,  $\phi = \phi(x, t) \in \mathbb{R}_+$  stands for the concentration of chemoattractant secreted by cells, the constant  $\mu > 0$  measures the intensity of cell response to the chemoattractant,  $\frac{1}{\varepsilon} \rho u$  is a damping term due to the interaction between cells and the underlying substratum with the friction coefficient  $\varepsilon \in (0, 1)$ ,  $a > 0$  and  $b > 0$  are two constants denoting the growth and death rates of the chemoattractant respectively, and the pressure  $P(\rho)$  is assumed to be a smooth function depending only on the density.

The System (HPC) was recently developed by Gamba et. al. in [25] and by Ambrosi, Bussolino and Preziosi in [1] to describe blood vessel networks from in vitro observations. In addition, the System (HPC) has also been derived from the fluid-dynamical approximation of a nonlinear mean-field Fokker-Planck equation in [9].

Without the effect of chemotaxis, the System (HPC) becomes the isentropic damped compressible Euler equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \frac{1}{\varepsilon} \rho u = 0, \end{cases} \quad x \in \mathbb{R}^d, \quad t > 0. \quad (\text{DCE})$$

Recently, there has been important progress made on the analysis of global solutions to the System (DCE). For small initial data in the Sobolev spaces  $H^s$  ( $s > \frac{d}{2} + 1$ ), the global well-posedness and time-asymptotical behaviors of classical solutions for (DCE) have been studied in many significant works [19, 47, 54, 57]. In particular, it was shown in [54, 57] that the damping term  $\rho u$  can prevent the formation of shock waves that would occurs in finite time without this term. Then, using the theory of partially dissipative hyperbolic system, Fang, Xu and Kawashima showed that the System (DCE) is globally well-posed for small initial data in inhomogeneous Besov space  $B_{2,1}^{\frac{d}{2}+1}$  in [60]. Very recently, inspired by the

works of Hoff and Haspot in [29, 31] for the compressible Navier-Stokes equations, the first author and Danchin in [13, 14] considered a new effective velocity (also called damped mode) in the low frequencies regime to recover new and useful properties on the solution. Then, motivated by Beauchard and Zuazua's approach in [5], they constructed a Lyapunov functional including lower-order terms to handle the high frequencies. These two elements allowed them to establish the global well-posedness and optimal time-decay rates of global solutions to (DCE) for small initial data in the critical homogeneous Besov space  $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ . Moreover, they established the global well-posedness in a  $L^p$  type hybrid space  $\dot{B}_{p,2}^{\frac{d}{p}, \frac{d}{2}+1}$  with some  $p \geq 2$  [15]. In addition, many important results have been shown concerning the global dynamics of more general hyperbolic systems or hyperbolic-parabolic systems, refer to [7, 14, 15, 38, 49, 53, 60–62] and references therein. To have a better understanding of this sequence of improvements, one must keep in mind the following embeddings:

$$H^s(s > \frac{d}{2} + 1) \hookrightarrow B_{2,1}^{\frac{d}{2}+1} \hookrightarrow \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1} \hookrightarrow \dot{B}_{p,2}^{\frac{d}{p}, \frac{d}{2}+1} (p > 2) \hookrightarrow C_b^1.$$

In this paper, we first establish the global well-posedness and optimal decay rates of solutions to the System (HPC) for small initial data in  $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ . Our study is in line with the work of Danchin concerning the study of the compressible Navier-Stokes system in critical spaces e.g. [16]. It is motivated by the fact that this space appears to be the largest space in which one is able to justify the existence of unique global-in-time solution for the system (HPC).

Next, we focus on the relaxation limit of the System (HPC). Let us recall some important works on the relaxation limit or large time asymptotic for the System (DCE). Considering the diffusive rescaling  $(\rho^\varepsilon, u^\varepsilon)(x, \tau) := (\rho, \frac{u}{\varepsilon})(x, t)$  for  $\tau = \varepsilon t$ , the System (DCE) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \varepsilon^2 \partial_t(\rho^\varepsilon u^\varepsilon) + \varepsilon^2 \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla P(\rho^\varepsilon) + \rho^\varepsilon u^\varepsilon = 0. \end{cases} \quad (\text{DCE}_\varepsilon)$$

As  $\varepsilon \rightarrow 0$ , the dynamics of the System (DCE<sub>ε</sub>) is formally governed by the porous media model

$$\begin{cases} \partial_t \rho^* - \Delta P(\rho^*) = 0, \\ \rho^* u^* = -\nabla P(\rho^*) \quad (\text{Darcy's law}). \end{cases} \quad (\text{PM})$$

The rigorous derivation of the relaxation limit of (DCE) to (PM) was studied in [12, 32, 40, 41, 59], and the time-asymptotical convergence of global solutions with vacuum for the system (DCE) to the Barenblatt profile of (PM) was made in [30, 39]. In [15], the authors justified rigorously the relaxation limit of (DCE) and also derived an explicit convergence rate of the process in the multi-dimensional setting. Their key point comes from a spectral analysis of the system which reveals that the suitable threshold between the low and high frequencies to study the system should be  $J_\varepsilon = \lfloor -\log_2 \varepsilon \rfloor + k$  with  $k$  a small enough constant. This threshold corresponds to the place where the 0-order terms and the 1-order terms have the

same weight (parameter included) and allow them to recover suitable uniform bounds for the relaxation limit. It is this approach that we are going to adapt to our system here.

A natural and important issue is to investigate the influences of the chemoattractant concentration  $\phi$  secreted by cells on the well-posedness and qualitative behaviors of solutions to the System (HPC). From the mathematical viewpoint, the main difficulties about this system are that it doesn't satisfy the usual symmetric conditions of the classical hyperbolic-parabolic systems nor the Shizuta-Kawashima (SK) condition, see [38, 49, 53, 62]. Therefore, even though it seems to have "enough dissipation", the classical methods for partially dissipative systems can not be applied to the System (HPC). These issues mainly arise from the term  $\mu\rho\nabla\phi$  which implies a reconsideration of the way to handle both the low and high frequencies regimes.

There is a lot of literature on the study of the System (HPC) [6, 22–24, 27, 36, 37, 45, 46, 52]. Among them, the numerical simulations of the System (HPC) can be found in [22, 24, 45, 46]. For the one-dimensional case, the existence and nonlinear time-stability of steady states of (HPC) were established in [6, 27], and the convergence to nonlinear diffusion waves of (HPC) in  $\mathbb{R}$  was proved in [36]. As for the multi-dimensional case, the linear stability of solutions of (HPC) was obtained under the assumption  $P'(\bar{\rho}) > \frac{a\mu}{b}\bar{\rho}$  in [52] with an additional viscous term  $\Delta u$  added in (HPC)<sub>2</sub>. The existence and time-decay estimates of global solutions of (HPC) around a constant equilibrium state  $(\bar{\rho}, 0, \frac{a}{b}\bar{\rho})$  in the Sobolev spaces  $H^s$  ( $s > \frac{d}{2} + 1$ ) have been proved by Russo and Sepe in [22, 23] under a smallness assumption on the background density  $\bar{\rho} > 0$ . Very recently, the large-time profile of small data solutions of (HPC) with Sobolev regularity in the three-dimensional whole space was derived by Liu, Peng and Wang in [37].

Concerning the justification of the relaxation limit (or large friction limit) of the System (HPC) as  $\varepsilon \rightarrow 0$ , again we introduce the rescaled time variable  $\tau = \varepsilon t$  and define

$$(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)(x, \tau) := (\rho, \frac{u}{\varepsilon}, \phi)(x, t). \quad (1.1)$$

Then  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  satisfies

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \varepsilon^2 \partial_t(\rho^\varepsilon u^\varepsilon) + \varepsilon^2 \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla P(\rho^\varepsilon) - \mu \rho^\varepsilon \nabla \phi^\varepsilon + \rho^\varepsilon u^\varepsilon = 0, \\ \varepsilon \partial_t \phi^\varepsilon - \Delta \phi^\varepsilon - a \rho^\varepsilon + b \phi^\varepsilon = 0. \end{cases} \quad (\text{HPC}_\varepsilon)$$

At the formal level, as  $\varepsilon \rightarrow 0$ , the solution  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  of the System (HPC<sub>ε</sub>) converges to the limit  $(\rho^*, u^*, \phi^*)$  solving the Keller-Segel model

$$\begin{cases} \partial_t \rho^* - \operatorname{div}(\nabla P(\rho^*) - \mu \rho^* \nabla \phi^*) = 0, \\ \rho^* u^* = -\nabla P(\rho^*) + \mu \rho^* \nabla \phi^*, \\ -\Delta \phi^* - a \rho^* + b \phi^* = 0, \end{cases} \quad (\text{KS})$$

which is a classical model in biology to describe the collective motion of cells or the evolution of a density of bacteria in models involving chemotaxis, cf. [3, 43, 48, 51].

The momentum formula  $(\text{KS})_2$  is analogous to the Darcy's law  $(\text{PM})_2$ . To our knowledge, the only result about the relaxation limit from  $(\text{HPC})$  to  $(\text{KS})$  is due to Francesco and Donatelli in [20]. They justified it in a two-dimensional periodic domain by using compensated compactness tools under the assumption that the solutions for  $(\text{HPC})$  exist globally in time and satisfy some uniform estimates. Let us also mention the recent work by Lattanzio and Tzavaras [33], in which they justified the relaxation limit from the damped Euler-Poisson equations (i.e., without  $\partial_t \phi$  in  $(\text{HPC})_3$ ) to  $(\text{KS})$  based on the gradient flows approach in Wasserstein distance.

The purpose of this paper is to study the Cauchy problem for the multi-dimensional System  $(\text{HPC})$  in a critical regularity framework. First, under the natural stability assumption  $P'(\bar{\rho}) > \frac{a\mu}{b}\bar{\rho}$ , we establish the global existence and uniqueness of classical solutions around the constant equilibrium state  $(\bar{\rho}, 0, \frac{a}{b}\bar{\rho})$  in  $L^2$ -type hybrid Besov spaces with different regularity exponents in low and high frequencies. Then, we derive the optimal time-decay rates of global solutions under an additional regularity assumption (weaker than  $L^1$ ) on the initial data. And afterward, we provide a rigorous justification of the relaxation limit from  $(\text{HPC}_\varepsilon)$  to  $(\text{KS})$  with an explicit convergence rate.

The rest of the paper is organized as follows. In Section 2, we state the main results and explain the strategies of this paper. In Section 3, we establish a-priori estimates and prove Theorem 2.1 concerning the global well-posedness for the System  $(\text{HPC})$ . In Section 4, we prove Theorem 2.2 concerning the optimal time-decay estimates. Section 5 is devoted to the proof of Theorems 2.3 and 2.4 about the global well-posedness for the Keller-Segel system and the relaxation limit process. Section 6 is an appendix regarding some tools in Besov spaces which are used throughout the paper.

## 2 Main results and strategy of proof

### 2.1 Functional spaces

Before stating our main results, we explain the notations and definitions used throughout this paper.  $C > 0$  denotes a constant independent of  $\varepsilon$  and time,  $f \lesssim g$  (resp  $f \gtrsim g$ ) means  $f \leq Cg$  (resp  $f \geq Cg$ ), and  $f \sim g$  stands for  $f \lesssim g$  and  $f \gtrsim g$ . For any Banach space  $X$  and the functions  $f, g \in X$ , let  $\|(f, g)\|_X := \|f\|_X + \|g\|_X$ . For any  $T > 0$  and  $1 \leq \varrho \leq \infty$ , we denote by  $L^\varrho(0, T; X)$  the set of measurable functions  $g : [0, T] \rightarrow X$  such that  $t \mapsto \|g(t)\|_X$  is in  $L^\varrho(0, T)$  and write  $\|\cdot\|_{L^\varrho(0, T; X)} := \|\cdot\|_{L^\varrho_T(X)}$ .

We recall the notations of the Littlewood-Paley decomposition and Besov spaces. The reader can refer to [4][Chapter 2] for a complete overview. Choose a smooth radial non-increasing function  $\chi(\xi)$

with compact supported in  $B(0, \frac{4}{3})$  and  $\chi(\xi) = 1$  in  $B(0, \frac{3}{4})$  such that

$$\varphi(\xi) := \chi(\frac{\xi}{2}) - \chi(\xi), \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1, \quad \text{Supp } \varphi \subset \{\xi \in \mathbb{R}^d \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}.$$

For any  $j \in \mathbb{Z}$ , the homogeneous dyadic blocks  $\dot{\Delta}_j$  and the low-frequency cut-off operator  $\dot{S}_j$  are defined by

$$\dot{\Delta}_j u := \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F}u), \quad \dot{S}_j u := \mathcal{F}^{-1}(\chi(2^{-j} \cdot) \mathcal{F}u),$$

where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  stand for the Fourier transform and its inverse. From now on, we use the shorthand notation  $\dot{\Delta}_j u = u_j$ .

Let  $\mathcal{S}'_h$  be the set of tempered distributions on  $\mathbb{R}^d$  such that every  $u \in \mathcal{S}'_h$  satisfies  $u \in \mathcal{S}'$  and  $\lim_{j \rightarrow -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$ . Then one has

$$u = \sum_{j \in \mathbb{Z}} u_j \quad \text{in } \mathcal{S}', \quad \dot{S}_j u = \sum_{j' \leq j-1} u_{j'}, \quad \forall u \in \mathcal{S}'_h,$$

With the help of these dyadic blocks, the homogeneous Besov space  $\dot{B}_{p,r}^s$  for any  $p, r \in [1, \infty]$  and  $s \in \mathbb{R}$  is defined by

$$\dot{B}_{p,r}^s := \{u \in \mathcal{S}'_h \mid \|u\|_{\dot{B}_{p,r}^s} := \|\{2^{js} \|u_j\|_{L^p}\}_{j \in \mathbb{Z}}\|_{l^r} < \infty\}.$$

We denote the Chemin-Lerner type space  $\tilde{L}^\varrho(0, T; \dot{B}_{p,r}^s)$  for any  $\varrho, r, q \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $T > 0$ :

$$\tilde{L}^\varrho(0, T; \dot{B}_{p,r}^s) := \{u \in L^\varrho(0, T; \mathcal{S}'_h) \mid \|u\|_{\tilde{L}^\varrho_T(\dot{B}_{p,r}^s)} := \|\{2^{js} \|u_j\|_{L_T^\varrho(L^p)}\}_{j \in \mathbb{Z}}\|_{l^r} < \infty\}.$$

By the Minkowski inequality, it holds that

$$\|u\|_{\tilde{L}^\varrho_T(\dot{B}_{p,r}^s)} \leq \|u\|_{L_T^\varrho(\dot{B}_{p,r}^s)} \quad \text{for } r \geq \rho \quad \text{and} \quad \|u\|_{\tilde{L}^\varrho_T(\dot{B}_{p,r}^s)} \geq \|u\|_{L_T^\varrho(\dot{B}_{p,r}^s)} \quad \text{for } r \leq \rho,$$

where  $\|\cdot\|_{L_T^\varrho(\dot{B}_{p,r}^s)}$  is the usual Lebesgue-Besov norm. Moreover, we write

$$\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{p,r}^s) := \{u \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{p,r}^s) \mid \|f\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{p,r}^s)} < \infty\}.$$

In order to restrict our Besov norms to the low and high frequencies regions, we set an integer  $J_\varepsilon$ , called threshold, to denote the following notations for  $p \in [1, \infty]$  and  $s \in \mathbb{R}$ :

$$\begin{aligned} \|u\|_{\dot{B}_{p,r}^s}^{\ell, J_\varepsilon} &:= \|\{2^{js} \|u_j\|_{L^p}\}_{j \leq J_\varepsilon}\|_{\ell^r}, & \|u\|_{\dot{B}_{p,r}^s}^{h, J_\varepsilon} &:= \|\{2^{js} \|\dot{\Delta}_j u\|_{L^p}\}_{j \geq J_\varepsilon-1}\|_{\ell^r}, \\ \|u\|_{\tilde{L}^\varrho_T(\dot{B}_{p,r}^s)}^{\ell, J_\varepsilon} &:= \|\{2^{js} \|\dot{\Delta}_j u\|_{L_T^\varrho(L^p)}\}_{j \leq J_\varepsilon}\|_{\ell^r}, & \|u\|_{\tilde{L}^\varrho_T(\dot{B}_{p,r}^s)}^{h, J_\varepsilon} &:= \|\{2^{js} \|\dot{\Delta}_j u\|_{L_T^\varrho(L^p)}\}_{j \geq J_\varepsilon-1}\|_{\ell^r}. \end{aligned}$$

For any  $u \in \mathcal{S}'_h$ , we also define the low-frequency part  $u^{\ell, J_\varepsilon}$  and the high-frequency part  $u^{h, J_\varepsilon}$  by

$$u^{\ell, J_\varepsilon} := \sum_{j \leq J_\varepsilon-1} u_j, \quad u^{h, J_\varepsilon} := u - u^{\ell, J_\varepsilon} = \sum_{j \geq J_\varepsilon} u_j.$$

It is easy to check for any  $s' > 0$  that

$$\|u^{\ell, J_\varepsilon}\|_{\dot{B}_{p,r}^s} \leq \|u\|_{\dot{B}_{p,r}^s}^{\ell, J_\varepsilon} \leq 2^{J_\varepsilon s'} \|u\|_{\dot{B}_{p,r}^{s-s'}}, \quad \|u^{h, J_\varepsilon}\|_{\dot{B}_{p,r}^s} \leq \|u\|_{\dot{B}_{p,r}^s}^{h, J_\varepsilon} \leq 2^{-(J_\varepsilon-1)s'} \|u\|_{\dot{B}_{p,r}^{s+s'}}. \quad (2.1)$$

Whenever the value of  $J_\varepsilon$  is clear from the context (which will also be the case after), the exponent  $J_\varepsilon$  will be omitted in the above notations for simplicity. Note also that  $J_\varepsilon$  will depend on  $\varepsilon$  which implies that all the frequency-restricted norms also depend on  $\varepsilon$ .

