Hyperbolic approximation : Hypocoercivity and hybrid Besov spaces

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$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + up) - \kappa \Delta T - \operatorname{div}(\tau \cdot u) = 0. \end{cases} \tag{1}$$

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A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

And such behavior contradicts the principle of causality.

An alternative

- To correct this unrealistic feature, which is known as the *paradox of heat conduction*, one solution is to use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

where the thermal relaxation characteristic time ε represents the time lag required to establish steady heat conduction in a volume element once a temperature gradient has been imposed across it.

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- We will now focus on this hyperbolic coupling and come back to this hyperbolic Navier-Stokes equations later.

First-order partially dissipative coupling

Porous media approximation

- During my Ph.D, in collaboration with Raphaël Danchin, we studied the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

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- To do so, we derive uniform a priori estimates using the Littlewood-Paley decomposition and tools from the hypocoercivity theory.

Toy-model analysis

Let us have a look at the one-dimensional damped p-system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \dfrac{v}{\varepsilon} = 0. \end{cases}$$

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- **First difficulty:** how to handle the partially dissipative structure? Indeed, standard energy estimates leads to:

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- Idea: Inspired by the hypocoercivity theory, consider the following perturbed functional

$$\mathcal{L}^2 = \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2 + \varepsilon \int_{\mathbb{R}} v\partial_x u.$$

Differentiating in time this functional, one obtains

$$\frac{d}{dt}\mathcal{L}^2 + \frac{1}{\varepsilon}\|v\|_{L^2}^2 + \varepsilon\|(\partial_x u, \partial_x v)\|_{L^2}^2 \leq 0.$$

Toy-model analysis (continued)

- **Second difficulty**: the decay rates depend on the frequencies and the relaxation parameter ε.

  From the previous estimate, one obtains

$$\frac{d}{dt}\|(u,v)\|_{L^2} + \min\left(\frac{1}{\varepsilon}, \varepsilon|\xi|^2\right)\|(u,v)\|_{L^2} \leq 0.$$

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- And, in high frequencies $|\xi| > \frac{1}{\varepsilon}$, the solution is exponentially damped.
- For $U = (u, v)$, one has

$$\|U^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|U_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|U^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq Ct^{-\frac{d}{2}}\|U_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)}$$

  where U^h and U^ℓ correspond, respectively, to the high and low frequencies of the solution.

A side note on hypocoercivity for general systems

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\forall \xi \in \mathbb{R}, \quad \ker B \cap \{\text{eigenvectors of } A\xi\} = \{0\}. \tag{SK}$$

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Inspired by this fact and the theories of hypocoercivity and hypoellipticity, Beauchard and Zuazua constructed the following Lyapunov functional to recover decay estimates:

$$\mathcal{L} \triangleq \|U\|_{H^1}^2 + \int_{\mathbb{R}^d} \mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(BA^{k-1}\widehat{U} \cdot BA^k \widehat{U}\big).$$

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Again, differentiating in time this functional leads to

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0.$$

- However, this *hyperbolic hypocoercivity* approach does not depict the full story for these systems.
- Concerning the study of the high frequency of the solution, such analysis is sufficient but the low frequency behavior of the solution is more involved.
- And, as we shall see, the distinction between these two regime is crucial to justify the relaxation process.

- Back to the damped p-system:

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A spectral analysis of the matrix associated to the system:

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"New" observations

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- The threshold between low and high frequencies is at $\frac{1}{\varepsilon}$
- $\rightarrow$ **The behavior of solution depend on the relation between ξ and ε and there is an extra property to use in low frequencies.**

- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon}$ $\rightarrow$ **Leads to crucial uniform estimates**

