Well-posedness and pressure-relaxation of multifluid systems[1]

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BENASQUE IX Partial differential equations, optimal design and numerics
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[1] Joint work with Cosmin Burtea, Jin Tan and Ling-Yun Shou

Modelling two-phase flows

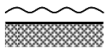
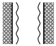
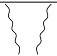
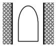
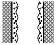
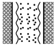
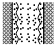
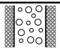
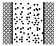
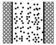
Class	Typical regimes	Geometry	Configuration	Examples
Separated flows	Film flow		Liquid film in gas Gas film in liquid	Film condensation Film boiling
	Annular flow		Liquid core and gas film Gas core and liquid film	Film boiling Boilers
	Jet flow		Liquid jet in gas Gas jet in liquid	Atomization Jet condenser
Mixed or Transitional flows	Cap, Slug or Churn-turbulent flow		Gas pocket in liquid	Sodium boiling in forced convection
	Bubbly annular flow		Gas bubbles in liquid film with gas core	Evaporators with wall nucleation
	Droplet annular flow		Gas core with droplets and liquid film	Steam generator
	Bubbly droplet annular flow		Gas core with droplets and liquid film with gas bubbles	Boiling nuclear reactor channel
Dispersed flows	Bubbly flow		Gas bubbles in liquid	Chemical reactors
	Droplet flow		Liquid droplets in gas	Spray cooling
	Particulate flow		Solid particles in gas or liquid	Transportation of powder

Figure: Ishii 1975, classification of two-phase flow

Modelling of dispersed two-phase flows

- One need to introduce two new unknowns: **volume fractions**, measuring how much space does one phase occupy at a given position in space.
- $\rightarrow$ In order to have a closed system, two extra equations are needed.
- In 1986, Baer and Nunziato proposed a model for which the volume fractions verify

$$\partial_t \alpha_\pm + v_I \cdot \nabla \alpha_\pm = \frac{P_\pm - P_\mp}{\varepsilon}, \tag{1}$$

- Later, Kapila et al. in 2001 proposed the following equations

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ P_+ = P_-. \end{cases} \tag{2}$$

- Clearly, the PDE closure equations (1) can be interpreted as a *pressure-relaxed* version of the algebraic closure equations (2).

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$\rightarrow$ In our works, it is this singular limit that we wish to justify rigorously.

Why? Importance of relaxation procedure

- **Numerically:** reduce the number of constraints imposed on a system and therefore simplifies its numerical study.
- **Theoretically:** establish relationship between different system (possibly under specific scaling)
- If one gets can can get an explicit convergence rate of the singular limit process $\rightarrow$ **one can use strong relaxation** type argument and may obtain new properties:

$$u = u^* + (u - u^*)$$

  where u is the solution of the relaxation system and u^* of the limit system.
- See the works of Danchin on the incompressible limit of the Navier-Stokes equations or Titi et al. on the primitive equation.

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- See the works of Danchin on the incompressible limit of the Navier-Stokes equations or Titi et al. on the primitive equation.
- Maybe in the future for the compressible Euler with nonlinear damping and doubly non-linear equation as Giesselmann and Egger have already established a convergence rate.

System

We consider the following damped Baer-Nunziato system with linear damping

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \frac{\alpha_+ \alpha_-}{\nu} \left(P_+ \left(\rho_+ \right) - P_- \left(\rho_- \right) \right), \\ \partial_t \left(\alpha_\pm \rho_\pm \right) + \operatorname{div} \left(\alpha_\pm \rho_\pm u \right) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P - \nu \Delta u + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = \alpha_+ P_+ \left(\rho_+ \right) + \alpha_- P_- \left(\rho_- \right) \end{cases} \tag{BN}$$

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- **Rigorous derivation of (BN) in the one-dimensional setting by Bresch-Burtea-Lagoutière in 2021, key ingredients:** homogenisation procedure from the mesoscopic scale to the macroscopic one using Young measures to study the propagation of oscillation. (Serre '92).