## 2.2 Main results

In this paper, the solution  $(\rho, u, \phi)$  to the System (HPC) is obtained as a small perturbation of the equilibrium state

$$(\bar{\rho}, 0, \bar{\phi}), \quad (2.2)$$

with the constant background density  $\bar{\rho} > 0$  and the constant background concentration  $\bar{\phi} := \frac{a}{b}\bar{\rho} > 0$ . We assume that the pressure law  $P(\rho)$  satisfies

$$P'(\rho) > 0 \text{ for } \rho \text{ close to } \bar{\rho}.$$

Then, introducing the variables

$$n := \int_{\bar{\rho}}^{\rho} \frac{P'(s)}{s} ds, \quad \psi := \phi - \bar{\phi},$$

we can reformulate the System (HPC) as

$$\begin{cases} \partial_t n + u \cdot \nabla n + c_0 \operatorname{div} u + G(n) \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \frac{1}{\varepsilon} u + \nabla n - \mu \nabla \psi = 0, \\ \partial_t \psi - \Delta \psi + b\psi - c_1 n - H(n) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \end{cases} \quad (\text{HPC-r})$$

where  $c_i$  ( $i = 1, 2$ ) are the constants

$$c_0 := P'(\bar{\rho}), \quad c_1 := a \partial_n \rho|_{n=0} = \frac{a\rho}{P'(\rho)} \Big|_{n=0} = \frac{a\bar{\rho}}{P'(\bar{\rho})},$$

and  $G(n)$  and  $H(n)$  are the nonlinear terms

$$G(n) := P'(\rho) - P'(\bar{\rho}), \quad H(n) := a\left(\rho - \bar{\rho} - \frac{\bar{\rho}}{P'(\bar{\rho})}n\right).$$

First, we establish the global well-posedness to the Cauchy problem for the System (HPC-r) in hybrid Besov spaces. We denote the total energy

$$\mathcal{X}(t) := \mathcal{X}_L(t) + \mathcal{X}_H(t), \quad (2.3)$$

with

$$\begin{aligned} \mathcal{X}_L(t) := & \| (n, u, \psi) \|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \| (n, \psi) \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \| u \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon^{-\frac{1}{2}} \| u \|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \| \partial_t \psi \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ & + \left\| \frac{1}{\varepsilon} u + \nabla n - \mu \nabla \psi \right\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \| \psi - (b - \Delta)^{-1}(c_1 n + H(n)) \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell, \end{aligned} \quad (2.4)$$

$$\begin{aligned} \mathcal{X}_H(t) := & \varepsilon \| (n, u, \nabla \psi) \|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \| (n, u) \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \| \psi \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+3})}^h + \varepsilon^{-\frac{1}{2}} \| u \|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \| \partial_t \psi \|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h. \end{aligned} \quad (2.5)$$

**Theorem 2.1.** (Global well-posedness for the HPC system) Let  $d \geq 1$ ,

$$P'(\bar{\rho}) > \frac{a\mu}{b}\bar{\rho} > 0 \quad (2.6)$$

and

$$J_\varepsilon = [-\log_2 \varepsilon] + k \quad (2.7)$$

for a sufficiently small constant  $k$  independent of  $\varepsilon$ . There exists a positive constant  $\alpha$  independent of  $\varepsilon$  such that if  $(n_0, u_0) \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ ,  $\psi_0 \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}$  and

$$\mathcal{X}_0 := \|(n_0, u_0, \psi_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|(n_0, u_0, \nabla \psi_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \leq \alpha, \quad (2.8)$$

then the System (HPC-r) supplemented with the initial data  $(n_0, u_0, \psi_0)$  admits a unique global solution  $(n, u, \psi)$  satisfying

$$\begin{cases} n^\ell \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+2}), & n^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1}), \\ u^\ell \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1}), & u^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1}), \\ \psi^\ell \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+2}), & \psi^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+2}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+3}), \end{cases} \quad (2.9)$$

and there exists a constant  $C > 0$  independent of time and  $\varepsilon$  such that

$$\mathcal{X}(t) \leq C\mathcal{X}_0, \quad t > 0. \quad (2.10)$$

**Remark 2.1.** Since that the map  $\rho \rightarrow \int_{\bar{\rho}}^\rho \frac{P'(s)}{s} ds$  is diffeomorphism from  $\mathbb{R}_+$  onto a small neighborhood of 0, one can show

$$\rho - \bar{\rho}, u \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}) \hookrightarrow \mathcal{C}(\mathbb{R}_+; \mathcal{C}^1), \quad \phi - \bar{\phi} \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}) \hookrightarrow \mathcal{C}(\mathbb{R}_+; \mathcal{C}^2).$$

Thus,  $(\rho, u, \phi)$  is indeed the unique global classical solution to the System (HPC).

**Remark 2.2.** As commented in [27, 52], (2.6) is a natural hypothesis for the linear stability of System (HPC). In this paper, it ensures the ellipticity of the operator  $\tilde{\Delta}_1$  given by (3.6)<sub>1</sub> in the low-frequency analysis and the coercivity estimates (3.53) in the high-frequency analysis. In addition, the condition (2.6) is also needed to prove the global well-posedness of the System (KS) in Theorem 2.3.

**Remark 2.3.** The choice of the threshold  $J_\varepsilon$  in (2.7) is used to control the high-order linear terms in the low frequencies region so as to derive the uniform-in- $\varepsilon$  estimates (2.10). It is motivated by a spectral analysis of the linear compressible Euler system in [15] where it is shown that the threshold between low and high frequencies should exactly be around  $\frac{1}{\varepsilon}$ . The presence of the  $\log_2$  function is only due to the fact that our computations are based on a dyadic decomposition of the frequency space.

**Remark 2.4.** The constant  $k$  in (2.7) can be computed explicitly in terms of  $d, \mu, a, b$  and  $P$ .



**Remark 2.5.** *It is shown that the effective velocity  $u + \varepsilon \nabla n - \varepsilon \mu \nabla \psi$  and the effective concentration  $\psi - (b - \Delta)^{-1}(c_1 n + H(n))$  have better regularity properties in (2.10) as they refer to purely damped mode in (3.5)<sub>2</sub>-(3.5)<sub>3</sub>. These properties are crucial to derive the quantitative error estimates of the relaxation process in Theorem 2.4.*

Next, we study the optimal time-decay rates of global solutions to the System (HPC-r).

**Theorem 2.2.** *(Optimal time-decay rates for the HPC system) Assume  $d \geq 1$  and that (2.6)-(2.7)-(2.8) hold. Let  $(n, u, \psi)$  be the corresponding global solution to the System (HPC-r) supplemented with the initial data  $(n_0, u_0, \psi_0)$  given by Theorem 2.1. If it further holds*

$$(n_0, u_0, \psi_0)^\ell \in \dot{B}_{2,\infty}^{\sigma_0} \quad \text{for } \sigma_0 \in \left[-\frac{d}{2}, \frac{d}{2}\right) \quad \text{uniformly in } \varepsilon, \quad (2.11)$$

then for some constant  $C > 0$  independent of  $\varepsilon$  and time,  $(n, u, \psi)$  satisfies

$$\|(n, u, \psi)\|_{L_t^\ell(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \leq C(\mathcal{X}_0 + \|(n_0, u_0, \psi_0)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell), \quad t > 0,$$

and

$$\begin{cases} \|(n, u, \psi)(t)\|_{\dot{B}_{2,1}^\sigma} \leq C(1 + \varepsilon t)^{-\frac{1}{2}(\sigma - \sigma_0)}, & \sigma \in (\sigma_0, \frac{d}{2}], \\ \|(n, u, \nabla \psi)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \leq \frac{C}{\varepsilon}(1 + \varepsilon t)^{-\frac{1}{2}(\frac{d}{2} - \sigma_0)}. \end{cases} \quad (2.12)$$

Furthermore, if  $d \geq 2$  and  $\sigma_0 \in [-\frac{d}{2}, \frac{d}{2} - 1)$ , then the following time-decay estimates hold:

$$\begin{cases} \|u(t)\|_{\dot{B}_{2,\infty}^{\sigma_0}} + \|(b\psi - c_1 n - H(n))(t)\|_{\dot{B}_{2,\infty}^{\sigma_0}} \leq \frac{C}{\varepsilon}(1 + \varepsilon t)^{-\frac{1}{2}}, \\ \|u(t)\|_{\dot{B}_{2,1}^\sigma} + \|(b\psi - c_1 n - H(n))(t)\|_{\dot{B}_{2,1}^\sigma} \leq \frac{C}{\varepsilon}(1 + \varepsilon t)^{-\frac{1}{2}(1 + \sigma - \sigma_0)}, \end{cases} \quad \sigma \in (\sigma_0, \frac{d}{2} - 1]. \quad (2.13)$$

**Remark 2.6.** *Due to the embedding  $L^1 \hookrightarrow \dot{B}_{2,\infty}^{-\frac{d}{2}}$ , the above assumption (2.11) with  $\sigma_0 = -\frac{d}{2}$  covers the classical  $L^1$  condition presented first by Matsumura-Nishida in [42]. Moreover, it should be noted that we don't require additional  $\dot{B}_{2,\infty}^{\sigma_0}$ -smallness assumption on the initial data. This is an improvement compared to previous researches in similar settings e.g. [18] where such smallness assumption is used to derive decay rates. Our approach is based on new time-weighted energy estimates and interpolation arguments, which is an adaptation of the previous approaches by Guo-Wang [26] and Xin-Xu [58] and enables us to deal with mixed  $L^1$ -in-time and  $L^2$ -in-time dissipation structures with different variables.*

**Remark 2.7.** *By (2.12)-(2.13) and standard interpolation inequalities, for  $r \geq 2$  and  $\Lambda = (-\Delta)^{\frac{1}{2}}$ , we have the following classical  $L^r$ -type time-decay rates of the global solution  $(\rho, u, \phi)$  to the System (HPC):*

$$\|\Lambda^\sigma(\rho - \bar{\rho}, u, \phi - \bar{\phi})(t)\|_{L^r} \leq C(1 + \varepsilon t)^{-\frac{1}{2}(\sigma + \frac{d}{2} - \frac{d}{r} - \sigma_0)}, \quad \sigma + \frac{d}{2} - \frac{d}{r} \in (\sigma_0, \frac{d}{2}]. \quad (2.14)$$

Moreover, as  $b\phi - a\rho = b\psi - c_1 n - H(n)$ , it holds for  $d \geq 2$  and  $\sigma_0 \in [-\frac{d}{2}, \frac{d}{2} - 1)$  that

$$\|\Lambda^\sigma u(t)\|_{L^r} + \|\Lambda^\sigma(b\phi - a\rho)(t)\|_{L^r} \leq \frac{C}{\varepsilon}(1 + \varepsilon t)^{-\frac{1}{2}(1 + \sigma + \frac{d}{2} - \frac{d}{r} - \sigma_0)}, \quad \sigma + \frac{d}{2} - \frac{d}{r} \in (\sigma_0, \frac{d}{2} - 1]. \quad (2.15)$$

Therefore, the time-decay estimates (2.14)-(2.15) improve the previous result of decay rates of global solutions to the System (HPC) in [23] subject to  $L^1$  assumption on the initial data.

**Remark 2.8.** Theorem 2.2 implies that the density  $\rho$  and the velocity  $u$  converge to  $\bar{\rho}$  and 0 respectively as  $t \rightarrow \infty$  at the same decay rates as the global solutions of compressible Euler equations with damping, refer to [13, 14, 54, 61] and references therein. In addition, it is proved that the coupling term  $b\phi - a\rho$  decays faster than the chemoattractant production  $a\rho$  and the chemoattractant consumption  $-b\phi$ , which, as far as we know, is a new phenomenon observed in this paper.

Next, we justify rigorously the relaxation limit from (HPC $_\varepsilon$ ) to (KS). To do this, we need the following global well-posedness result for the System (KS).

**Theorem 2.3.** (Global well-posedness for the Keller-Segel system) Let  $d \geq 1$  and  $p \in [1, \infty]$ . Suppose that (2.6) holds. There exists a positive constant  $\alpha^*$  such that if

$$\|\rho_0^* - \bar{\rho}\|_{\dot{B}_{p,1}^{\frac{d}{p}}} \leq \alpha^*, \quad (2.16)$$

then a unique solution  $\rho^*$  to the System (KS) supplemented with the initial data  $\rho_0^*$  exists globally in time and satisfies

$$\|\rho^* - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} + \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \leq C\|\rho_0^* - \bar{\rho}\|_{\dot{B}_{p,1}^{\frac{d}{p}}}, \quad t > 0, \quad (2.17)$$

where  $C > 0$  is a constant independent of time.

**Remark 2.9.** For  $d \geq 2$  and  $P(\rho) = \rho^m$  ( $m \geq 1$ ), it is well-known that the solutions of the Keller-Segel equations with vacuum have different behaviors depending on  $m$ . For  $m > 2 - \frac{2}{d}$ , the global solutions exist for all mass sizes, see e.g. [55]. On the other hand, when  $1 \leq m \leq 2 - \frac{2}{d}$ , the solutions exhibit finite time blow-up phenomena for some classes of initial data and exist globally if the initial mass is small, refer to [21, 56] and references therein. Theorem 2.3 shows that under the assumption (2.6), the Cauchy problem for the System (KS) is well-posed globally for general pressure laws covering  $P(\rho) = \rho^m$  ( $m \geq 1$ ) if the initial density is close enough to the far from vacuum constant  $\bar{\rho} > 0$ .

Then, we show the quantitative error estimates between global solutions of (HPC $_\varepsilon$ ) and (KS).

**Theorem 2.4.** (Relaxation limit) Assume  $d \geq 1$ , (2.6)-(2.8) and that (2.16) is satisfied with  $p = 2$ . Let  $(\rho, u, \phi)$  be the global solution to the System (HPC) associated to the initial data  $(\rho_0, u_0, \phi_0)$  obtained with Theorem 2.1, and  $(\rho^*, u^*, \phi^*)$  be the global solution to the System (KS) supplemented with the initial data  $\rho_0^*$  given by Theorem 2.3. Then, for any  $t > 0$ , there exists a constant  $C > 0$  independent of  $\varepsilon$  and time such that

$$\|\nabla P(\rho^\varepsilon) - \mu\rho^\varepsilon\nabla\phi^\varepsilon + \rho^\varepsilon u^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\Delta\phi^\varepsilon - a\rho^\varepsilon + b\phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \leq C\varepsilon, \quad (2.18)$$

where  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  are defined in (1.1). If it further holds

$$\|\rho_0 - \rho_0^*\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} = \mathcal{O}(\varepsilon), \quad (2.19)$$

then for any  $t > 0$ , we have the quantitative error estimates

$$\|\rho^\varepsilon - \rho^*\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\rho^\varepsilon - \rho^*\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|u^\varepsilon - u^*\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\phi^\varepsilon - \phi^*\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})} \leq C\varepsilon. \quad (2.20)$$

Consequently, as  $\varepsilon \rightarrow 0$ ,  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  converges to  $(\rho^*, u^*, \phi^*)$  in the following sense:

$$\begin{cases} \rho^\varepsilon \rightarrow \rho^* & \text{strongly in } \tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}), \\ u^\varepsilon \rightarrow u^* & \text{strongly in } L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}), \\ \phi^\varepsilon \rightarrow \phi^* & \text{strongly in } L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}). \end{cases} \quad (2.21)$$

**Remark 2.10.** To the best of our knowledge, Theorem 2.4 is the first result concerning uniform-in-time quantitative error estimates of the relaxation process from the HPC model to the Keller-Segel equations.

## 2.3 Strategy of proofs

Before explaining the strategies and ingredients to prove the above theorems, let us illustrate the main difficulties. First, the analysis of System (HPC) can not be included in the classical theory for partial dissipative hyperbolic systems [14, 60–62] or hyperbolic-parabolic systems [49, 53]. In fact, to avoid the smallness assumption on the constant state  $\bar{\rho}$ , we should not study the System (HPC-r) as a damped compressible Euler equations and a damped heat equation with additional source terms (i.e. by treating  $\mu \nabla \psi$  and  $c_1 n$  as source terms). And due to the difference between the hyperbolic scaling in (HPC-r)<sub>1</sub>-(HPC-r)<sub>2</sub> and the parabolic scaling in (HPC-r)<sub>3</sub>, we are not able to perform a rescaling as in [13–15] to reduce the proofs to the case  $\varepsilon = 1$  which allowed the authors to easily recover the exact dependency with respect to the parameter  $\varepsilon$  thanks to the homogeneity of the norms. In addition, compared with usual nonlinear terms in compressible flows, there is no any spatial derivative in the nonlinear term  $H(n)$  in (HPC-r)<sub>3</sub>, and the classical composition estimates may not be applied to  $H(n)$  because of the quadratic property. Overcoming these differences requires delicate energy estimates with mixed  $L^1$ -time and  $L^2$ -time regularities of dissipation terms in both high and low frequencies.

### Global well-posedness for the HPC system

The first step is to establish a-priori estimates. In the low frequencies region, we introduce two new effective variables

$$\varphi := \psi - (b - \Delta)^{-1}(c_1 n + H(n)) \quad \text{and} \quad v := u + \varepsilon \nabla n - \varepsilon \mu \nabla \psi,$$

so that  $(n, \varphi, v)$  satisfies the equations

$$\begin{cases} \partial_t n - \tilde{\Delta}_1 n = L_1 + R_1, \\ \partial_t \varphi - \tilde{\Delta}_2 \varphi = L_2 + R_2, \\ \partial_t v + \varepsilon^{-1} v = L_3 + R_3, \end{cases}$$

where the operators  $\tilde{\Delta}_i$  ( $i = 1, 2$ ) satisfy the dissipative properties (3.13)-(3.14), the high-order linear terms  $L_i$  ( $i = 1, 2, 3$ ) will be absorbed if the threshold  $J_\varepsilon$  takes the form (2.7) for a  $k$  chosen small enough, and the nonlinear terms  $R_i$  ( $i = 1, 2, 3$ ) will be estimated by some non so-classical composition lemmas and product estimates, see Subsection 3.1. Both these new unknowns can be considered as low-frequency damped modes for which we can recover optimal dissipative properties. The second unknown is reminiscent of the effective velocity introduced by Hoff and Haspot in [29, 31] to treat the high frequencies for the compressible Navier-Stokes system and by Crin-Barat and Danchin in [13] to analyze the low frequencies for the compressible Euler equations with damping.

In the high frequencies region, we are able to derive the Lyapunov type inequality

$$\frac{d}{dt} \mathcal{L}_j(t) + \mathcal{H}_j(t) \lesssim (\text{nonlinear terms}) \sqrt{\mathcal{L}_j(t)}, \quad j \geq J_\varepsilon - 1,$$

where  $\mathcal{L}_j(t)$  is the nonlinear energy functional

$$\begin{aligned} \mathcal{L}_j(t) := \varepsilon \int & \left[ \frac{1}{2} |n_j|^2 + \frac{K 2^{-2j}}{2} |H(n)_j|^2 + \frac{1}{2} w_j |u_j|^2 + \frac{\mu b}{2c_1} |\psi_j|^2 + \frac{\mu}{2c_1} |\nabla \psi_j|^2 \right. \\ & \left. - \mu n_j \psi_j - H(n)_j \psi_j \right] dx + \eta_0 2^{-2j} \int \left( \frac{\mu}{2c_1} |\nabla \psi_j|^2 + u_j \cdot \nabla n_j \right) dx, \quad (2.22) \end{aligned}$$

and  $\mathcal{H}_j(t)$  is the corresponding dissipation

$$\begin{aligned} \mathcal{H}_j(t) := \varepsilon \int & \left( \frac{1}{\varepsilon} w_j |u_j|^2 + |\partial_t \psi_j|^2 \right) dx + \eta_0 2^{-2j} \int \left( |\nabla n_j|^2 + \frac{\mu b}{c_1} |\nabla \psi_j|^2 + \frac{\mu}{c_1} |\Delta \psi_j|^2 \right. \\ & \left. - 2\mu \nabla n_j \cdot \nabla \psi_j - w_j |\operatorname{div} u_j|^2 + \frac{1}{\varepsilon} u_j \cdot \nabla n_j \right) dx, \end{aligned}$$

with  $K$  and  $\eta_0$  two well-chosen constants and the weight function  $w_j := c_0 + \dot{S}_{j-1} G(n)$ . The construction of the Lyapunov functional with lower-order terms is motivated by the previous work by Beauchard and Zuazua in [5] and the derivation of basic energy equality of the HPC system. The weight function satisfying  $w_j \sim 1$  by the smallness of energy is used to get rid of the nonlinear term  $G(n) \operatorname{div} u$  which breaks the symmetry in (HPC-r)<sub>1</sub>. It should be noted that to compensate  $\mu \nabla \psi$  in (HPC-r)<sub>2</sub>, one has to take the  $L^2$  inner of the equation for  $\psi_j$  with  $\partial_t \psi_j$ . Consequently, we need to control

$$\int H(n)_j \partial_t \psi_j dx = \frac{d}{dt} \int 2^{-j} H(n)_j 2^j \psi_j dx - \int 2^{-j} \partial_t H(n)_j 2^j \psi_j dx,$$

and thus  $2^{-2j} \|H(n)_j\|_{L^2}^2$  is added into the energy functional  $\mathcal{L}_j(t)$  to avoid difficulties caused by additional nonlinearities. It is also noted that the dissipation term  $\|\partial_t \psi_j\|_{L^2}^2$  in  $\mathcal{H}_j(t)$  plays a key role in deriving the

dissipation term  $2^{-2j} \|H(n)_j\|_{L^2}^2$ . In addition, we show new composition estimates in Lemma 6.6 to deal with the quadratic nonlinear term  $H(n)$ . With the help of these observations, we are able to establish the expected estimates in the high frequencies region. The details will be presented in Subsection 3.2.

### Optimal time-decay rates

To show Theorem 2.2, we adapt the works by Guo-Wang [26] and Xin-Xu [58] about optimal time-decay rates for the compressible Navier-Stokes equations without additional smallness assumptions. However, the strategy here is slightly different due to the specific dissipative structures in low frequencies and in high frequencies of our system.