Insights from the spectral analysis

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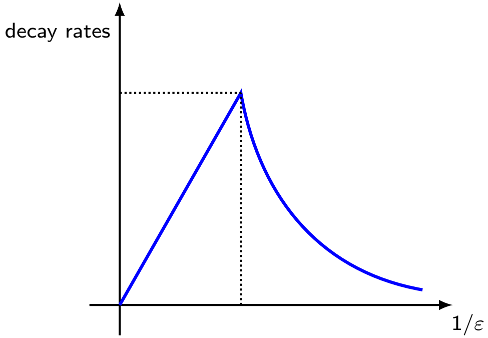
- The asymptotic behaviour of the solution when $\varepsilon \to 0$ is not so intuitive.
  - Naively, we expect that as the damping coefficient becomes larger the dissipation becomes more dominant.
  - However, the so-called *overdamping* effect occurs: **the decay rates are related to** $(\varepsilon, 1/\varepsilon)$**.**



- To handle this phenomenon, not mixing the frequencies and having the threshold between both frequency regime set at $\frac{1}{\varepsilon}$ is crucial.

New goal: replicate what the spectral analysis tells us at the level of the a priori estimates.

Low frequencies

- To be able to take into account that the behavior of the solution depends on the frequency region under the scope, we work with the following hybrid homogeneous Besov norms:

$$\|f\|^{h}_{\dot{\mathbb{B}}^{s}_{2,1}} \triangleq \sum_{j\geq \frac{1}{\varepsilon}} 2^{js}\|\dot{\Delta}_j f\|_{L^2} \quad \text{and} \quad \|f\|^{\ell}_{\dot{\mathbb{B}}^{s'}_{p,1}} \triangleq \sum_{j\leq \frac{1}{\varepsilon}} 2^{js'}\|\dot{\Delta}_j f\|_{L^p}$$

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- And to recover the "missing" property in low frequency, let us look again at damped p-system:

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$$\begin{cases} \partial_t u - \varepsilon\partial^2_{xx} u = -\partial_x w \\ \partial_t w + \dfrac{w}{\varepsilon} = -\varepsilon\partial^2_{xx} v. \end{cases}$$

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$\rightarrow$ **Highlights the behavior observed in the spectral analysis, not just the heat effect as depicted in the previous references.**

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

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- It is possible to study the two equations in a decoupled way without losing derivatives. Indeed, thanks to the Bernstein inequality, the source terms can be absorbed in the low-frequency regime:

$$\|\partial_x f\|^{\ell}_{B^s_{p,1}} \leq \frac{1}{\varepsilon} \|f\|^{\ell}_{B^s_{p,1}}.$$

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

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- One easily obtains

$$\|(u,w)\|^{\ell}_{B^s_{p,1}} + \varepsilon \|u\|^{\ell}_{L^1_T(B^{s+2}_{p,1})} + \frac{1}{\varepsilon} \|w\|^{\ell}_{L^1_T(B^s_{p,1})} \leq \|(u_0, w_0, v_0)\|^{\ell}_{B^s_{p,1}}$$

- Due to the non-zero imaginary part of the eigenvalues in high frequencies, such L^p procedure is not available in this region.
- Nevertheless, we can still perform our analysis in such hybrid $L^2 - L^p$ framework.

The system reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

For this system, the damped mode verifying better properties in low frequencies is $w = u + \dfrac{\nabla P(\rho)}{\rho}$ which is associated to the Darcy law. Inserting it in the above equation one recovers:

$$\partial_t \rho - \Delta P(\rho) = \operatorname{div} w.$$

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Then, using that $\|w\|_{B^s_{p,1}} = \mathcal{O}(\varepsilon)$ (as it solves a purely damped equation), in the error estimates we can deduce that ρ converge strongly, at the rate ε, toward the solution of the porous media equation.

Relaxation result

Theorem (Danchin, C-B, Math. Ann. 2022)

Let $d \geq 1$, $p \in [2,4]$ *and* $\varepsilon > 0$.