First, we assume the additional regularity on the initial data (2.11) which is a weaker assumption than the usual  $L^1$  assumption, and we show that this regularity is propagated to the solution constructed in the existence theorem, see (4.1). Then, we introduce a new time-weighted functional  $\mathcal{D}^\theta(t)$ , defined in (2.3) with the time weight  $t^\theta$  for  $\theta > 1 + \frac{1}{2}(\frac{d}{2} - \sigma_0)$ . Since it involves the higher decay information  $t^{-\theta}$ , one needs to bound  $\mathcal{D}^\theta(t)$  by  $t^{\theta - \frac{1}{2}(\frac{d}{2} - \sigma_0)}$  in which  $\frac{1}{2}(\frac{d}{2} - \sigma_0)$  matches the index of the time-decay rate for  $\sigma = \frac{d}{2}$  in (2.12)<sub>1</sub>. To derive the time-weighted estimates in low frequencies, we multiply the equations of  $(n, v, \varphi)$  by  $t^\theta$  to obtain

$$\begin{cases} \partial_t(t^\theta n) - \tilde{\Delta}_1(t^\theta n) = \theta t^{\theta-1} n + t^\theta L_1 + t^\theta R_1, \\ \partial_t(t^\theta \varphi) - \tilde{\Delta}_2(t^\theta \varphi) = \theta t^{\theta-1} \varphi + t^\theta L_2 + t^\theta R_2, \\ \partial_t(t^\theta v) + \varepsilon^{-1}(t^\theta v) = \theta t^{\theta-1} v + t^\theta L_3 + t^\theta R_3. \end{cases}$$

Here the right-hand terms of the above equations can be estimated by similar arguments as in the low-frequency analysis to prove the global existence. The key point here is to make use of an interpolation inequality in both time and space, e.g.,

$$\int_0^t \tau^{\theta-1} \|n^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \lesssim (\|n^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})} t^{\theta - \frac{1}{2}(\frac{d}{2} - \sigma_0)})^{\frac{2}{\frac{d}{2} + 2 - \sigma_0}} (\varepsilon \|n^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2} + 2})})^{\frac{\frac{d}{2} - \sigma_0}{\frac{d}{2} + 2 - \sigma_0}}.$$

For the high frequencies, multiplying the Lyapunov type inequality by  $t^\theta$  yields

$$\frac{d}{dt}(t^\theta \mathcal{L}_j(t)) + t^\theta \mathcal{H}_j(t) \lesssim t^{\theta-1} \mathcal{L}_j(t) + t^\theta (\text{nonlinear terms}) \sqrt{\mathcal{L}_j(t)}, \quad j \geq J_\varepsilon - 1.$$

Arguing similarly as in the high-frequency analysis of global existence to control the nonlinear terms and taking advantage of the inequality

$$\int_0^t \tau^{\theta-1} \|n\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}}^h d\tau \lesssim (\|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2} + 1})}^h t^{\theta - \frac{1}{2}(\frac{d}{2} - \sigma_0)})^{\frac{2}{\frac{d}{2} + 2 - \sigma_0}} (\|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2} + 1})}^h)^{\frac{\frac{d}{2} - \sigma_0}{\frac{d}{2} + 2 - \sigma_0}},$$

we can prove the high-frequency time-weighted estimates. Adding these time-weighted estimates in both low and high frequencies, we are able to get (2.12). Then, the faster time-decay rates in (2.12)-(2.13) are shown by isolating and capturing the damping effects in the equations (HPC-r)<sub>2</sub> and (HPC-r)<sub>3</sub>.

### Relaxation limit

To justify the relaxation limit, we first establish the global well-posedness for the System (KS). This can be done by using the reformulation

$$\partial_t \rho^\varepsilon - \tilde{\Delta}_* \rho^\varepsilon = \text{nonlinear terms}, \quad (2.23)$$

where  $\tilde{\Delta}_*$  is the differential operator defined by  $\tilde{\Delta}_* := (P'(\bar{\rho}) - \mu a \bar{\rho} (b - \Delta)^{-1}) \Delta$  which behaves like the Laplacian  $\Delta$  under the condition  $P'(\bar{\rho}) > \frac{\mu a}{b} \bar{\rho}$ . Thus, treating (2.23) as a heat equation, we can obtain the expected a-priori estimates, see Subsection 5.1.

Finally, to justify the strong convergence and derive a convergence rate of the relaxation process, we estimate the error equations

$$\begin{cases} \partial_t \delta \rho^\varepsilon - \tilde{\Delta}_* \delta \rho^\varepsilon = \mathcal{G}^\varepsilon, \\ \delta \phi^\varepsilon = (b - \Delta)^{-1} (-\varepsilon \partial_t \phi^\varepsilon + a \delta \rho^\varepsilon), \end{cases}$$

with  $\delta \rho^\varepsilon = \rho^\varepsilon - \rho^*$  and  $\delta \phi^\varepsilon = \phi^\varepsilon - \phi^*$  for  $(\rho^\varepsilon, \phi^\varepsilon)$  defined in (1.1). Thanks to the uniform-in- $\varepsilon$  estimates (2.10) on the effective velocity  $\frac{1}{\varepsilon}(u^\varepsilon + \frac{1}{\rho^\varepsilon} \nabla P(\rho^\varepsilon) - \mu \nabla \psi^\varepsilon)$  and the time derivative  $\partial_t \phi^\varepsilon$ , we are able to show that  $\|\mathcal{G}^\varepsilon\|_{L_t^1(B_{2,1}^{\frac{d}{2}-1})} = \mathcal{O}(\varepsilon)$ . Then, handling the first error equation as an heat equation, we derive the estimate for  $\delta \rho^\varepsilon$  and justify the relaxation limit of global classical solutions from (HPC $_\varepsilon$ ) to (KS) with the convergence rate  $\varepsilon$ .

### 3 Global well-posedness for the HPC system

In this section, we prove Theorem 2.1 pertaining to the global well-posedness of classical solutions to the Cauchy problem for the HPC system (HPC-r). The key points are to derive the a-priori estimates for any classical solution  $(n, u, \psi)$  uniformly with respect to time and then apply a bootstrap argument. To this end, we assume that we are given a  $(n, u, \psi)$  is a smooth solution of (HPC-r) and we do the following a-priori assumption:

$$\mathcal{X}(t) \leq C_0 \mathcal{X}_0 \leq C_0 \alpha, \quad (3.1)$$

where  $C_0 > 0$  and  $\alpha > 0$  are two constants which will be determined later. This assumption will enable us to use freely the composition estimates.

#### 3.1 Low-frequency analysis

In the low frequencies region, we introduce two new effective unknowns

$$\varphi := \psi - (b - \Delta)^{-1} (c_1 n + H(n)) \quad \text{and} \quad v := u + \varepsilon \nabla n - \varepsilon \mu \nabla \psi, \quad (3.2)$$

so as to obtain the fine structure

$$\begin{cases} \partial_t u + u \cdot \nabla u + \frac{1}{\varepsilon} v = 0, \\ \partial_t \psi + B \varphi - \Delta \varphi = 0. \end{cases} \quad (3.3)$$

Making use of (3.3) and substituting

$$\begin{cases} \psi = \varphi + (b - \Delta)^{-1}(c_1 n + H(n)), \\ u = v - \varepsilon \nabla(1 - \mu c_1 (b - \Delta)^{-1})n + \varepsilon \mu \nabla \varphi + \varepsilon \mu \nabla (b - \Delta)^{-1} H(n) \end{cases} \quad (3.4)$$

into (HPC-r), we can verify that  $(n, \varphi, v)$  satisfies the equations

$$\begin{cases} \partial_t n - \tilde{\Delta}_1 n = L_1 + R_1, \\ \partial_t \varphi - \tilde{\Delta}_2 \varphi = L_2 + R_2, \\ \partial_t v + \varepsilon^{-1} v = L_3 + R_3, \end{cases} \quad (3.5)$$

where  $\tilde{\Delta}_i$  ( $i = 1, 2$ ) are the differential operators

$$\begin{cases} \tilde{\Delta}_1 := \varepsilon c_0 (1 - \mu c_1 \Delta (b - \Delta)^{-1}) \Delta, \\ \tilde{\Delta}_2 := -b + (1 + \varepsilon \mu c_0 c_1 (b - \Delta)^{-1}) \Delta, \end{cases} \quad (3.6)$$

$L_i$  ( $i = 1, 2, 3$ ) are the linear terms

$$\begin{cases} L_1 := -\varepsilon \mu c_0 \Delta \psi - c_0 \operatorname{div} v, \\ L_2 := -\varepsilon c_0 c_1 (b - \Delta)^{-1} (1 - c_1 (b - \Delta)^{-1}) \Delta n + c_0 c_1 (b - \Delta)^{-1} \operatorname{div} v, \\ L_3 := \varepsilon^2 c_0 \nabla (1 - c_1 \Delta (b - \Delta)^{-1}) \Delta n - \varepsilon^2 \mu c_0 \nabla \Delta \varphi + \varepsilon \mu \nabla (b - \Delta) \varphi - \varepsilon c_0 \nabla \operatorname{div} v, \end{cases} \quad (3.7)$$

and  $R_i$  ( $i = 1, 2, 3$ ) are the nonlinear terms

$$\begin{cases} R_1 := -u \cdot \nabla n - G(n) \operatorname{div} u - \varepsilon \mu c_0 \Delta (b - \Delta)^{-1} H(n), \\ R_2 := -(b - \Delta)^{-1} (c_1 R_1 + \partial_t H(n)), \\ R_3 := \varepsilon \nabla R_1 - u \cdot \nabla u. \end{cases} \quad (3.8)$$

Before estimating (3.5), we consider the following linear problems

$$\begin{cases} \partial_t f - \tilde{\Delta}_1 f = g, & x \in \mathbb{R}^d, \quad t > 0, \\ f(x, 0) = f_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (3.9)$$

and

$$\begin{cases} \partial_t f - \tilde{\Delta}_2 f = g, & x \in \mathbb{R}^d, \quad t > 0 \\ f(x, 0) = f_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (3.10)$$

The following lemma will be very useful to obtain our a-priori estimates.

**Lemma 3.1.** *Let  $r \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $T > 0$ . Assume  $u_0 \in \dot{B}_{2,r}^s$  and  $g \in L^1(0, T; \dot{B}_{2,r}^s)$ .*

(1) *If  $P'(\bar{\rho}) > \frac{\mu a}{b} \bar{\rho}$  holds and  $f$  is a solution to the problem (3.9), then there exists a universal constant  $C > 0$  such that*

$$\|f\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \varepsilon \|f\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} \leq C(\|f_0\|_{\dot{B}_{2,r}^s} + \|g\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)}), \quad 0 < t < T. \quad (3.11)$$

(2) If  $f$  is a solution to the problem (3.10), then it holds that

$$\|f\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|f\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s \cap \dot{B}_{2,r}^{s+2})} \leq C(\|f_0\|_{\dot{B}_{2,r}^s} + \|g\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)}), \quad 0 < t < T. \quad (3.12)$$

**Proof.** Under the assumption  $P'(\bar{\rho}) > \frac{\mu a}{b} \bar{\rho}$ , it can be verified with the Plancherel formula that  $\tilde{\Delta}_1$  satisfies the dissipative property

$$- \int \tilde{\Delta}_1 f f dx = \varepsilon c_0 \int (1 - \mu c_1 b^{-1} + \frac{\mu c_1}{b + |\xi|^2}) |\xi|^2 |\mathcal{F}(f)|^2 d\xi \geq \varepsilon c_0 (1 - \mu c_1 b^{-1}) \|\nabla f\|_{L^2}^2. \quad (3.13)$$

In addition, one has

$$- \int \tilde{\Delta}_2 f f dx = \int (b + (1 + \frac{\varepsilon \mu c_0 c_1}{b + |\xi|^2}) |\xi|^2) |\mathcal{F}(f)|^2 dx \geq b \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2. \quad (3.14)$$

Thus, performing energy estimates, we can show (3.11)-(3.12) using (2.6) and the definition of  $c_1$ .  $\square$

The key point in the low frequencies region is the following a-priori estimates of  $(n, \varphi, v)$ .

**Lemma 3.2.** Let  $T > 0$  be any given time and  $J_\varepsilon = [-\log_2 \varepsilon] + k$  for a sufficiently small constant  $k$ . Then, it holds under the assumptions (2.6) and (3.1) that

$$\begin{aligned} \|(n, \varphi, v)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ \leq C(\mathcal{X}_0 + \mathcal{X}^2(t)), \quad 0 < t < T. \end{aligned} \quad (3.15)$$

**Proof.** First, by (3.5)<sub>1</sub> and (3.11), we have

$$\begin{aligned} \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \\ \lesssim \|n_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|L_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|R_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ \lesssim \|n_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|R_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ \lesssim \|n_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+3})}^\ell + \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|R_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell, \end{aligned} \quad (3.16)$$

where in the last inequality one has used the embedding  $\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+3} \hookrightarrow \dot{B}_{2,1}^{\frac{d}{2}+2}$ . It is clear that the Bessel potential  $(b - \Delta)^{-1}$  satisfies

$$\|(b - \Delta)^{-1} f\|_{\dot{B}_{p,r}^s \cap \dot{B}_{p,r}^{s+2}} \lesssim \|f\|_{\dot{B}_{p,r}^s}, \quad \forall f \in \dot{B}_{p,r}^s, \quad s \in \mathbb{R}, \quad p, r \in [1, \infty]. \quad (3.17)$$

This combined with (3.5)<sub>2</sub>, (3.11) and (3.16) implies

$$\begin{aligned} \|\varphi\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \\ \lesssim \|\varphi|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|L_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|R_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ \lesssim \|\varphi|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|R_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ \lesssim \|(n, \varphi)|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+3})}^\ell + \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|(R_1, R_2)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell. \end{aligned} \quad (3.18)$$



Similarly, it also holds by (3.5)<sub>3</sub> and (3.18) that

$$\begin{aligned}
& \|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
& \lesssim \|v|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|L_3\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|R_3\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
& \lesssim \|v|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon^2 \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+3})}^\ell + \varepsilon \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+3})}^\ell + \varepsilon \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|R_3\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell.
\end{aligned} \tag{3.19}$$

Combining (3.18)-(3.19) together and using (2.1), we have

$$\begin{aligned}
& \|(n, \varphi, v)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
& \lesssim \|(n, \varphi, v)|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon^2 \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+3})}^\ell + \varepsilon \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+3})}^\ell \\
& \quad + \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|(R_1, R_2, R_3)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
& \leq \|(n, \varphi, v)|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon^2 2^{J_\varepsilon} \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \varepsilon 2^{J_\varepsilon} \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \\
& \quad + \varepsilon 2^{J_\varepsilon} (1 + \varepsilon 2^{J_\varepsilon}) \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|(R_1, R_2, R_3)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell.
\end{aligned} \tag{3.20}$$

If set  $J_\varepsilon = [-\log_2 \varepsilon] + k$ , then we conclude from (3.20) and  $\varepsilon 2^{J_\varepsilon} \sim 2^k$  that there is a sufficiently small constant  $k$  such that

$$\begin{aligned}
& \|(n, \varphi, v)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
& \lesssim \|(n, \varphi, v)|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|(R_1, R_2, R_3)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell.
\end{aligned} \tag{3.21}$$

By (2.1), we obtain for  $p, r \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $s' > 0$  that

$$\|f^\ell\|_{\dot{B}_{p,r}^s} \leq \|f\|_{\dot{B}_{p,r}^s}^\ell \lesssim \varepsilon^{-s'} \|f\|_{\dot{B}_{p,r}^{s-s'}}^\ell, \quad \|f^h\|_{\dot{B}_{p,r}^s} \leq \|f\|_{\dot{B}_{p,r}^s}^h \leq \varepsilon^{s'} \|f\|_{\dot{B}_{p,r}^{s+s'}}^h. \tag{3.22}$$

According to (3.17), (3.22) and (6.6), the first term on the right-hand side of (3.21) satisfies

$$\|(n, \varphi, v)|_{t=0}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell \lesssim \|(n_0, u_0, \psi_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|n_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h. \tag{3.23}$$

The nonlinear terms  $R_i$  ( $i = 1, 2, 3$ ) can be estimated as follows. By (6.3) and standard interpolations, one has

$$\begin{aligned}
& \|u \cdot \nabla n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
& \lesssim \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|n\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\
& \lesssim \varepsilon^{-\frac{1}{2}} \left( \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^h \right) \left( (\|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell)^{\frac{1}{2}} (\varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell)^{\frac{1}{2}} \right. \\
& \quad \left. + (\varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h)^{\frac{1}{2}} (\|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h)^{\frac{1}{2}} \right) \lesssim \mathcal{X}^2(t).
\end{aligned}$$

From (3.22) and the embedding  $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$ , we also get

$$\|n\|_{L_t^\infty(L^\infty)} \lesssim \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}(t), \quad (3.24)$$

which together with (3.1), (6.3) and (6.6) yields

$$\begin{aligned} & \|G(n) \operatorname{div} u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ & \lesssim \|G(n)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & \lesssim (\|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) (\|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \lesssim \mathcal{X}^2(t). \end{aligned}$$

In addition, we derive from (3.24) and the composition estimate (6.8) that

$$\varepsilon \|H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \lesssim (\|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) (\varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \lesssim \mathcal{X}^2(t). \quad (3.25)$$

It thus holds that

$$\|R_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \mathcal{X}^2(t). \quad (3.26)$$

Similarly to (3.26), it follows from (HPC-r)<sub>1</sub>, (3.24), (6.3) and (6.6) that

$$\begin{aligned} & \|\partial_t H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ & \lesssim \|u \cdot \nabla H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|(H(n) + c_0 H'(n) + G(n)H'(n)) \operatorname{div} u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \mathcal{X}^2(t). \end{aligned}$$

The above estimate and (3.26) imply

$$\|R_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \|R_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|\partial_t H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \mathcal{X}^2(t). \quad (3.27)$$

Finally, one deduces by (3.22), (3.26) and (6.3) that

$$\|R_3\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \|R_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + (\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \lesssim \mathcal{X}^2(t). \quad (3.28)$$

Substituting the above estimates (3.23)-(3.28) into (3.21), we end up with (3.15). The proof of Lemma 3.2 is complete.  $\square$

The estimates of  $(n, \varphi, v)$  implies the expected low-frequency estimates on the original unknowns  $(n, \psi, u)$ . This is illustrated by the following lemma.

**Lemma 3.3.** *Let  $T > 0$  be any given time and  $J_\varepsilon = [-\log_2 \varepsilon] + k$  for a sufficiently small constant  $k$ . Then, under the assumptions (2.6) and (3.1), we have*

$$\begin{aligned} & \|(u, \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon^{-\frac{1}{2}} \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ & + \varepsilon \|\psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\partial_t \psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \leq C(\mathcal{X}_0 + C\mathcal{X}^2(t)), \quad 0 < t < T. \end{aligned} \quad (3.29)$$

**Proof.** Since  $u$  satisfies (3.4)<sub>2</sub>, applying (3.15), (3.17), (3.22), (3.25) and (6.6), we obtain

$$\left\{ \begin{array}{l} \|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|(n, H(n))\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon \|\varphi\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell \\ \lesssim \|(n, v, \varphi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0 + \mathcal{X}^2(t), \\ \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell \lesssim \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon \|(n, H(n))\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \varepsilon \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell \\ \lesssim \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|(n, H(n))\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \mathcal{X}_0 + \mathcal{X}^2(t). \end{array} \right. \quad (3.30)$$

In addition, as (3.3)<sub>2</sub>, (3.4)<sub>1</sub>, (3.15), (3.25) and (6.6), it holds that

$$\left\{ \begin{array}{l} \|\psi\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \|\varphi\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|(n, H(n))\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \mathcal{X}_0 + \mathcal{X}^2(t), \\ \varepsilon \|\psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \lesssim \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \varepsilon \|(n, H(n))\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \lesssim \mathcal{X}_0 + \mathcal{X}^2(t), \\ \|\partial_t \psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \lesssim \|\varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \lesssim \mathcal{X}_0 + \mathcal{X}^2(t). \end{array} \right. \quad (3.31)$$

The combination of (3.15), (3.31) and  $\varepsilon^{-\frac{1}{2}}u = \varepsilon^{-\frac{1}{2}}v - \varepsilon^{\frac{1}{2}}(\nabla n - \mu \nabla \psi)$  gives rise to

$$\begin{aligned} \varepsilon^{-\frac{1}{2}} \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell &\lesssim \varepsilon^{-\frac{1}{2}} \|v\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon^{\frac{1}{2}} \|(n, \psi)\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell \\ &\lesssim (\|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell)^{\frac{1}{2}} \left( \frac{1}{\varepsilon} \|v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \right)^{\frac{1}{2}} + (\|(n, \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell)^{\frac{1}{2}} (\varepsilon \|(n, \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell)^{\frac{1}{2}} \\ &\lesssim \mathcal{X}_0 + \mathcal{X}^2(t). \end{aligned} \quad (3.32)$$

By (3.15) and (3.30)-(3.32), (3.29) follows.  $\square$

This concludes the low-frequency analysis to prove the global well-posedness.

### 3.2 High-frequency analysis

In this subsection, we establish the corresponding a-priori estimates in high frequencies regime. To do so, we will differentiate in time the energy functional  $\mathcal{L}_j(t)$  defined through (2.22).

First, based on some new observations about hyperbolic-parabolic dissipative structures for the System (HPC-r), we localize in frequencies and derive the following nonlinear energy estimates.

**Lemma 3.4.** *For any  $j \in \mathbb{Z}$  and the constant  $K > 0$  to be chosen later, it holds that*

$$\begin{aligned} &\frac{d}{dt} \int \left[ \frac{1}{2} |n_j|^2 + \frac{K}{2} 2^{-2j} |H(n)_j|^2 + \frac{1}{2} w_j |u_j|^2 + \frac{\mu b}{2c_1} |\psi_j|^2 + \frac{\mu}{2c_1} |\nabla \psi_j|^2 - \mu n_j \psi_j - H(n)_j \psi_j \right] dx \\ &+ \int \left( \frac{1}{\varepsilon} w_j |u_j|^2 + |\partial_t \psi_j|^2 \right) dx \\ &\lesssim (1 + \|w_j\|_{L^\infty}) (\|\operatorname{div} u\|_{L^\infty} \|(n_j, u_j)\|_{L^2} + \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2}) \|(n_j, u_j, \psi_j)\|_{L^2} \\ &+ \|\partial_t w_j\|_{L^\infty} \|u_j\|_{L^2}^2 + \|\nabla w_j\|_{L^\infty} (1 + \|u\|_{L^\infty}) \|u_j\|_{L^2} \|(n_j, u_j, \psi_j)\|_{L^2} \\ &+ 2^{-j} \|\partial_t H(n)_j\|_{L^2} (2^j \|\psi_j\|_{L^2} + 2^{-j} \|H(n)_j\|_{L^2}) + \|u\|_{L^\infty} 2^{-j} \|\nabla(n_j, H(n)_j)\|_{L^2} 2^j \|\psi_j\|_{L^2}, \end{aligned} \quad (3.33)$$

where  $w_j$  is the the weight function

$$w_j := c_0 + \dot{S}_{j-1}G(n), \quad (3.34)$$

and  $\mathcal{R}_{i,j}$  ( $i = 1, 2$ ) are the commutator terms

$$\begin{cases} \mathcal{R}_{1,j} := \dot{S}_{j-1}u \cdot \nabla n_j - \dot{\Delta}_j(u \cdot \nabla n) + \dot{S}_{j-1}G(n) \operatorname{div} u_j - \dot{\Delta}_j(G(n) \operatorname{div} u), \\ \mathcal{R}_{2,j} := \dot{S}_{j-1}u \cdot \nabla u_j - \dot{\Delta}_j(u \cdot \nabla u). \end{cases} \quad (3.35)$$

**Proof.** Applying the operator  $\dot{\Delta}_j$  to the equations (HPC-r), we have

$$\begin{cases} \partial_t n_j + \dot{S}_{j-1}u \cdot \nabla n_j + w_j \operatorname{div} u_j = \mathcal{R}_{1,j}, \\ \partial_t u_j + \dot{S}_{j-1}u \cdot \nabla u_j + \frac{1}{\varepsilon} u_j + \nabla n_j - \mu \nabla \psi_j = \mathcal{R}_{2,j}, \\ \partial_t \psi_j - \Delta \psi_j + b \psi_j - c_1 n_j - H(n)_j = 0. \end{cases} \quad (3.36)$$

We take the  $L^2$  inner product of (3.36)<sub>1</sub> with  $n_j$  to obtain

$$\frac{d}{dt} \int \frac{1}{2} |n_j|^2 dx + \int w_j \operatorname{div} u_j n_j dx = \int (\frac{1}{2} \dot{S}_{j-1} \operatorname{div} u |n_j|^2 + \mathcal{R}_{1,j} n_j) dx. \quad (3.37)$$

Meanwhile, taking the  $L^2$  inner product of (3.36)<sub>2</sub> with  $w_j u_j$ , we get

$$\begin{aligned} & \frac{d}{dt} \int \frac{1}{2} w_j |u_j|^2 dx + \int [\frac{1}{\varepsilon} w_j |u_j|^2 - w_j n_j \operatorname{div} u_j + \mu w_j \psi_j \operatorname{div} u_j] dx \\ &= \int [\frac{1}{2} w_j \dot{S}_{j-1} \operatorname{div} u |u_j|^2 + w_j \mathcal{R}_{2,j} \cdot u_j + \frac{1}{2} \partial_t w_j |u_j|^2 + \nabla w_j \cdot (\frac{1}{2} \dot{S}_{j-1} u |u_j|^2 + n_j u_j - \mu \psi_j u_j)] dx. \end{aligned} \quad (3.38)$$

In addition, one derives after multiplying (3.36)<sub>3</sub> by  $\partial_t \psi_j$  and integrating the resulting equation over  $\mathbb{R}^d$  that

$$\begin{aligned} & \frac{d}{dt} \int (\frac{b}{2} |\psi_j|^2 + \frac{1}{2} |\nabla \psi_j|^2) dx + \int |\partial_t \psi_j|^2 dx - c_1 \int n_j \partial_t \psi_j dx \\ &= \int H(n)_j \partial_t \psi_j dx = \frac{d}{dt} \int H(n)_j \psi_j dx - \int \partial_t H(n)_j \psi_j dx. \end{aligned} \quad (3.39)$$

Due to (3.36)<sub>1</sub>, it holds that

$$- \int n_j \partial_t \psi_j dx = - \frac{d}{dt} \int n_j \psi_j dx - \int w_j \operatorname{div} u_j \psi_j dx + \int (-\dot{S}_{j-1}u \cdot \nabla n_j + \mathcal{R}_{1,j}) \psi_j dx. \quad (3.40)$$

Finally, we have

$$\frac{d}{dt} \int \frac{2^{-2j}}{2} |H(n)_j|^2 dx \leq 2^{-j} \|\partial_t H(n)_j\|_{L^2} 2^{-j} \|H(n)_j\|_{L^2}. \quad (3.41)$$

Thus the combination of (3.37)-(3.41) leads to (3.33). The proof of Lemma 3.4 is complete.  $\square$

**Remark 3.1.** Note that the crucial point in the previous proof is the cancellation of the term

$\int w_j \operatorname{div} u_j \psi_j dx$  that appears in (3.37) thanks to the same term with the opposite sign and different coefficients appearing in (3.40).

**Remark 3.2.** *The main difficulty is to deal with the nonlinear term  $H(n)$ . At first glance, we need*

$$\int H(n)_j \partial_t \psi_j dx \leq \frac{1}{4} \|\partial_t \psi_j\|_{L^2}^2 + \|H(n)_j\|_{L^2}^2. \quad (3.42)$$

However, the property (3.42) may not be applied to derive the  $L^1$ -time type estimates since we can only obtain  $2^{-2j} \|H(n)_j\|_{L^2}^2$  from the dissipation  $\mathcal{H}_j(t)$ , see Lemma 3.7. The main ingredient here is that we insert the nonlinear term  $2^{-2j} \|H(n)_j\|_{L^2}^2$  into the energy functional to cancel the term  $\int H(n)_j \partial_t \psi_j dx$ . It should be noted that (3.42) will be useful to show the uniqueness in Subsection 3.4.

Next, we differentiate in time the terms of lower order in the Lyapunov functional. It gives us the dissipation estimates for the non-directly damped unknowns  $n$  and  $\psi$ .

**Lemma 3.5.** *For any  $j \in \mathbb{Z}$ , we have*

$$\begin{aligned} & \frac{d}{dt} \int \left( \frac{\mu}{2c_1} |\nabla \psi_j|^2 + u_j \cdot \nabla n_j \right) dx \\ & + \int \left[ |\nabla n_j|^2 + \frac{\mu b}{c_1} |\nabla \psi_j|^2 + \frac{\mu}{c_1} |\Delta \psi_j|^2 - 2\mu \nabla n_j \cdot \nabla \psi_j - w_j |\operatorname{div} u_j|^2 + \frac{1}{\varepsilon} u_j \cdot \nabla n_j \right] dx \\ & \lesssim \|\nabla u\|_{L^\infty} \|\nabla n_j\|_{L^2} \|u_j\|_{L^2} + \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2} \|\nabla(n_j, u_j)\|_{L^2} + \|H(n)_j\|_{L^2} \|\Delta \psi_j\|_{L^2}. \end{aligned} \quad (3.43)$$

**Proof.** One concludes from (3.36)<sub>1</sub>-(3.36)<sub>2</sub> that

$$\begin{aligned} & \frac{d}{dt} \int u_j \cdot \nabla n_j dx + \int \left[ |\nabla n_j|^2 - w_j |\operatorname{div} u_j|^2 + \frac{1}{\varepsilon} u_j \cdot \nabla n_j - \mu \nabla n_j \cdot \nabla \psi_j \right] dx \\ & = \int \left[ -(\nabla \dot{S}_{j-1} u \cdot \nabla n_j) \cdot u_j - \dot{S}_{j-1} \operatorname{div} u u_j \cdot \nabla n_j - \mathcal{R}_{1,j} \operatorname{div} u_j + \mathcal{R}_{2,j} \cdot \nabla n_j \right] dx, \end{aligned} \quad (3.44)$$

where we has used

$$\begin{aligned} & \int \left[ \nabla(\dot{S}_{j-1} u \cdot \nabla n_j) \cdot u_j + (\dot{S}_{j-1} u \cdot \nabla u_j) \cdot \nabla n_j \right] dx \\ & = \int \left[ (\nabla \dot{S}_{j-1} u \cdot \nabla n_j) \cdot u_j - \dot{S}_{j-1} \operatorname{div} u u_j \cdot \nabla n_j \right] dx. \end{aligned}$$

Thence we get after multiplying (3.36)<sub>3</sub> by  $-\psi_j$  and integrating the resulting equation by parts that

$$\frac{d}{dt} \int \frac{1}{2} |\nabla \psi_j|^2 dx + \int (b |\nabla \psi_j|^2 + |\psi_j|^2 - c_1 \nabla n_j \cdot \nabla \psi_j + H(n)_j \psi_j) dx = 0. \quad (3.45)$$

By (3.44)-(3.45), (3.43) follows.  $\square$

Then, we have the following high-frequency estimates:

**Lemma 3.6.** *Let  $T > 0$  be any given time, and  $J_\varepsilon = [-\log_2 \varepsilon] + k$  for a sufficiently small constant  $k$ . Then, under the assumptions (2.6) and (3.1), we have*

$$\mathcal{X}_H(t) \leq C(\mathcal{X}_0 + C\mathcal{X}^2(t)), \quad 0 < t < T. \quad (3.46)$$

**Proof.** Let the constants  $\alpha$ ,  $\eta_0$  and  $K$  be chosen later. We recall that

$$\begin{cases} \mathcal{L}_j(t) := \varepsilon \int \left[ \frac{1}{2} |n_j|^2 + \frac{K2^{-2j}}{2} |H(n)_j|^2 + \frac{1}{2} w_j |u_j|^2 + \frac{\mu b}{2c_1} |\psi_j|^2 + \frac{\mu}{2c_1} |\nabla \psi_j|^2 \right. \\ \quad \left. - \mu n_j \psi_j - H(n)_j \psi_j \right] dx + \eta_0 2^{-2j} \int \left( \frac{\mu}{2c_1} |\nabla \psi_j|^2 + u_j \cdot \nabla n_j \right) dx, \\ \mathcal{H}_j(t) := \varepsilon \int \left( \frac{1}{\varepsilon} w_j |u_j|^2 + |\partial_t \psi_j|^2 \right) dx + \eta_0 2^{-2j} \int \left( |\nabla n_j|^2 + \frac{\mu b}{c_1} |\nabla \psi_j|^2 + \frac{\mu}{c_1} |\psi_j|^2 \right. \\ \quad \left. - 2\mu \nabla n_j \cdot \nabla \psi_j - w_j |\operatorname{div} u_j|^2 + \frac{1}{\varepsilon} u_j \cdot \nabla n_j \right) dx. \end{cases}$$

By (3.1) and (3.34), there is a constant  $C_* > 0$  independent of time and  $\varepsilon$  such that

$$\frac{c_0}{2} \leq c_0 - C_* \mathcal{X}_0 \leq w_j(x, t) \leq c_0 + C_* \mathcal{X}_0 \leq \frac{3c_0}{2}, \quad (x, t) \in \mathbb{R}^d \times [0, t], \quad j \in \mathbb{Z}, \quad (3.47)$$

provided

$$\mathcal{X}_0 \leq \alpha \leq \frac{C_*}{2}. \quad (3.48)$$

One gets from (3.22) that

$$\|u\|_{L_t^\infty(L^\infty)} + \varepsilon \|\nabla u\|_{L_t^\infty(L^\infty)} \lesssim \|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}(t). \quad (3.49)$$

In accordance with (3.33), (3.43), (3.47), (3.49), the Bernstein inequality and the fact  $2^{-j} \lesssim \varepsilon$  for any  $j \geq J_\varepsilon - 1$ , we have

$$\begin{aligned} & \frac{d}{dt} \mathcal{L}_j(t) + \mathcal{H}_j(t) \\ & \lesssim \varepsilon (\|\nabla u\|_{L^\infty} \|(n_j, u_j)\|_{L^2} + \|u\|_{L^\infty} \|(n_j, H(n)_j)\|_{L^2} + \|(\partial_t G(n), \nabla G(n))\|_{L^\infty} \|u_j\|_{L^2} \\ & \quad + \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2} + \|H(n)_j\|_{L^2} + 2^{-j} \|\partial_t H(n)_j\|_{L^2}) (\|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2} + 2^{-j} \|H(n)_j\|_{L^2}). \end{aligned} \quad (3.50)$$

We now show the following lemma.

**Lemma 3.7.** *Let the assumptions of Lemma 3.6 be in force. Then it holds for any  $j \geq J_\varepsilon - 1$  that*

$$\begin{cases} \mathcal{L}_j(t) \sim \varepsilon \|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2}^2 + \varepsilon 2^{-2j} \|H(n)_j\|_{L^2}^2, \\ \mathcal{H}_j(t) \gtrsim \frac{1}{\varepsilon} \mathcal{L}_j(t). \end{cases} \quad (3.51)$$

**Proof.** Note that we have

$$|n_j|^2 + \frac{\mu b}{c_1} |\psi_j|^2 - 2\mu n_j \psi_j = (n_j, \psi_j) \mathbb{M} (n_j, \psi_j)^\top, \quad \mathbb{M} := \begin{pmatrix} 1 & -\mu \\ -\mu & \frac{\mu b}{c_1} \end{pmatrix}.$$

It is easy to verify that  $\mathbb{M}$  is a positive definite matrix under the condition  $\frac{\mu c_1}{b} < 1$ , i.e.  $\frac{\mu a}{b} \bar{\rho} < P'(\bar{\rho})$ .

Hence, there holds

$$|n_j|^2 + \frac{\mu b}{c_1} |\psi_j|^2 - 2\mu n_j \psi_j \gtrsim |n_j|^2 + |\psi_j|^2. \quad (3.52)$$

Similarly, one has

$$|\nabla n_j|^2 + \frac{\mu b}{c_1} |\nabla \psi_j|^2 - 2\mu \nabla n_j \cdot \nabla \psi_j \gtrsim |\nabla n_j|^2 + |\nabla \psi_j|^2. \quad (3.53)$$

It follows by (3.47), (3.52), the Bernstein inequality and  $2^{-j} \leq \varepsilon$  for any  $j \geq J_\varepsilon - 1$  that

$$\begin{cases} \mathcal{L}_j(t) \leq \varepsilon \int [C(1 - \eta_0)(|n_j|^2 + |u_j|^2 + |\psi_j|^2 + 2^{2j}|\psi_j|^2) + 2^{-2j}(K + C)|H(n)_j|^2] dx, \\ \mathcal{L}_j(t) \geq \varepsilon \int [\frac{1}{C}(1 - \eta_0)(|n_j|^2 + |u_j|^2 + |\psi_j|^2 + 2^{2j}|\psi_j|^2) + 2^{-2j}(K - C)|H(n)_j|^2] dx. \end{cases} \quad (3.54)$$

Noting that it holds

$$|\Delta \psi_j|^2 \geq \frac{1}{2} |H(n)_j + c_1 n_j - b \psi_j|^2 - |\partial_t \psi_j|^2,$$

we deduce from (3.53) and the Bernstein inequality that

$$\begin{aligned} \mathcal{H}_j(t) &\geq \frac{1}{C} \int [(1 - \eta_0)|u_j|^2 + \varepsilon(1 - \eta_0)|\partial_t \psi_j|^2 \\ &\quad + \eta_0(|n_j|^2 + |\psi_j|^2 + 2^{2j}|\psi_j|^2 + 2^{-2j}|H(n)_j + c_1 n_j - b \psi_j|^2)] dx \\ &\geq \frac{1}{C} \int [\frac{1}{\varepsilon}(1 - \eta_0)|u_j|^2 + (1 - \eta_0)|\partial_t \psi_j|^2 \\ &\quad + \frac{\eta_0}{C}(|n_j|^2 + |\psi_j|^2 + 2^{2j}|\psi_j|^2 + 2^{-2j}|H(n)_j|^2)] dx. \end{aligned} \quad (3.55)$$

Choosing  $K = 2C$  and a suitably small constant  $\eta_0 \in (0, 1)$  in (3.54)-(3.55), we derive (3.51).  $\square$

We now resume the proof of Lemma 3.6. Combining (3.50)-(3.51) together yields

$$\begin{aligned} &\frac{d}{dt} \mathcal{L}_j(t) + \frac{1}{\varepsilon} \mathcal{L}_j(t) \\ &\lesssim \left( \|\nabla u\|_{L^\infty} \varepsilon \|(n_j, u_j)\|_{L^2} + \|u\|_{L^\infty} \varepsilon \|n_j\|_{L^2} + \|(\partial_t G(n), \nabla G(n))\|_{L^\infty} \varepsilon \|u_j\|_{L^2} \right. \\ &\quad \left. + \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2} + \|H(n)_j\|_{L^2} + 2^{-j} \|\partial_t H(n)_j\|_{L^2} \right) \sqrt{\frac{1}{\varepsilon} \mathcal{L}_j(t)}, \quad j \geq J_\varepsilon - 1. \end{aligned} \quad (3.56)$$