- *Let* $\bar{\rho}$ *be a strictly positive constant and* $(\rho - \bar{\rho}, v)$ *be the solution of the compressible Euler system with damping (constructed with the previous arguments)*
- *Let* $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$ *be the unique solution associated to the Cauchy problem:*

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \end{cases}$$

If we assume that

$$\|\widetilde{\rho}_0^{\varepsilon} - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^\infty(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^{\varepsilon})}{\widetilde{\rho}^{\varepsilon}} + \widetilde{v}^{\varepsilon} \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Remarks

- Performing a similar analysis with Sobolev spaces does not allow to exhibit an explicit convergence rate.

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- Performing a similar analysis with Sobolev spaces does not allow to exhibit an explicit convergence rate.
- One has to be careful when justifying the limit $\varepsilon \to 0$. Indeed, recall that the threshold between low and high frequencies is situated at $\frac{1}{\varepsilon}$.
- Therefore, when $\varepsilon \to 0$, the low frequencies recovers the whole space of frequency and the high frequencies disappear.
- And the behavior of the solution in low frequencies is similar to the one of the limit system.

Application to a hyperbolic Navier-Stokes system

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε, by the following hyperbolic system

$$\begin{cases} \partial_t u + \operatorname{div} v = 0 \\ \varepsilon \partial_t v + \nabla u + v = 0. \end{cases}$$

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- Our aim is now to understand to what extent this approximation can be used to approximate systems modelling physical phenomena.

Performing such approximation for the compressible Navier-Stokes system, one has

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t (\rho T) + \operatorname{div}(\rho u T + up) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \partial_t q + q + \kappa \nabla T = 0, \end{cases} \tag{NSCC}$$

Let us now see how to justify that the solution of this system converge to the solution of the classical Navier-Stokes.

Frequency splitting

- First of all, knowledge about the limit system is necessary. Danchin showed the existence of global-in-time strong solutions by highlighting that the solution satisfy different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.

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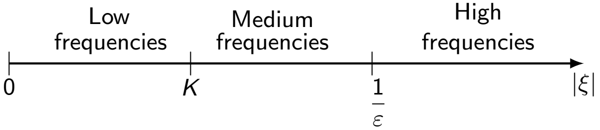
Therefore, in order to obtain the complete picture, it appears natural to divide the frequency space as follows



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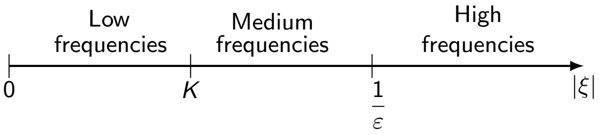

Formally, when $\varepsilon \to 0$, it means that:

- The low frequency regime is not modified.
- The mid-frequency regime becomes larger and larger and recovers the high-frequency regime.
- The high frequency regime disappears.

And, in the limit, we retrieve the behavior of the compressible Navier-Stokes system.

In order to justify this preliminary analysis concretely, it is therefore necessary to introduce functional spaces associated with each regimes: For $p \in [1, \infty]$ and $s \in \mathbb{R}$, we define the homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|^{\ell}_{\dot{B}^s_{p,1}} := \sum_{j \le J_0} 2^{js}\|f_j\|_{L^2}, \qquad \|f\|^{m,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{J_0 \le j \le J_\varepsilon} 2^{js}\|f_j\|_{L^2}$$

$$\text{and} \quad \|f\|^{h,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{j \ge J_\varepsilon - 1} 2^{js}\|f_j\|_{L^2}$$

where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\log_2(\varepsilon)$.

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Then, in each regime, different methods have to be developed to derive a priori estimates. Again, with a combination of hypocoercivity and efficient unknowns but also tools similar to the one that were used to deal with the underlying limit system.

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$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0$$

can be approximated by the 0-th order Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 \boldsymbol{b} = 0, \\ \varepsilon \partial_t \boldsymbol{b} + \mathcal{R}_1 \rho + \boldsymbol{b} = 0. \end{cases} \tag{2DB}$$

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  Such justification involves anisotropic Besov spaces so as to recover crucial L^∞ bounds on the gradient of the solution.
- Quid of general conditions on a given operator for such an approximation to hold.

Thank you for your attention!

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