After dividing both sides of (3.56) by  $(\frac{1}{\varepsilon} \mathcal{L}_j(t) + \eta^2)^{\frac{1}{2}}$  for  $\eta > 0$ , integrating the resulting inequality over  $[0, t]$  and then taking the limit as  $\eta \rightarrow 0$ , we get

$$\begin{aligned} &\varepsilon \|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2} + \int_0^t \|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2} d\tau \\ &\leq \varepsilon \|(n_j, u_j, \psi_j, \nabla \psi_j)(0)\|_{L^2} + \varepsilon 2^{-j} \|H(n)_j(0)\|_{L^2} \\ &\quad + \int_0^t \left( \varepsilon \|\nabla u\|_{L^\infty} \|(n_j, u_j)\|_{L^2} + \varepsilon \|u\|_{L^\infty} \|(n_j, H(n)_j)\|_{L^2} \right. \\ &\quad \left. + \varepsilon \|(\nabla G(n), \partial_t G(n))\|_{L^\infty} \|u_j\|_{L^2} + \varepsilon \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2} + \varepsilon \|H(n)_j\|_{L^2} + 2^{-j} \|\partial_t H(n)_j\|_{L^2} \right) d\tau. \end{aligned} \quad (3.57)$$

After multiplying (3.57) by  $2^{j(\frac{d}{2}+1)}$  and summing over  $j \geq J_\varepsilon - 1$ , we infer that

$$\begin{aligned}
& \varepsilon \|(n, u, \psi, \nabla \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|(n, u, \psi, \nabla \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
& \lesssim \mathcal{X}_0 + \|\nabla u\|_{L_t^1(L^\infty)} \varepsilon \|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|u\|_{L_t^2(L^\infty)} \|(n, H(n))\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
& \quad + \varepsilon \|(\nabla G(n), \partial_t G(n))\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \varepsilon \sum_{j \geq J_\varepsilon - 1} 2^{j(\frac{d}{2}+1)} \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L_t^1(L^2)} \\
& \quad + \|H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\partial_t H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h.
\end{aligned} \tag{3.58}$$

The nonlinear terms on the right-hand side of (3.58) can be estimated as follows. First, it is easy to see that

$$\|\nabla u\|_{L_t^1(L^\infty)} \varepsilon \|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \leq (\|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \varepsilon \|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}^2(t). \tag{3.59}$$

Due to (3.24), (6.6) and standard interpolations, one has

$$\begin{aligned}
& \|u\|_{L_t^2(L^\infty)} \|(n, H(n))\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
& \lesssim \varepsilon^{-\frac{1}{2}} (\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \varepsilon^{\frac{1}{2}} (\|n\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|n\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \lesssim \mathcal{X}^2(t).
\end{aligned} \tag{3.60}$$

It also holds by (HPC-r)<sub>1</sub>, (3.22) and (3.24) that

$$\begin{aligned}
& \varepsilon \|(\nabla G(n), \partial_t G(n))\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
& \lesssim (\varepsilon \|\nabla n\|_{L_t^\infty(L^\infty)} (1 + \|u\|_{L_t^\infty(L^\infty)}) + \varepsilon \|\nabla u\|_{L_t^2(L^\infty)} (1 + \|n\|_{L_t^\infty(L^\infty)})) \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
& \lesssim (\|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon \|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}^2(t).
\end{aligned} \tag{3.61}$$

We turn to estimate the commutator terms. It follows from (3.22) and (6.5) that

$$\begin{aligned}
& \varepsilon \sum_{j \geq J_\varepsilon - 1} 2^{j(\frac{d}{2}+1)} (\|\dot{S}_{j-1} u \cdot \nabla n_j - \dot{\Delta}_j (u \cdot \nabla n)\|_{L_t^1(L^2)} + \|\dot{S}_{j-1} u \cdot \nabla u_j - \dot{\Delta}_j (u \cdot \nabla u)\|_{L_t^1(L^2)}) \\
& \lesssim \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \varepsilon \|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\
& \lesssim (\|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) (\|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon \|(n, u)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \lesssim \mathcal{X}^2(t).
\end{aligned}$$

And we obtain by using (3.24), (6.6) and  $G(0) = 0$  that

$$\varepsilon \|G(n)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})} \lesssim \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}(t),$$

which together with (2.1) and (6.5) leads to

$$\begin{aligned}
& \varepsilon \sum_{j \geq J_\varepsilon - 1} 2^{j(\frac{d}{2}+1)} \|\dot{S}_{j-1} G(n) \operatorname{div} u_j - \dot{\Delta}_j (G(n) \operatorname{div} u)\|_{L_t^1(L^2)} \\
& \lesssim \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \varepsilon \|G(n)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})} \lesssim \mathcal{X}^2(t).
\end{aligned}$$



Therefore, the commutator terms can be controlled by

$$\varepsilon \sum_{j \geq J_\varepsilon - 1} 2^{j(\frac{d}{2}+1)} \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L_t^1(L^2)} \lesssim \mathcal{X}^2(t). \quad (3.62)$$

With the help of the composition estimate (6.9) for the quadratic function  $H(n)$ , it holds that

$$\|H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim (\|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) (\varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \lesssim \mathcal{X}^2(t). \quad (3.63)$$

Finally, one deduces by (HPC-r)<sub>1</sub>, (3.24), (3.22), (6.3) and (6.6) that

$$\begin{aligned} & \|\partial_t H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \\ & \lesssim (\|H'(n) - H'(0)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + H'(0)) \|\partial_t n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\ & \lesssim \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} (\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|n\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})} + (1 + \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}) \lesssim \mathcal{X}^2(t). \end{aligned} \quad (3.64)$$

Inserting the above estimates (3.59)-(3.64) into (3.58), we derive

$$\varepsilon \|(n, u, \psi, \nabla \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|(n, u, \psi, \nabla \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0 + \mathcal{X}^2(t). \quad (3.65)$$

Furthermore, since  $u$  satisfies

$$\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \lesssim \varepsilon \|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h, \quad \frac{1}{\varepsilon} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \lesssim \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h,$$

one infers from (3.65) that

$$\varepsilon^{-\frac{1}{2}} \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^h \lesssim (\|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h)^{\frac{1}{2}} \left( \frac{1}{\varepsilon} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \right)^{\frac{1}{2}} \lesssim \mathcal{X}_0 + \mathcal{X}^2(t). \quad (3.66)$$

According to (3.22), (3.63)-(3.65) and maximal regularity estimates for the heat equation (HPC-r)<sub>3</sub> (cf. [4, 11]), it follows that

$$\begin{aligned} & \|\psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+3})}^h + \|\partial_t \psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\ & \lesssim \|\psi_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|-b\psi + c_1 n + H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\ & \lesssim \varepsilon \|\psi_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^h + \|(n, \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0 + \mathcal{X}^2(t). \end{aligned} \quad (3.67)$$

Collecting (3.65)-(3.67), we have (3.46) and finish the proof of Lemma 3.6.  $\square$

### 3.3 Global existence

In this subsection, we construct a local Friedrichs approximation (see, e.g., [4, Page 440]) and extend the local approximate sequence to a global one by the a-priori estimates established in Subsections 3.1-3.2. Then we show the convergence of the approximate sequence to the expected global solution to the Cauchy problem for the System (HPC-r).

**Proof of Theorem 2.1:** Denote the Friedrichs projector

$$\dot{\mathbb{E}}_q f := \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_q} \mathcal{F} f), \quad \forall f \in L_q^2,$$

where  $L_q^2$  is the set of  $L^2$  functions spectrally supported in the annulus  $\mathcal{C}_q := \{\xi \in \mathbb{R}^d \mid \frac{1}{q} \leq |\xi| \leq q\}$  endowed with the standard  $L^2$  topology, and  $\mathbf{1}_{\mathcal{C}_q}$  is the characteristic function on the annulus  $\mathcal{C}_q$ .

Then, we are ready to solve the following approximate problem for  $q \geq 1$ :

$$\begin{cases} \partial_t n^q + \dot{\mathbb{E}}_q(u^q \cdot \nabla n^q) + c_0 \operatorname{div} u^q + \dot{\mathbb{E}}_q(G(n^q) \operatorname{div} u^q) = 0, \\ \partial_t u^q + \dot{\mathbb{E}}_q(u^q \cdot \nabla u^q) + \frac{1}{\varepsilon} u^q + \nabla n^q - \mu \nabla \psi^q = 0, \\ \partial_t \psi^q - \Delta \psi^q + b \psi^q - c_1 n^q - \dot{\mathbb{E}}_q H(n^q) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\ (n^q, u^q, \psi^q)(x, 0) = (\dot{\mathbb{E}}_q n_0, \dot{\mathbb{E}}_q u_0, \dot{\mathbb{E}}_q \psi_0)(x), \quad x \in \mathbb{R}^d. \end{cases} \quad (3.68)$$

It is classical to show that  $(\dot{\mathbb{E}}_q n_0, \dot{\mathbb{E}}_q u_0, \dot{\mathbb{E}}_q \psi_0)$  satisfies (2.8) uniformly with respect to  $q \geq 1$  and converges to  $(n_0, u_0, \psi_0)$  strongly in the sense (2.8). Since all the Sobolev norms are equivalent in (3.68) due to the Bernstein inequality, we can check that (3.68) is a system of ordinary differential equations in  $L_q^2 \times L_q^2 \times L_q^2$  and locally Lipschitz with respect to the variable  $(n^q, u^q, \psi^q)$  for every  $q \geq 1$ . By virtue of the Cauchy-Lipschitz theorem in [4, Page 124], there exists a maximal time  $T_q^* > 0$  such that the problem (3.68) admits a unique solution  $(n^q, u^q, \psi^q) \in C([0, T_q^*]; L_q^2)$  on  $[0, T_q^*]$ .

Let  $\mathcal{X}(n, u, \psi) := \mathcal{X}(t)$  be given by (2.3). Define the maximal time

$$T_q := \sup \{t \geq 0 \mid \text{there exists a unique solution } (n^q, u^q, \psi^q) \text{ to the Cauchy problem (3.68) on } [0, t] \text{ satisfying } \mathcal{X}(n^q, u^q, \psi^q) \leq C_0 \mathcal{X}_0\}. \quad (3.69)$$

By the time continuity of  $(n^q, u^q, \psi^q)$ , we have  $0 < T_q \leq T_q^*$ .

We claim  $T_q = T_q^*$ . To prove it, we use a contradiction argument and assume that  $T_q < T_q^*$ . Since  $(n^q, u^q, \psi^q) = \dot{\mathbb{E}}_n(n^q, u^q, \psi^q)$ , the orthogonal projector  $\dot{\mathbb{E}}_n$  has no effect on the energy estimates established in Lemmas 3.2-3.3 and 3.6. By virtue of (3.15), (3.29), (3.46) and (3.69), as long as  $\mathcal{X}_0$  satisfies (3.48), we have

$$\mathcal{X}(n^q, u^q, \psi^q)(t) \leq C \mathcal{X}_0 + C(\mathcal{X}(n^q, u^q, \psi^q))^2(t), \quad 0 < t < T_q. \quad (3.70)$$

By (3.70), we first choose  $C_0 = 4c$  and then take

$$\mathcal{X}_0 \leq \alpha := \min\left\{\frac{1}{16C^2}, \frac{C_*}{2}\right\}$$

to derive

$$\mathcal{X}(n^q, u^q, \psi^q)(t) \leq \frac{1}{2} C_0 \mathcal{X}_0, \quad 0 < t < T_q. \quad (3.71)$$

By (3.71) and the time continuity property,  $T_q$  is not the maximal time such that  $\mathcal{X}(n^q, u^q, \psi^q)(t) \leq C_0 \mathcal{X}_0$  holds. This contradicts the definition of  $T_q$ .

If  $T_q^* < \infty$ , by (3.71) and  $T_q = T_q^*$ , we can take  $(n^q, u^q, \psi^q)(t)$  for  $t$  sufficiently close to  $T_q^*$  as the new initial data and obtain the existence from  $t$  to some  $t + \eta^* > T_q^*$  with a suitably small constant  $\eta^* > 0$  by the Cauchy-Lipschitz theorem, which contradicts the definition of  $T_q^*$ . Therefore, we have  $T_q^* = \infty$ , and  $(n^q, u^q, \psi^q)$  is indeed a global solution to the Cauchy problem (3.68).

From the uniform estimates  $\mathcal{X}_q(t) \leq C_0 \mathcal{X}_0$  and the equations (3.68), one can estimate the time derivatives  $(\partial_t n^q, \partial_t u^q, \partial_t \psi^q)$  in a suitable sense uniform with respect to  $q$ . According to these uniform estimates, the Aubin-Lions lemma and the Cantor diagonal process, there is a limit  $(n, u, \psi)$  such that as  $n \rightarrow \infty$ , it holds, up to a subsequence (still denoted by  $(n^q, u^q, \psi^q)$ ), that

$$(\chi n^q, \chi u^q, \chi \psi^q) \rightarrow (\chi n, \chi u, \chi \psi) \quad \text{strongly in } L^2(0, T; \dot{B}_{2,1}^{\frac{d}{2}}), \quad \forall T > 0, \quad \forall \chi \in \mathcal{D}(\mathbb{R}^d \times (0, T)).$$

Thus, it is easy to prove that the limit  $(n, u, \psi)$  solves the System (HPC-r) with the initial data  $(n_0, u_0, \psi_0)$  in the sense of distributions, and thanks to  $\mathcal{X}_q(t) \leq C_0 \mathcal{X}_0$ , the global solution  $(n, u, \psi)$  to the System (HPC-r) is indeed a classical one and satisfies the properties (2.9)-(2.10). To finish the proof of Theorem 2.1, we show that the solution constructed in this section is unique.

### 3.4 Uniqueness

Suppose that (2.6)-(2.8) of Theorem 2.1 are in force. Without loss of generation, we assume  $\varepsilon = 1$  in this subsection. For any given time  $T > 0$ , let  $(n_i, u_i, \psi_i)$  ( $i = 1, 2$ ) be two solutions to the System (HPC-r) satisfying (2.9)-(2.10) with the same initial data  $(n_0, u_0, \psi_0)$  on  $[0, T]$ . Denote the discrepancy

$$(\tilde{n}, \tilde{u}, \tilde{\psi}) := (n_1 - n_2, u_1 - u_2, \psi_1 - \psi_2).$$

Then it is easy to verify that  $(\tilde{n}_j, \tilde{u}_j, \tilde{\psi}_j)$  satisfies the equations

$$\begin{cases} \partial_t \tilde{n}_j + u_1 \cdot \nabla \tilde{n}_j + (c_0 + G(n_1)) \operatorname{div} \tilde{u}_j = \tilde{\mathcal{R}}_{1,j}, \\ \partial_t \tilde{u}_j + u_1 \cdot \nabla \tilde{u}_j + \frac{1}{\varepsilon} \tilde{u}_j + \nabla \tilde{n}_j - \mu \nabla \tilde{\psi}_j = \tilde{\mathcal{R}}_{2,j}, \\ \partial_t \tilde{\psi}_j - \Delta \tilde{\psi} + B \tilde{\psi}_j - c_1 \tilde{n}_j = H(n_1)_j - H(n_2)_j, \end{cases} \quad (3.72)$$

where  $w_{1,j}$  is the weight function  $w_{1,j} = c_0 + \dot{S}_{j-1} G(n_1)$ , and  $\tilde{\mathcal{R}}_{i,j}$  ( $i = 1, 2$ ) are the nonlinear terms

$$\begin{cases} \tilde{\mathcal{R}}_{1,j} := \dot{S}_{j-1} u_1 \cdot \nabla \tilde{n}_j - \dot{\Delta}_j (u_1 \cdot \nabla \tilde{n}) + \dot{S}_{j-1} G(n_1) \operatorname{div} \tilde{u}_j - \dot{\Delta}_j (G(n_1) \operatorname{div} \tilde{u}_j) \\ \quad - \dot{\Delta}_j (\tilde{u} \cdot \nabla n_2) - \dot{\Delta}_j ((G(n_1) - G(n_2)) \operatorname{div} u_2), \\ \tilde{\mathcal{R}}_{2,j} := \dot{S}_{j-1} u_1 \cdot \nabla \tilde{u}_j - \dot{\Delta}_j (u_1 \cdot \nabla \tilde{u}) - \dot{\Delta}_j (\tilde{u} \cdot \nabla u_2). \end{cases}$$

From (2.9), there holds

$$\|(n_i, u_i)\|_{L^\infty} + \|(\nabla n_i, \nabla u_i)\|_{L^\infty} + \|(n_i, u_i)\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \lesssim 1, \quad t \in [0, T], \quad i = 1, 2. \quad (3.73)$$

By similar arguments as used in (3.36)-(3.40), one can obtain

$$\begin{aligned}
& \frac{d}{dt} \int \left[ \frac{1}{2} |\tilde{n}_j|^2 + \frac{1}{2} w_{1,j} |\tilde{u}_j|^2 + \frac{\mu b}{2c_1} |\tilde{\psi}_j|^2 + \frac{\mu}{2c_1} |\nabla \tilde{\psi}_j|^2 - \mu \tilde{n}_j \tilde{\psi}_j \right] dx + \int \left( \frac{1}{\varepsilon} w_{1,j} |\tilde{u}_j|^2 + |\partial_t \tilde{\psi}_j|^2 \right) dx \\
&= \int \left( \frac{1}{2} \dot{S}_{j-1} \operatorname{div} u_1 |\tilde{n}_j|^2 + \tilde{\mathcal{R}}_{1,j} \tilde{n}_j + \frac{1}{2} w_{1,j} \dot{S}_{j-1} \operatorname{div} u_1 |\tilde{u}_j|^2 + w_{1,j} \tilde{\mathcal{R}}_{2,j} \cdot \tilde{u}_j \right. \\
&\quad \left. + \frac{1}{2} \partial_t w_{1,j} |\tilde{u}_j|^2 + \nabla w_{1,j} \cdot \left( \frac{1}{2} u_1 |\tilde{u}_j|^2 + \tilde{n}_j \tilde{u}_j - \mu \tilde{\psi}_j \tilde{u}_j \right) \right. \\
&\quad \left. + \partial_t \tilde{\psi}_j (H(n_1)_j - H(n_2)_j) - c_1 (-\dot{S}_{j-1} u_1 \cdot \nabla \tilde{n}_j + \tilde{\mathcal{R}}_{1,j}) \tilde{\psi}_j \right) dx, \quad j \in \mathbb{Z}.
\end{aligned} \tag{3.74}$$

Here we omit the details for the sake of simplicity. Owing to (3.47)-(3.48), we have

$$\frac{c_0}{2} \leq w_{1,j}(x, t) \leq \frac{3c_0}{2}, \quad (x, t) \in \mathbb{R}^d \times (0, T), \quad j \in \mathbb{Z}. \tag{3.75}$$

By (2.6), (2.10), (3.74)-(3.75) and the fact

$$\int \partial_t \tilde{\psi}_j (H(n_1)_j - H(n_2)_j) dx \leq \frac{1}{4} \|\partial_t \tilde{\psi}_j\|_{L^2}^2 + \|H(n_1)_j - H(n_2)_j\|_{L^2}^2,$$

we deduce after direct computations that

$$\begin{aligned}
& \|(\tilde{n}, \tilde{u}, \tilde{\psi}, \nabla \tilde{\psi})\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \\
& \leq C \int_0^t \left( \|(\tilde{n}, \tilde{u}, \tilde{\psi}, \nabla \tilde{\psi})\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \sum_{j \in \mathbb{Z}} 2^{\frac{d}{2}j} \|(\tilde{\mathcal{R}}_{1,j}, \tilde{\mathcal{R}}_{2,j})\|_{L^2} \|(\tilde{n}, \tilde{u}, \tilde{\psi}, \nabla \tilde{\psi})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|H(n_1) - H(n_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \right) d\tau.
\end{aligned} \tag{3.76}$$

One gets from (3.73), (6.3), (6.5) and (6.7) that

$$\begin{aligned}
& \sum_{j \in \mathbb{Z}} 2^{j\frac{d}{2}} \|\tilde{\mathcal{R}}_{1,j}\|_{L^2} \\
& \lesssim \sum_{j \in \mathbb{Z}} 2^{\frac{d}{2}j} \left( \|\dot{S}_{j-1} u_1 \cdot \nabla \tilde{n}_j - \dot{\Delta}_j (u_1 \cdot \nabla \tilde{n})\|_{L^2} + \|\dot{S}_{j-1} G(n_1) \operatorname{div} \tilde{u}_j - \dot{\Delta}_j (G(n_1) \operatorname{div} \tilde{u}_j)\|_{L^2} \right) \\
& \quad + \|\tilde{u} \cdot \nabla n_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|(G(n_1) - G(n_2)) \operatorname{div} u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\
& \lesssim \|u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|G(n_1)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|n_2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\
& \quad + \|u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|G(n_1) - G(n_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|(\tilde{n}, \tilde{u})\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.
\end{aligned} \tag{3.77}$$

Similarly, it holds that

$$\sum_{j \in \mathbb{Z}} 2^{j\frac{d}{2}} \|\tilde{\mathcal{R}}_{2,j}\|_{L^2} \lesssim \|(u_1, u_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \tag{3.78}$$

Finally, using the composition estimate (6.7), we derive

$$\|H(n_1) - H(n_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \lesssim \|\tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2. \tag{3.79}$$

Combining (3.76)-(3.79) together and employing the Grönwall inequality, we prove  $(\tilde{n}, \tilde{u}, \tilde{\psi}) = 0$  a.e. in  $\mathbb{R}^d \times (0, T)$ .

## 4 Optimal time-decay rates

### 4.1 The evolution of negative Besov norms

We analyze the time-decay estimates of the global solution given by Theorem 2.1 without additional smallness condition on the initial data. The following lemma shows that the additional low-frequency regularity assumption (2.11) is propagated in time.

**Lemma 4.1.** *Let  $\sigma_0 \in [-\frac{d}{2}, \frac{d}{2}]$ . Then, under the assumptions (2.6), (2.8) and (2.11), the following inequality holds:*

$$\mathcal{X}_L^{\sigma_0}(t) \leq C(\mathcal{X}_0 + \|(n_0, u_0, \psi_0)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell), \quad t > 0, \quad (4.1)$$

where

$$\mathcal{X}_L^{\sigma_0}(t) := \|(n, u, \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \varepsilon \|(n, \psi)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \varepsilon^{-\frac{1}{2}} \|u\|_{\tilde{L}_t^2(\dot{B}_{2,\infty}^{\sigma_0})}^\ell. \quad (4.2)$$

**Proof.** Let the effective unknowns  $(\varphi, v)$  be denoted by (3.2). Owing to (3.5)<sub>1</sub> and (3.11), we have

$$\begin{aligned} & \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell \\ & \lesssim \|n_0\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \varepsilon \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \|R_1\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ & \lesssim \|n_0\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \varepsilon \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1} \cap \dot{B}_{2,\infty}^{\sigma_0+3})}^\ell + \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \|R_1\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell, \end{aligned} \quad (4.3)$$

which together with (3.5)<sub>2</sub>, (3.11) and (3.17) gives

$$\begin{aligned} & \|\varphi\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0} \cap \dot{B}_{2,\infty}^{\sigma_0+2})}^\ell \\ & \lesssim \|\varphi|_{t=0}\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \varepsilon \|n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \|R_2\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ & \lesssim \|(n, \varphi)|_{t=0}\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \varepsilon \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1} \cap \dot{B}_{2,\infty}^{\sigma_0+3})}^\ell + \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \|(R_1, R_2)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell. \end{aligned} \quad (4.4)$$

Making use of (3.5)<sub>3</sub> and (4.4), we also get

$$\begin{aligned} & \|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \frac{1}{\varepsilon} \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ & \lesssim \|v|_{t=0}\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \varepsilon^2 \|n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+3})}^\ell + \varepsilon \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1} \cap \dot{B}_{2,\infty}^{\sigma_0+3})}^\ell + \varepsilon \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|R_3\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell. \end{aligned} \quad (4.5)$$

The combination of (2.1) and (4.3)-(4.5) implies that for  $J_\varepsilon$  given by (2.7) with a sufficiently small constant  $k$ , we have

$$\begin{aligned} & \|(n, \varphi, v)\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0} \cap \dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \frac{1}{\varepsilon} \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ & \lesssim \|(n, \varphi, v)|_{t=0}\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \|(R_1, R_2, R_3)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell. \end{aligned} \quad (4.6)$$

We turn to estimate the nonlinear terms  $R_i$  ( $i = 1, 2, 3$ ). One infers from (3.24), (3.22) and the composition estimate (6.10) that

$$\begin{aligned} \varepsilon \|H(n)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell &\lesssim (\|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) (\varepsilon \|n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \\ &\lesssim \mathcal{X}(t) \mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}^2(t). \end{aligned} \quad (4.7)$$

And due to (3.24), (3.22) and (6.6), we get

$$\begin{aligned} &\|u \cdot \nabla u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \|u \cdot \nabla n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \|u \cdot \nabla G(n)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \|\partial_t H(n)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ &\lesssim \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \|(n, u)\|_{\tilde{L}_t^2(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \|u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell \\ &\lesssim \varepsilon^{-\frac{1}{2}} \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \left( (\|(n, u)\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell)^{\frac{1}{2}} (\varepsilon \|(n, u)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell)^{\frac{1}{2}} \right. \\ &\quad \left. + (\varepsilon \|(n, u)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h)^{\frac{1}{2}} (\|(n, u)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h)^{\frac{1}{2}} \right) \\ &\quad + (\|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) (\|u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+1})}^\ell + \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h) \\ &\lesssim \mathcal{X}(t) \mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}^2(t). \end{aligned} \quad (4.8)$$

One deduces by (3.8) and (4.7)-(4.8) that

$$\begin{aligned} \|(R_1, R_2, R_3)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell &\lesssim \varepsilon \|H(n)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|u \cdot \nabla n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \|u \cdot \nabla u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ &\quad + \|G(n) \operatorname{div} u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \|\partial_t H(n)\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ &\lesssim \mathcal{X}(t) \mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}^2(t). \end{aligned} \quad (4.9)$$

Combining (4.6) and (4.9) together, we gain

$$\begin{aligned} &\|(n, \varphi, v)\|_{\tilde{L}_t^\infty(\dot{B}_{2,\infty}^{\sigma_0})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \|\varphi\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0} \cap \dot{B}_{2,\infty}^{\sigma_0+2})}^\ell + \frac{1}{\varepsilon} \|v\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^{\sigma_0})}^\ell \\ &\lesssim \mathcal{X}_0 + \|(n_0, u_0, \psi_0)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \mathcal{X}(t) \mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}^2(t). \end{aligned}$$

Therefore, by similar computations as in (3.30)-(3.32), one has

$$\mathcal{X}_L^{\sigma_0}(t) \lesssim \mathcal{X}_0 + \|(n_0, u_0, \psi_0)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \mathcal{X}(t) \mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}^2(t).$$

This together with  $\mathcal{X}(t) \lesssim \mathcal{X}_0 \ll 1$  leads to (4.1).  $\square$

## 4.2 New time-weighted estimates

We apply a modified energy argument to prove the optimal time-decay rates (2.12)-(2.13) of global solutions to the Cauchy problem of the System (HPC-r). For a suitably large constant  $\theta$ , we introduce the time-weighted functional

$$\mathcal{D}^\theta(t) := \mathcal{D}_L^\theta(t) + \mathcal{D}_H^\theta(t), \quad (4.10)$$

with

$$\begin{aligned}\mathcal{D}_L^\theta(t) &:= \|\tau^\theta(n, u, \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon\|\tau^\theta(n, \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\tau^\theta u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell, \\ \mathcal{D}_H^\theta(t) &:= \varepsilon\|\tau^\theta(n, u, \nabla\psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\tau^\theta(n, u, \nabla\psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h.\end{aligned}$$

Compared with  $\mathcal{X}(t)$ , there are some time-weighted dissipation estimates missing in  $\mathcal{D}^\theta(t)$ . Here we omit these since  $\mathcal{D}^\theta(t)$  is sufficient to derive the desired time-decay rates.

In low frequencies, we have the following estimates.

**Lemma 4.2.** *Let  $\sigma_0 \in [-\frac{d}{2}, \frac{d}{2})$  and  $\theta > 1 + \frac{1}{2}(\frac{d}{2} - \sigma_0)$ . Then, under the assumptions of (2.6), (2.8) and (2.11), it holds for any constant  $\zeta > 0$  that*

$$\mathcal{D}_L^\theta(t) \leq \frac{C(\mathcal{X}(t) + \mathcal{X}_L^{\sigma_0}(t))}{\zeta} t^\theta (\varepsilon t)^{-\frac{1}{2}(\frac{d}{2} - \sigma_0)} + C(\zeta + \mathcal{X}(t))\mathcal{D}^\theta(t), \quad t > 0. \quad (4.11)$$

**Proof.** Let  $(\varphi, v)$  be the effective unknowns defined in (3.2). Multiplying the equations (3.5) by  $t^\theta$ , we have

$$\begin{cases} \partial_t(t^\theta n) - \tilde{\Delta}_1(t^\theta n) = \theta t^{\theta-1} n + t^\theta L_1 + t^\theta R_1, \\ \partial_t(t^\theta \varphi) - \tilde{\Delta}_2(t^\theta \varphi) = \theta t^{\theta-1} \varphi + t^\theta L_2 + t^\theta R_2, \\ \partial_t(t^\theta v) + \varepsilon^{-1}(t^\theta v) = \theta t^{\theta-1} v + t^\theta L_3 + t^\theta R_3, \end{cases}$$

with  $\tilde{\Delta}_i$  ( $i = 1, 2$ ),  $L_i$  ( $i = 1, 2, 3$ ) and  $R_i$  ( $i = 1, 2, 3$ ) given by (3.6), (3.7) and (3.8) respectively. By similar calculations as used in (3.16)-(3.28), we can derive

$$\begin{aligned}& \|\tau^\theta(n, v, \psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon\|\tau^\theta n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\tau^\theta \varphi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \frac{1}{\varepsilon}\|\tau^\theta v\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ & \lesssim \int_0^t \tau^{\theta-1} \|(n, v, \varphi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell d\tau + \mathcal{X}(t)\mathcal{D}^\theta(t).\end{aligned} \quad (4.12)$$

It can be proved as in (3.30)-(3.32) that the estimates (4.12) implies

$$\mathcal{D}_L^\theta(t) \lesssim \int_0^t \tau^{\theta-1} (\|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|H(n)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell) d\tau + \mathcal{X}(t)\mathcal{D}^\theta(t). \quad (4.13)$$

We omit the details of the above estimates and focus on controlling the first term on the right-hand side of (4.13) which is the key point. It follows from the interpolation inequality (6.1) and the Young inequality that

$$\begin{aligned}& \int_0^t \tau^{\theta-1} \|(n^\ell, u^\ell, \psi^\ell)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \\ & \lesssim \int_0^t \tau^{\theta-1} \left( \|(n^\ell, u^\ell, \psi^\ell)\|_{\dot{B}_{2,\infty}^{\sigma_0}} \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \|(n^\ell, u^\ell, \psi^\ell)\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} d\tau \\ & \lesssim \left( \mathcal{X}_L^{\sigma_0}(t) \int_0^t \tau^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)-1} d\tau \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \frac{1}{\varepsilon} (\varepsilon\|\tau^\theta(n, \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\tau^\theta u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell) \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} \\ & \lesssim \frac{\mathcal{X}_L^{\sigma_0}(t)}{\zeta} t^\theta (\varepsilon t)^{-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + \zeta \mathcal{D}_L^\theta(t).\end{aligned} \quad (4.14)$$

It also holds by (3.22) that

$$\begin{aligned}
& \int_0^t \tau^{\theta-1} \|(n^h, u^h, \psi^h)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \\
& \lesssim \int_0^t \tau^{\theta-1} \left( \varepsilon \|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} d\tau \\
& \lesssim \left( \mathcal{X}(t) \int_0^t \tau^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)-1} d\tau \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \|\tau^\theta(n, u, \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} \\
& \lesssim \frac{\mathcal{X}(t)}{\zeta} t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + \zeta \mathcal{D}_H^\theta(t).
\end{aligned} \tag{4.15}$$

In addition, we obtain from (4.14)-(4.15) and the composition estimate (6.6) that

$$\int_0^t \tau^{\theta-1} \|H(n)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell d\tau \lesssim \int_0^t \tau^{\theta-1} (\|n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h) dx \lesssim \frac{\mathcal{X}(t)}{\zeta} t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + \zeta \mathcal{D}_H^\theta(t). \tag{4.16}$$

One combines (4.14)-(4.16) together to obtain

$$\begin{aligned}
\int_0^t \tau^{\theta-1} \|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell d\tau & \lesssim \int_0^t \tau^{\theta-1} (\|(n^\ell, u^\ell, \psi^\ell)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|(n^h, u^h, \psi^h)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau \\
& \lesssim \frac{\mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}(t)}{\zeta} t^\theta (\varepsilon t)^{-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + \zeta \mathcal{D}^\theta(t).
\end{aligned} \tag{4.17}$$

Inserting (4.17) into (4.13), we prove (4.11).  $\square$

Then, we have the time-weighted estimates in high frequencies.

**Lemma 4.3.** *Let  $\sigma_0 \in [-\frac{d}{2}, \frac{d}{2})$  and  $\theta > 1 + \frac{1}{2}(\frac{d}{2} - \sigma_0)$ . Then, under the assumptions of (2.6), (2.8) and (2.11), it holds for any constant  $\zeta > 0$  that*

$$\mathcal{D}_H^\theta(t) \leq \frac{C\mathcal{X}(t)}{\zeta} t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + C(\zeta + \mathcal{X}(t))\mathcal{D}^\theta(t), \quad t > 0. \tag{4.18}$$

**Proof.** Multiplying the inequality (3.56) by  $t^\theta$ , we obtain for any  $j \geq J_\varepsilon - 1$  that

$$\begin{aligned}
& \frac{d}{dt} (t^\theta \mathcal{L}_j(t)) + \frac{1}{\varepsilon} t^\theta \mathcal{L}_j(t) \\
& \lesssim t^{\theta-1} \mathcal{L}_j(t) \\
& + t^\theta \left( \|\nabla u\|_{L^\infty} \varepsilon \|(n_j, u_j)\|_{L^2} + \|u\|_{L^\infty} \varepsilon \|(n_j, H(n)_j)\|_{L^2} + \|(\partial_t G(n), \nabla G(n))\|_{L^\infty} \varepsilon \|u_j\|_{L^2} \right. \\
& \left. + \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2} + \|H(n)_j\|_{L^2} + 2^{-j} \|\partial_t H(n)_j\|_{L^2} \right) \sqrt{\frac{1}{\varepsilon} \mathcal{L}_j(t)},
\end{aligned} \tag{4.19}$$

which together with (3.51)<sub>1</sub> and  $t^\theta \sqrt{\mathcal{L}_j(t)}|_{t=0} = 0$  gives rise to

$$\begin{aligned}
& \varepsilon t^\theta \|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2} + \int_0^t \tau^\theta \|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2} d\tau \\
& \lesssim \int_0^t \tau^{\theta-1} (\|(n_j, u_j, \psi_j, \nabla \psi_j)\|_{L^2} + 2^{-j} \|H(n)_j\|_{L^2}) d\tau \\
& + \int_0^t \tau^\theta \left( \varepsilon \|\nabla u\|_{L^\infty} (\|(n_j, u_j)\|_{L^2} + 2^{-j} \|H(n)_j\|_{L^2}) + \varepsilon \|u\|_{L^\infty} \|(n_j, H(n)_j)\|_{L^2} \right. \\
& \left. + \varepsilon (\|\nabla G(n)\|_{L^\infty} \|u_j\|_{L^2} + \varepsilon \|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j})\|_{L^2} + \|H(n)_j\|_{L^2} + 2^{-j} \|\partial_t H(n)_j\|_{L^2}) \right) d\tau.
\end{aligned}$$



Performing similar arguments as in (3.55)-(3.65), we get

$$\begin{aligned} & \varepsilon \|\tau^\theta(n, u, \psi, \nabla\psi)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\tau^\theta(n, u, \psi, \nabla\psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\ & \lesssim \varepsilon \int_0^t \tau^{\theta-1} (\|(n, u, \psi, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|H(n)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau + \mathcal{X}(t)\mathcal{D}^\theta(t). \end{aligned} \quad (4.20)$$

The details are omitted for brevity. In view of the dissipation estimates in high frequencies, the first term on the right-hand side of (4.20) can be estimated by

$$\begin{aligned} & \varepsilon \int_0^t \tau^{\theta-1} \|(n, u, \psi, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h d\tau \\ & \lesssim \int_0^t \tau^{\frac{2}{\frac{d}{2}+2-\sigma_0}(\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0))} \left( \varepsilon \|(n, u, \psi, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \|(n, u, \psi, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} d\tau \\ & \lesssim \left( \mathcal{X}(t) t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \mathcal{D}_H^\theta(t) \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} \\ & \lesssim \frac{\mathcal{X}(t)}{\zeta} t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + \zeta \mathcal{D}_H^\theta(t). \end{aligned} \quad (4.21)$$

In accordance with (3.22) and (6.6), it follows that

$$\begin{aligned} \varepsilon \int_0^t \tau^{\theta-1} \|H(n)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h d\tau & \lesssim \int_0^t \tau^{\theta-1} \varepsilon (\|n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \|n\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h) d\tau \\ & \lesssim \left( \mathcal{X}_L^{\sigma_0}(t) t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \varepsilon \|\tau^\theta n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} \\ & \quad + \left( \mathcal{X}(t) t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} \right)^{\frac{2}{\frac{d}{2}+2-\sigma_0}} \left( \|\tau^\theta n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{\frac{d}{2}-\sigma_0}{\frac{d}{2}+2-\sigma_0}} \\ & \lesssim \frac{\mathcal{X}_L^{\sigma_0}(t) + \mathcal{X}(t)}{\zeta} t^{\theta-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + \zeta \mathcal{D}^\theta(t). \end{aligned} \quad (4.22)$$

The combination of (4.20)-(4.22) gives rise to (4.11).  $\square$

**Proof of Theorem 2.2:** Assume that (2.6), (2.8) and (2.11) holds. Let  $(n, u, \psi)$  be the global solution to the System (HPC-r) given by Theorem 2.1, and  $\mathcal{X}(t)$ ,  $\mathcal{X}_L^{\sigma_0}(t)$  and  $\mathcal{D}^\theta(t)$  be defined by (2.3), (4.2) and (4.10) respectively. For any constants  $\theta > 1 + \frac{1}{2}(\frac{d}{2} - \sigma_0)$  and  $\zeta > 0$ , it follows from Lemmas 4.2-4.3 that

$$\mathcal{D}^\theta(t) \lesssim \frac{\mathcal{X}(t) + \mathcal{X}_L^{\sigma_0}(t)}{\zeta} t^\theta (\varepsilon t)^{-\frac{1}{2}(\frac{d}{2}-\sigma_0)} + (\zeta + \mathcal{X}(t))\mathcal{D}^\theta(t), \quad t > 0. \quad (4.23)$$

By (2.10), (4.1) and (4.23), we deduce after choosing a suitably small constant  $\zeta > 0$  that

$$t^\theta (\|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|(n, u, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h) \lesssim \mathcal{D}^\theta(t) \lesssim t^\theta (\varepsilon t)^{-\frac{1}{2}(\frac{d}{2}-\sigma_0)}. \quad (4.24)$$

Using (3.22), (4.24) and  $\mathcal{X}(t) \lesssim \mathcal{X}_0$ , we infer that

$$\|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \varepsilon \|(n, u, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \lesssim (1 + \varepsilon t)^{-\frac{1}{2}(\frac{d}{2}-\sigma_0)}. \quad (4.25)$$

Thence, it holds by (3.22), (4.1), (4.25) and (6.1) that

$$\begin{aligned}
\|(n, u, \psi)\|_{\dot{B}_{2,1}^{\sigma}} &\lesssim \|(n, u, \psi)\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-\sigma_0}}^{\frac{\frac{d}{2}-\sigma}{\frac{d}{2}-\sigma_0}} \|(n, u, \psi)\|_{\dot{B}_{2,1}^{\frac{\sigma-\sigma_0}{\frac{d}{2}-\sigma_0}}} \\
&\lesssim (\|(n, u, \psi)\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-\sigma_0}}^{\ell} + \varepsilon \|(n, u, \nabla \psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h)^{\frac{\frac{d}{2}-\sigma}{\frac{d}{2}-\sigma_0}} (1 + \varepsilon t)^{-\frac{\sigma-\sigma_0}{\frac{d}{2}-\sigma_0} \frac{1}{2}(\frac{d}{2}-\sigma_0)} \\
&\lesssim (1 + \varepsilon t)^{-\frac{1}{2}(\sigma-\sigma_0)}, \quad \sigma \in (\sigma_0, \frac{d}{2}).
\end{aligned}$$

This together with (4.25) yields the time-decay estimates (2.12).

We turn to prove (2.13) for  $d \geq 2$  and  $\sigma_0 \in [-\frac{d}{2}, \frac{d}{2} - 1)$ . Note that the equation (HPC-r) can be rewritten by

$$u = e^{-\frac{1}{\varepsilon}t}u_0 + \int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}(-c_0\nabla n + \mu\nabla\psi - u \cdot \nabla u)d\tau, \quad (4.26)$$

which gives

$$\|u\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} \lesssim e^{-\frac{1}{\varepsilon}t}\|u_0\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} + \int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}(\|(n, \psi)\|_{\dot{B}_{2,\infty}^{\sigma_0+1}}^{\ell} + \|u \cdot \nabla u\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell})d\tau. \quad (4.27)$$

Owing to  $\sigma_0 \leq \frac{d}{2} - 1$  and (2.12)<sub>1</sub>, we get

$$\|(n, \psi)\|_{\dot{B}_{2,\infty}^{\sigma_0+1}}^{\ell} \lesssim (1 + \varepsilon t)^{-\frac{1}{2}}, \quad (4.28)$$

It also holds by (2.12), (6.4) and  $\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \mathcal{X}(t) \lesssim \mathcal{X}_0$  that

$$\|u \cdot \nabla u\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} \lesssim \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,\infty}^{\sigma_0+1}} \lesssim (1 + \varepsilon t)^{-\frac{1}{2}}. \quad (4.29)$$

Inserting (4.28)-(4.29) into (4.27) shows

$$\|u\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} \lesssim \|u\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \lesssim e^{-\frac{1}{\varepsilon}t} + \int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}(1 + \varepsilon\tau)^{-\frac{1}{2}}d\tau \lesssim (1 + \varepsilon t)^{-\frac{1}{2}},$$

where one has used (4.25) and the estimate

$$\begin{aligned}
&\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}(1 + \varepsilon\tau)^{-\frac{1}{2}}d\tau \\
&\lesssim e^{-\frac{1}{2\varepsilon}t} \int_0^{\frac{t}{2}} (1 + \varepsilon\tau)^{-\frac{1}{2}}d\tau + (1 + \frac{\varepsilon}{2}t)^{-\frac{1}{2}} \int_{\frac{t}{2}}^t e^{-\frac{1}{\varepsilon}(t-\tau)}d\tau \lesssim \frac{1}{\varepsilon}(1 + \varepsilon t)^{-\frac{1}{2}}.
\end{aligned}$$

In addition, it is easy to verify that  $\tilde{\varphi} := b\psi - c_1n$  satisfies

$$\partial_t \tilde{\varphi} + b\tilde{\varphi} = b\Delta\psi + c_1c_0 \operatorname{div} u + bH(n) + c_1u \cdot \nabla u + c_1G(n) \operatorname{div} u. \quad (4.30)$$

Similarly, by virtue of (2.12), (3.22), (4.30), (6.4) and (6.6), we get

$$\begin{aligned}
\|\tilde{\varphi}\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} &\lesssim e^{-bt}\|(n_0, \psi_0)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell} + \int_0^t e^{-b(t-\tau)}\left(\frac{1}{\varepsilon}\|\psi\|_{\dot{B}_{2,\infty}^{\sigma_0+1}}^{\ell} + \|u\|_{\dot{B}_{2,\infty}^{\sigma_0+1}}^{\ell}\right. \\
&\quad \left. + \|(n, u)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}(\|(n, u)\|_{\dot{B}_{2,\infty}^{\sigma_0+1}}^{\ell} + \|(n, u)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h) + \|H(n)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^{\ell}\right)dx.
\end{aligned} \quad (4.31)$$

Owing to (3.22), (4.1), (4.25) and the composition estimates (6.6) and (6.10), there holds

$$\begin{cases} \|H(n)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim (1 + \varepsilon t)^{-\frac{1}{2}(\frac{d}{2} - \sigma_0)} \lesssim (1 + \varepsilon t)^{-\frac{1}{2}}, \\ \|H(n)\|_{\dot{B}_{2,\infty}^{\sigma_0}} \lesssim \|n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (\|n\|_{\dot{B}_{2,\infty}^{\sigma_0}} + \varepsilon \|n\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \lesssim (1 + \varepsilon t)^{-\frac{1}{2}(\frac{d}{2} - \sigma_0)} \lesssim (1 + \varepsilon t)^{-\frac{1}{2}}. \end{cases} \quad (4.32)$$

Combining (3.22), (4.25) and (4.31)-(4.32) together, we derive

$$\begin{aligned} & \|B\psi - c_1 n - H(n)\|_{\dot{B}_{2,\infty}^{\sigma_0}} \\ & \lesssim \|\tilde{\varphi}\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \|H(n)\|_{\dot{B}_{2,\infty}^{\sigma_0}}^\ell + \|(n, \nabla\psi)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \|H(n)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \lesssim \frac{1}{\varepsilon} (1 + \varepsilon t)^{-\frac{1}{2}}. \end{aligned}$$

Finally, we take the  $\dot{B}_{2,1}^\sigma$ -norm of (4.26) for any  $\sigma \in (\sigma_0, \frac{d}{2} - 1]$  to have

$$\begin{aligned} \|u\|_{\dot{B}_{2,1}^\sigma}^\ell & \lesssim e^{-\frac{1}{\varepsilon}t} \|u_0\|_{\dot{B}_{2,1}^\sigma}^\ell + \int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} (\|(n, \psi)\|_{\dot{B}_{2,1}^{\sigma+1}}^\ell + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^{\sigma+1}}) d\tau \\ & \lesssim \frac{1}{\varepsilon} (1 + \varepsilon t)^{-\frac{1}{2}(1+\sigma-\sigma_0)}, \end{aligned}$$

where one has used  $\|(n, u, \psi)\|_{\dot{B}_{2,1}^{\sigma+1}}^\ell \lesssim (1 + \varepsilon t)^{-\frac{1}{2}(1+\sigma-\sigma_0)}$  derived from (2.12)<sub>1</sub>. Therefore, the following decay rate holds:

$$\|u\|_{\dot{B}_{2,1}^\sigma} \lesssim \|u\|_{\dot{B}_{2,1}^\sigma}^\ell + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \lesssim \frac{1}{\varepsilon} (1 + \varepsilon t)^{-\frac{1}{2}(1+\sigma-\sigma_0)}.$$

Similarly, we can show

$$\|B\psi - c_1 n - H(n)\|_{\dot{B}_{2,1}^\sigma} \lesssim \frac{1}{\varepsilon} (1 + \varepsilon t)^{-\frac{1}{2}(1+\sigma-\sigma_0)}.$$

The details are omitted here. The proof of Theorem 2.2 is complete.

## 5 Relaxation limit

### 5.1 Global well-posedness problem for the Keller-Segel equations

In this subsection, we prove Theorem 2.3 related to the global well-posedness of strong solutions to the Cauchy problem for the System (KS). Thanks to the Taylor formula, there exists a function  $G_1$  vanishing at  $\bar{\rho}$  such that

$$P(\rho^*) - P(\bar{\rho}) = P'(\bar{\rho})(\rho^* - \bar{\rho}) + G_1(\rho^*)(\rho^* - \bar{\rho}). \quad (5.1)$$

We also notice that

$$\Delta\phi^* = \mu a \bar{\rho} (b - \Delta)^{-1} \Delta\rho^*. \quad (5.2)$$

Substituting (5.1)-(5.2) into (KS)<sub>1</sub>, we can rewrite the System (KS) as

$$\begin{cases} \partial_t \rho^* - \tilde{\Delta}_* \rho^* = \Delta(G_1(\rho^*)(\rho^* - \bar{\rho})) - \mu \operatorname{div}((\rho^* - \bar{\rho})\nabla \phi^*), \\ \phi^* - \bar{\phi} = a(b - \Delta)^{-1}(\rho^* - \bar{\rho}), \end{cases} \quad (5.3)$$

where  $\tilde{\Delta}_*$  is the differential operator

$$\tilde{\Delta}_* := (P'(\bar{\rho}) - \mu a \bar{\rho}(b - \Delta)^{-1})\Delta. \quad (5.4)$$

Consider the linear problem

$$\begin{cases} \partial_t f - \tilde{\Delta}_* f = g, & x \in \mathbb{R}^d, \quad t > 0, \\ f(x, 0) = f_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (5.5)$$

**Lemma 5.1.** *Let  $P'(\bar{\rho}) > \frac{\mu a}{b}\bar{\rho}$ . For  $s \in \mathbb{R}$  and  $p \in [1, \infty]$ , assume  $u_0 \in \dot{B}_{p,1}^s$  and  $g \in L^1(0, T; \dot{B}_{p,1}^s)$ . If  $f$  is a solution to the problem (5.5), then there exists a universal constant  $C > 0$  such that*

$$\|f\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^s)} + \|f\|_{L_t^1(\dot{B}_{p,1}^{s+2})} \leq C(\|f_0\|_{\dot{B}_{p,1}^s} + \|g\|_{L_t^1(\dot{B}_{p,1}^s)}), \quad 0 < t < T. \quad (5.6)$$

**Proof.** Let the semi-group operator  $e^{\tilde{\Delta}_* t}$  be defined as

$$e^{\tilde{\Delta}_* t} f = \mathcal{F}^{-1}\left(e^{-(P'(\bar{\rho}) - \frac{\mu a}{b}\bar{\rho})|\xi|^2 t - \frac{\mu a \bar{\rho}|\xi|^4}{b+|\xi|^2} t} \mathcal{F}(f)(\xi)\right).$$

By similar calculations as in [4, 11], one can show that

$$\operatorname{Supp} \mathcal{F}(f) \subset \lambda \mathbf{C} \Rightarrow \|e^{\tilde{\Delta}_* t} f\|_{L^p} \leq C e^{-C(P'(\bar{\rho}) - \frac{\mu a}{b}\bar{\rho})\lambda^2} \|f\|_{L^p}. \quad (5.7)$$

Thus, since  $P'(\bar{\rho}) > \frac{\mu a}{b}\bar{\rho}$ , we are able to obtain (5.6) by virtue of usual energy estimates similar to the heat equation.  $\square$

**Proof of Theorem 2.4:** It follows from (5.3) and Lemma 5.1 that

$$\begin{aligned} & \|\rho^* - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} + \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \\ & \lesssim \|\rho_0^* - \bar{\rho}\|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \|G_1(\rho^*)(\rho^* - \bar{\rho})\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} + \|(\rho^* - \bar{\rho})\nabla \phi^*\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+1})}. \end{aligned} \quad (5.8)$$

Making use of (5.3)<sub>2</sub> and the properties (3.17) of the Bessel potential  $(b - \Delta)^{-1}$  yield

$$\begin{cases} \|\phi^* - \bar{\phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \leq \|\rho - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})}, \\ \|\phi^* - \bar{\phi}\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{d}{p}+1})} \lesssim \|\rho - \bar{\rho}\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{d}{p}+1})} \lesssim \|\rho^* - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} + \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})}. \end{cases} \quad (5.9)$$

We employ (5.9), the product law (6.2) and the composition estimate (6.6) to have

$$\begin{aligned}
& \|G_1(\rho^*)(\rho^* - \bar{\rho})\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} + \|(\rho^* - \bar{\rho})\nabla\phi^*\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+1})} \\
& \lesssim \|\rho^* - \bar{\rho}\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} + \|\rho^* - \bar{\rho}\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|\phi^* - \bar{\phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \\
& \quad + \|\rho^* - \bar{\rho}\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}+1})} \|\phi^* - \bar{\phi}\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}+1})} \\
& \lesssim \left( \|\rho^* - \bar{\rho}\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} + \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \right)^2.
\end{aligned} \tag{5.10}$$

By (5.8) and (5.10), one obtains

$$\|\rho^* - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} + \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \lesssim \|\rho_0^* - \bar{\rho}\|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \left( \|\rho^* - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} + \|\rho^* - \bar{\rho}\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+2})} \right)^2. \tag{5.11}$$

Hence, in view of the Friedrichs approximation and bootstrap argument, the left-hand side of (5.11) can be bounded for all  $T > 0$  provided that (2.16) is satisfied with  $\alpha^*$  small enough. Then it is easy to deduce the global existence of strong solutions to the Cauchy problem for the System (KS). And the uniqueness in the space  $\tilde{L}^\infty(0, T; \dot{B}_{p,1}^{\frac{d}{p}-1}) \cap L^1(0, T; \dot{B}_{p,1}^{\frac{d}{p}+1})$  can be proved in a simple way. We omit the details for brevity.

## 5.2 Quantitative error estimates

This subsection is devoted to the proof of Theorem 2.4 pertaining to the relaxation limit from the HPC system to the Keller-Segel model. Recall that  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  is defined through the following diffusive scaling (1.1) and it satisfies the System (HPC $_\varepsilon$ ).

From the bounds obtained with Theorem 2.1, we can immediately deduce the following lemma concerning the unknowns  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  under the diffusive scaling.

**Corollary 5.1.** *Let the assumptions (2.6), (2.8) and (2.16) hold and let  $(n, u, \psi)$  be the global solution to the System (HPC-r) given by Theorem 2.1. Then  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  satisfies*

$$\left\{ \begin{array}{l}
\varepsilon \|u^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon^2 \|u^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|u^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|u^\varepsilon\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \leq C\mathcal{X}_0, \\
\|\phi^\varepsilon - \bar{\phi}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|\phi^\varepsilon - \bar{\phi}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+2})}^h + \|\phi^\varepsilon - \bar{\phi}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\phi^\varepsilon - \bar{\phi}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+3})}^h \leq C\mathcal{X}_0, \\
\|\rho^\varepsilon - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|\rho^\varepsilon - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\rho^\varepsilon - \bar{\rho}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \varepsilon^{-1} \|\rho^\varepsilon - \bar{\rho}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \leq C\mathcal{X}_0, \\
\|\rho^\varepsilon - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|(\rho^\varepsilon - \bar{\rho}, \phi^\varepsilon - \bar{\phi})\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})} \leq C\mathcal{X}_0, \\
\|v^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \varepsilon \|\partial_t \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \leq C\mathcal{X}_0\varepsilon,
\end{array} \right. \tag{5.12}$$

with  $v^\varepsilon = u^\varepsilon + \frac{1}{\rho^\varepsilon} \nabla P(\rho^\varepsilon) - \mu \nabla \phi^\varepsilon$ .

**Remark 5.1.** *Note that (5.12)<sub>4</sub>-(5.12)<sub>5</sub> are the main ingredients to show the relaxation limit.*

**Proof.** The inequality (5.12)<sub>1</sub>-(5.12)<sub>2</sub> can be derived by (2.10) and the scaling (1.1) directly. Recall that in (HPC-r), we have  $a(\rho - \bar{\rho}) = c_1 n + H(n)$ . Thus, we are able to apply (1.1), (3.22) and the composition estimates (6.8)-(6.9) to have

$$\begin{aligned}
& \|\rho^\varepsilon - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|\rho^\varepsilon - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&= \|\rho - \bar{\rho}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|\rho - \bar{\rho}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&\lesssim \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|H(n)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \varepsilon \|H(n)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&\lesssim \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0.
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
& \|\rho^\varepsilon - \bar{\rho}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \frac{1}{\varepsilon} \|\rho^\varepsilon - \bar{\rho}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&= \varepsilon \|\rho - \bar{\rho}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|\rho - \bar{\rho}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&\lesssim \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \varepsilon \|H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \varepsilon \|H(n)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&\lesssim \varepsilon \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell + \|n\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0.
\end{aligned}$$

Thence, (5.12)<sub>4</sub> follows by (5.12)<sub>2</sub>-(5.12)<sub>3</sub> and standard interpolation inequalities. We also obtain from (1.1), (2.10), (3.22) and (5.12)<sub>2</sub> that

$$\begin{aligned}
\|v^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} &= \varepsilon \left\| \frac{1}{\varepsilon} u + \nabla n - \mu \nabla \psi \right\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\
&\lesssim \varepsilon \left\| \frac{1}{\varepsilon} u + \nabla n - \mu \nabla \psi \right\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|(n, u, \nabla \psi)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0 \varepsilon.
\end{aligned}$$

Finally, it holds from (2.10) and  $\varepsilon \partial_\tau \phi^\varepsilon(x, \tau) = \partial_t \psi(x, \varepsilon t)$  that

$$\|\partial_t \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} = \|\partial_t \psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \|\partial_t \psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \varepsilon \|\partial_t \psi\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \lesssim \mathcal{X}_0.$$

□

***Proof of Theorem 2.4:*** Let the assumptions (2.6)-(2.8) hold and let  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  be the global solution to the System (HPC<sub>ε</sub>) subject to the initial data  $(\rho_0, u_0, \phi_0)$  given by Theorem 2.1. According to the equations (HPC<sub>ε</sub>) and Corollary 5.1, we have the uniform estimates

$$\begin{cases} \|\rho^\varepsilon v^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim (\|\rho^\varepsilon - \bar{\rho}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \bar{\rho}) \|v^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \mathcal{X}_0 \varepsilon, \\ \|- \Delta \phi^\varepsilon - a \rho^\varepsilon + b \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} = \varepsilon \|\partial_t \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \mathcal{X}_0 \varepsilon, \end{cases} \quad (5.13)$$

which yields (2.18).

Assume further that (2.19) holds. With these bounds in hand, we can derive expected quantitative error estimates. From the Taylor formula, there exists a smooth function  $G_1$  vanishing at  $\bar{\rho}$  such that

$$P(\rho^\varepsilon) - P(\bar{\rho}) = P'(\bar{\rho})(\rho^\varepsilon - \bar{\rho}) + G_1(\rho^\varepsilon)(\rho^\varepsilon - \bar{\rho}).$$

It is easy to verify that  $\rho^\varepsilon$  satisfies

$$\begin{cases} \partial_t \rho^\varepsilon - \tilde{\Delta}_* \rho^\varepsilon = \mathcal{R}^\varepsilon + \Delta(G_1(\rho^\varepsilon)(\rho^\varepsilon - \bar{\rho})) - \mu \operatorname{div}((\rho^\varepsilon - \bar{\rho})\nabla \phi^\varepsilon), \\ \phi^\varepsilon = (b - \Delta)^{-1}(-\varepsilon \partial_t \phi^\varepsilon + a \rho^\varepsilon). \end{cases} \quad (5.14)$$

with

$$\mathcal{R}^\varepsilon := -\operatorname{div}(\rho^\varepsilon v^\varepsilon) + \varepsilon \mu \bar{\rho} \Delta(b - \Delta)^{-1} \partial_t \phi^\varepsilon.$$

Introducing the error variables

$$\delta \rho^\varepsilon := \rho^\varepsilon - \rho^*, \quad \delta u^\varepsilon := u^\varepsilon - u^*, \quad \delta \phi^\varepsilon := \phi^\varepsilon - \phi^*,$$

we deduce from (5.3) and (5.14) that  $\delta \rho^\varepsilon$  and  $\delta \phi^\varepsilon$  satisfies the error equations

$$\begin{cases} \partial_t \delta \rho^\varepsilon - \tilde{\Delta}_* \delta \rho^\varepsilon = \mathcal{R}^\varepsilon + \Delta((G_1(\rho^\varepsilon) - G_1(\rho^*))(\rho^\varepsilon - \bar{\rho})) + \Delta(\delta \rho^\varepsilon G_1(\rho^*)) \\ \quad - \mu \operatorname{div}(\delta \rho^\varepsilon \nabla \phi^\varepsilon) - \mu \operatorname{div}((\rho^* - \bar{\rho})\nabla \delta \phi^\varepsilon), \\ \delta \phi^\varepsilon = (b - \Delta)^{-1}(-\varepsilon \partial_t \phi^\varepsilon + a \delta \rho^\varepsilon). \end{cases} \quad (5.15)$$

with the operator  $\tilde{\Delta}_*$  defined by (5.4). It follows by Lemma 5.1 and (2.19) that

$$\begin{aligned} & \|\delta \rho^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta \rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & \lesssim \varepsilon + \|\mathcal{R}^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|(G_1(\rho^\varepsilon) - G_1(\rho^*))(\rho^\varepsilon - \bar{\rho})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & \quad + \|\delta \rho^\varepsilon G_1(\rho^*)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\delta \rho^\varepsilon \nabla \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \|(\rho^* - \bar{\rho})\nabla \delta \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}. \end{aligned} \quad (5.16)$$

Using (5.13), we can immediately get that  $\mathcal{R}^\varepsilon$  is a  $\mathcal{O}(\varepsilon)$  in  $L_t^1(B_{2,1}^{d/2-1})$ :

$$\|\mathcal{R}^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \lesssim \|\rho^\varepsilon v^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \varepsilon \|\partial_t \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \mathcal{X}_0 \varepsilon.$$

In accordance with (5.12), the product law (6.2) and the composition estimate (6.7) from the Appendix, one has

$$\begin{aligned} \|(G_1(\rho^\varepsilon) - G_1(\rho^*))(\rho^\varepsilon - \bar{\rho})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} & \leq \|\delta \rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \|\rho^\varepsilon - \bar{\rho}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\delta \rho^\varepsilon\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\rho^\varepsilon - \bar{\rho}\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & \lesssim \mathcal{X}_0 (\|\delta \rho^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta \rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}). \end{aligned}$$

Similarly, it also holds by (2.17), (5.12) and (6.2) that

$$\begin{aligned} \|\delta \rho^\varepsilon G_1(\rho^*)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} & \lesssim \|\delta \rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \|\rho^* - \bar{\rho}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\delta \rho^\varepsilon\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\rho^* - \bar{\rho}\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & \lesssim \|\rho_0^* - \bar{\rho}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (\|\delta \rho^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta \rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}). \end{aligned}$$

One deduces from (3.17), (5.12) and (5.15)<sub>2</sub> that

$$\begin{aligned} \|\delta\phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})} &\lesssim \varepsilon \|(b - \Delta)^{-1} \partial_t \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})} + \|(b - \Delta)^{-1} \delta\rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})} \\ &\lesssim \varepsilon + \|\delta\rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}, \end{aligned} \quad (5.17)$$

which together with (2.17) and (5.12) yields

$$\begin{aligned} &\|\delta\rho^\varepsilon \nabla \phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \|(\rho^* - \bar{\rho}) \nabla \delta\phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\lesssim \|\delta\rho^\varepsilon\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\phi^\varepsilon - \bar{\phi}\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\rho^* - \bar{\rho}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\delta\phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ &\lesssim \mathcal{X}_0 (\|\delta\rho^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta\rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}) + \varepsilon. \end{aligned}$$

Thence, gathering all the above estimates, we obtain

$$\begin{aligned} &\|\delta\rho^\varepsilon\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|(\delta\rho^\varepsilon, \delta\phi^\varepsilon)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ &\lesssim (\mathcal{X}_0 + \|\rho_0^* - \bar{\rho}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) (\|\delta\rho^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta\rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}) + \varepsilon. \end{aligned}$$

Thus, with the help of the smallness of  $\mathcal{X}_0 + \|\rho_0^* - \bar{\rho}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$ , we conclude that

$$\|\delta\rho^\varepsilon\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta\rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\delta\phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2})} \lesssim \varepsilon. \quad (5.18)$$

Furthermore, using (5.12)<sub>3</sub>, (5.17)-(5.18), (6.7) and  $\delta u^\varepsilon = v^\varepsilon - \nabla \int_{\rho_*}^{\rho^\varepsilon} \frac{P'(s)}{s} ds + \mu \nabla \delta\phi^\varepsilon$ , we get

$$\|\delta u^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \|v^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\delta\rho^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\delta\phi^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \lesssim \varepsilon. \quad (5.19)$$

By (5.18)-(5.19), (2.20) holds. Consequently, as  $\varepsilon \rightarrow 0$ , the global solution  $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)$  for the System (HPC<sub>ε</sub>) converges strongly to the global solution  $(\rho^*, u^*, \phi^*)$  for the System (KS) in the sense (2.21).

## 6 Appendix

We recall some basic properties of Besov spaces and product estimates which will be used repeatedly in this paper. The reader can refer to [4, Chapters 2-3] for more details. Remark that all the properties remain true for the Chemin–Lerner type spaces, up to the modification of the regularity exponent  $s$  according to Hölder’s inequality for the time variable.

The first lemma pertains to the so-called Bernstein inequalities.

**Lemma 6.1** ([4]). *Let  $0 < r < R$ ,  $1 \leq p \leq q \leq \infty$  and  $k \in \mathbb{N}$ . For any function  $u \in L^p$  and  $\lambda > 0$ , it holds*

$$\begin{cases} \text{Supp } \mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq \lambda R\} \Rightarrow \|D^k u\|_{L^q} \lesssim \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \\ \text{Supp } \mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d \mid \lambda r \leq |\xi| \leq \lambda R\} \Rightarrow \|D^k u\|_{L^p} \sim \lambda^k \|u\|_{L^p}. \end{cases}$$



Due to the Bernstein inequalities, the Besov spaces have many useful properties:

**Lemma 6.2** ([4]). *The following properties hold:*

- For any  $s \in \mathbb{R}$ ,  $1 \leq p_1 \leq p_2 \leq \infty$  and  $1 \leq r_1 \leq r_2 \leq \infty$ , it holds

$$\dot{B}_{p_1, r_1}^s \hookrightarrow \dot{B}_{p_2, r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})};$$

- For any  $1 \leq p \leq q \leq \infty$ , we have the following chain of continuous embedding:

$$\dot{B}_{p,1}^0 \hookrightarrow L^p \hookrightarrow \dot{B}_{p,\infty}^0 \hookrightarrow \dot{B}_{q,\infty}^\sigma \quad \text{for } \sigma = -d(\frac{1}{p} - \frac{1}{q}) < 0;$$

- The following real interpolation property is satisfied for all  $1 \leq p \leq \infty$ ,  $s_1 < s_2$  and  $\theta \in (0, 1)$ :

$$\|u\|_{\dot{B}_{p,1}^{\theta s_1 + (1-\theta)s_2}} \lesssim \frac{1}{\theta(1-\theta)(s_2 - s_1)} \|u\|_{\dot{B}_{p,\infty}^{s_1}}^\theta \|u\|_{\dot{B}_{p,\infty}^{s_2}}^{1-\theta}. \quad (6.1)$$

- Let  $\Lambda = (-\Delta)^{\frac{1}{2}}$  be denoted by  $(\Lambda^\sigma u)(x) = \mathcal{F}^{-1}(|\xi|^\sigma \mathcal{F}u)(x)$  for  $\sigma \in \mathbb{R}$ , then  $\Lambda^\sigma$  is an isomorphism from  $\dot{B}_{p,r}^s$  to  $\dot{B}_{p,r}^{s-\sigma}$ ;

- Let  $1 \leq p_1, p_2, r_1, r_2 \leq \infty$ ,  $s_1 \in \mathbb{R}$  and  $s_2 \in \mathbb{R}$  satisfy

$$s_2 < \frac{d}{p_2} \quad \text{or} \quad s_2 = \frac{d}{p_2} \quad \text{and} \quad r_2 = 1.$$

The space  $\dot{B}_{p_1, r_1}^{s_1} \cap \dot{B}_{p_2, r_2}^{s_2}$  endowed with the norm  $\|\cdot\|_{\dot{B}_{p_1, r_1}^{s_1}} + \|\cdot\|_{\dot{B}_{p_2, r_2}^{s_2}}$  is a Banach space and satisfies weak compact and Fatou properties: If  $u_n$  is a uniformly bounded sequence of  $\dot{B}_{p_1, r_1}^{s_1} \cap \dot{B}_{p_2, r_2}^{s_2}$ , then an element  $u$  of  $\dot{B}_{p_1, r_1}^{s_1} \cap \dot{B}_{p_2, r_2}^{s_2}$  and a subsequence  $u_{n_k}$  exist such that

$$\lim_{k \rightarrow \infty} u_{n_k} = u \quad \text{in } \mathcal{S}' \quad \text{and} \quad \|u\|_{\dot{B}_{p_1, r_1}^{s_1} \cap \dot{B}_{p_2, r_2}^{s_2}} \lesssim \liminf_{n_k \rightarrow \infty} \|u_{n_k}\|_{\dot{B}_{p_1, r_1}^{s_1} \cap \dot{B}_{p_2, r_2}^{s_2}}.$$

The following Morse-type product estimates in Besov spaces play a fundamental role in our analysis of nonlinear terms:

**Lemma 6.3** ([4]). *The following statements hold:*

- Let  $p, r \in [1, \infty]$  and  $s > 0$ . Then  $\dot{B}_{p,r}^s \cap L^\infty$  is a algebra and

$$\|uv\|_{\dot{B}_{p,r}^s} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{\dot{B}_{p,r}^s}, \quad (6.2)$$

- For  $p, r \in [1, \infty]$  and  $s \in (-\min\{\frac{d}{p}, \frac{d(p-1)}{p}\}, \frac{d}{p}]$ , there holds

$$\|uv\|_{\dot{B}_{p,r}^s} \lesssim \|u\|_{\dot{B}_{p,1}^{\frac{d}{p}}} \|v\|_{\dot{B}_{p,r}^s}; \quad (6.3)$$

- If  $p \in [1, \infty]$  and  $s \in (-\min\{\frac{d}{p}, \frac{d(p-1)}{p}\}, \frac{d}{p}]$ , then it holds

$$\|uv\|_{\dot{B}_{p,\infty}^s} \lesssim \|u\|_{\dot{B}_{p,1}^{\frac{d}{p}}} \|v\|_{\dot{B}_{p,\infty}^s}; \quad (6.4)$$

The following commutator estimates are used to control some nonlinearities in high frequencies:

**Lemma 6.4** ([4]). *Let  $p \in [1, \infty]$  and  $s \in [-\frac{d}{p} - 1, \frac{d}{p} + 1]$ . Then it holds*

$$\sum_{j \in \mathbb{Z}} 2^{js} \|\dot{S}_{j-1} u \mathcal{A} v - \dot{\Delta}_j (u \mathcal{A} v)\|_{L^p} \lesssim \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|v\|_{\dot{B}_{p,1}^s}, \quad (6.5)$$

where  $\mathcal{A}$  is either  $\nabla$  or  $\text{div}$ .

We recall the classical estimates about the continuity for composition of functions:

**Lemma 6.5** ([4, 17]). *Let  $p, r \in [1, \infty]$ ,  $s > 0$  and  $F \in C^\infty(\mathbb{R})$ . Then for any smooth function  $F(f)$ , there exists a constant  $C_f > 0$  depending only on  $\|f\|_{L^\infty}$ ,  $F$ ,  $s$ ,  $p$  and  $d$  such that*

$$\|F(f) - F(0)\|_{\dot{B}_{p,r}^s} \leq C_f \|f\|_{\dot{B}_{p,r}^s}. \quad (6.6)$$

In addition, if  $-\frac{d}{p} < s \leq \frac{d}{p}$  and  $f_1, f_2 \in \dot{B}_{p,r}^s \cap \dot{B}_{p,1}^{\frac{d}{2}}$ , then we have

$$\|F(f_1) - F(f_2)\|_{\dot{B}_{p,1}^s} \leq C_{f_1, f_2} (1 + \|(f_1, f_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|f_1 - f_2\|_{\dot{B}_{p,1}^s}, \quad (6.7)$$

where the constant  $C_{f_1, f_2} > 0$  depends only on  $\|(f_1, f_2)\|_{L^\infty}$ ,  $F$ ,  $s$ ,  $p$  and  $d$ .

In order to control the nonlinear term  $H(n)$ , we prove some estimates concerning the continuity of composition by a quadratic function.

**Lemma 6.6.** *Let  $J_\varepsilon$  be given by (2.7). For any smooth function  $F(f)$ , it holds that*

$$\|F(f) - F(0) - F'(0)f\|_{\dot{B}_{2,1}^s}^\ell \leq C_f \|f\|_{L^\infty} (\|f\|_{\dot{B}_{2,1}^s}^\ell + \varepsilon^{\sigma-s} \|f\|_{\dot{B}_{2,1}^h}^\ell), \quad s > 0, \quad \sigma \geq 0, \quad (6.8)$$

$$\|F(f) - F(0) - F'(0)f\|_{\dot{B}_{2,1}^s}^h \leq C_f \|f\|_{L^\infty} (\varepsilon^{\sigma-s} \|f\|_{\dot{B}_{2,1}^\sigma}^\ell + \|f\|_{\dot{B}_{2,1}^s}^h), \quad s > 0, \quad \sigma \in \mathbb{R}, \quad (6.9)$$

where  $C_f > 0$  denotes a constant dependent of  $\|f\|_{L^\infty}$ ,  $F''$ ,  $s$ ,  $\sigma$  and  $d$ .

Furthermore, we have

$$\|F(f) - F(0) - F'(0)f\|_{\dot{B}_{2,\infty}^s}^\ell \leq C_f \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (\|f\|_{\dot{B}_{2,\infty}^s}^\ell + \varepsilon^{\frac{d}{2}+1-s} \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h), \quad s \geq -\frac{d}{2}. \quad (6.10)$$

**Proof.** First, we adapt a similar proof as in [10] to prove (6.8). Using Taylor formula at order 2 yields

$$\begin{aligned} F(f) - F(0) - F'(0)f &= \sum_{k' \in \mathbb{Z}} (F(\dot{S}_{k'+1} f) - F(\dot{S}_{k'} f)) - F'(0)f \\ &= \sum_{k' \in \mathbb{Z}} (m_{1,k'} \dot{S}_{k'} f \dot{\Delta}_{k'} f + m_{2,k'} (\dot{\Delta}_{k'} f)^2), \end{aligned} \quad (6.11)$$

with

$$m_{1,k'} := \int_0^1 \int_0^1 F''(\theta(\dot{S}_{k'} f + \tau \dot{\Delta}_{k'} f)) d\tau d\theta, \quad m_{2,k'} := \int_0^1 \int_0^1 F''(\theta(\dot{S}_{k'} f + \tau \dot{\Delta}_{k'} f)) \tau d\tau d\theta.$$

Owing to the Bernstein lemma, we deduce for any  $q \geq 0$  that

$$\|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \leq C(1 + \|f\|_{L^\infty})^q \|f\|_{L^\infty} 2^{(k'-k)q} \|\dot{\Delta}_{k'}f\|_{L^2}. \quad (6.12)$$

It holds by (6.12) that

$$\begin{aligned} & \sum_{k \leq J_\varepsilon, k' \in \mathbb{Z}} 2^{ks} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \\ & \leq \left( \sum_{k' \leq k \leq J_\varepsilon} + \sum_{k \leq k' \leq J_\varepsilon} + \sum_{k \leq J_\varepsilon \leq k'} \right) 2^{ks} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \\ & \leq C_f \|f\|_{L^\infty} \left( \sum_{k' \leq J_\varepsilon} 2^{k's} \|\dot{\Delta}_{k'}f\|_{L^2} \sum_{k' \geq k} 2^{(k-k')s} + \sum_{k' \leq J_\varepsilon} 2^{k's} \|\dot{\Delta}_{k'}f\|_{L^2} \sum_{k' \leq k} 2^{(k-k')(s-[s]-1)} \right. \\ & \quad \left. + \sum_{k \leq J_\varepsilon} 2^{ks} \sum_{k' \geq J_\varepsilon} 2^{-k'\sigma} 2^{k'\sigma} \|\dot{\Delta}_{k'}f\|_{L^2} \right) \\ & \leq C_f \|f\|_{L^\infty} (\|f\|_{\dot{B}_{2,1}^s}^\ell + \varepsilon^{\sigma-s} \|f\|_{\dot{B}_{2,1}^s}^h). \end{aligned}$$

The part of  $m_{2,k'}$  can be controlled similarly. Thus, (6.8) holds.

We turn to the proof of (6.9). Similarly, we begin with the decomposition (6.10). One obtains by (6.12) that

$$\begin{aligned} & \left( \sum_{J_\varepsilon-1 \leq k \leq k'} + \sum_{J_\varepsilon-1 \leq k' \leq k} \right) 2^{ks} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \\ & \leq C_f \|f\|_{L^\infty} \left( \sum_{k' \geq J_\varepsilon-1} 2^{k's} \|\dot{\Delta}_{k'}f\|_{L^2} \sum_{k' \geq k} 2^{(k-k')s} + \sum_{k' \geq J_\varepsilon-1} 2^{k's} \|\dot{\Delta}_{k'}f\|_{L^2} \sum_{k' \leq k} 2^{(k-k')(s-[s]-1)} \right) \quad (6.13) \\ & \leq C_f \|f\|_{L^\infty} \|f\|_{\dot{B}_{2,1}^s}^h. \end{aligned}$$

Concerning the summation on  $\{(k, k') \mid k' \leq J_\varepsilon - 1 \leq k\}$ , we derive

$$\begin{aligned} & \sum_{k' \leq J_\varepsilon-1 \leq k} 2^{ks} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \\ & \leq C_f \|f\|_{L^\infty} \sum_{k' \leq J_\varepsilon-1} 2^{k'(s-\sigma)} 2^{k'\sigma} \|\dot{\Delta}_{k'}f\|_{L^2} \sum_{k' \leq k} 2^{(k-k')(s-[s]-1)} \leq C_f \varepsilon^{\sigma-s} \|f\|_{L^\infty} \|f\|_{\dot{B}_{2,1}^s}^\ell, \end{aligned} \quad (6.14)$$

provided  $s \geq \sigma$ . For  $0 < s < \sigma$ , it also holds that

$$\begin{aligned} & \sum_{k' \leq J_\varepsilon-1 \leq k} 2^{ks} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \\ & \leq 2^{(J_\varepsilon-1)(s-\sigma)} \sum_{k' \leq J_\varepsilon-1 \leq k} 2^{k\sigma} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \leq C_f \varepsilon^{\sigma-s} \|f\|_{L^\infty} \|f\|_{\dot{B}_{2,1}^s}^\ell. \end{aligned} \quad (6.15)$$

Due to (6.13)-(6.15), one has

$$\sum_{k \geq J_\varepsilon-1, k' \in \mathbb{Z}} 2^{ks} \|\dot{\Delta}_k(m_{1,k'}\dot{S}_{k'}f\dot{\Delta}_{k'}f)\|_{L^2} \leq C_f \|f\|_{L^\infty} (\varepsilon^{\sigma-s} \|f\|_{\dot{B}_{2,1}^s}^\ell + \|f\|_{\dot{B}_{2,1}^s}^h).$$

Since the the second part of (6.11) can be estimated similarly, (6.9) follows.

We are left with proving (6.10) for any  $s \geq -\frac{d}{2}$ . The case of  $s > 0$  can be derived by similarly arguments as used in (6.8) and the embedding  $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$ . For  $-\frac{d}{2} \leq s \leq 0$ , one concludes from (3.22), (6.4), (6.6), the Bernstein inequality and the fact  $F(f) - F(0) - F'(0)f = \tilde{F}(f)f$  for some smooth function  $\tilde{F}(f)$  vanishing at 0 that

$$\|F(f) - F(0) - F'(0)f\|_{\dot{B}_{2,\infty}^s}^\ell \leq \|\tilde{F}(f)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|f\|_{\dot{B}_{2,\infty}^s} \leq C_f \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (\|f\|_{\dot{B}_{2,\infty}^s}^\ell + \varepsilon^{\frac{d}{2}+1-s} \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h).$$

The proof of Lemma 6.6 is complete.  $\square$

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