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- Notice that the pressure-relaxation coefficient depends on the viscosity!
- Goal: study the well-posedness of (BN) close to constant equilibrium and the pressure-relaxation and vanishing viscosity limit $\nu\to 0$ (and $\eta\to\infty$).
- When $\nu\to 0$ (BN) converges formally to the Kapila system:

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_t\left(\alpha_\pm\rho_\pm\right) + \operatorname{div}\left(\alpha_\pm\rho_\pm u\right) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P + \eta\rho u = 0, \\ \rho = \alpha_+\rho_+ + \alpha_-\rho_-, \\ P = P_+\left(\rho_+\right) = P_-\left(\rho_-\right), \end{cases} \qquad (K)$$

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When $\eta \to \infty$, one expects that the solution of (K), under the following rescaling:

$$(\rho^\eta, v^\eta)(t,x) \triangleq (\rho, \eta v)(\eta t, x),$$

converges to the coupled porous media type equations:

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_s(\alpha_\pm \varrho_\pm) - \mathrm{div} \left(\frac{\alpha_\pm \varrho_\pm}{\alpha_+ \varrho_+ + \alpha_- \varrho_-} \nabla \Pi \right) = 0, \\ \nabla \Pi + \varrho v = 0 \\ \Pi = P_+(\varrho_+) = P_-(\varrho_-). \end{cases} \qquad (\mathrm{PM})$$

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$\to$ Similar to the convergence from compressible Euler with damping to the porous media equation (that we get "for free" here as our estimate are uniform in η).

Sum-up of the relaxation processes

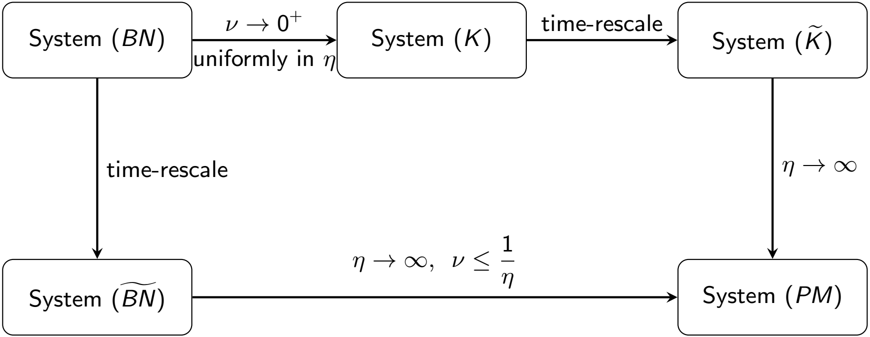

Figure: Relaxation limits diagram.

Equilibrium/Behaviour at infinity

We are concerned with solutions $(\alpha_\pm, \rho_\pm, u)$ close to a constant equilibrium $\left(\bar{\alpha}_\pm, \bar{\rho}_\pm, \overrightarrow{0}\right)$ with far field behaviour

$$\alpha_\pm(t,x) \to \bar{\alpha}_\pm, \quad \rho_\pm(t,x) \to \bar{\rho}_\pm, \quad u(t,x) \to \overrightarrow{0} \quad \text{as } |x| \to \infty, \tag{3}$$

where

$$0 < \bar{\alpha}_\pm < 1, \; 0 < \bar{\rho}_\pm$$

with

$$\bar{\alpha}_+ + \bar{\alpha}_- = 1 \tag{4}$$

while , $P_\pm(\rho_\pm) = A_\pm \rho_\pm^{\gamma_\pm}$ and

$$P_+(\bar{\rho}_+) = P_-(\bar{\rho}_-) \overset{not.}{=} \bar{P}.. \tag{5}$$

Where the last assumption is there to avoid the apparition of initial time-layers.

Difficulties

Back to the Baer-Nunziato system:

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Difficulties:

- Lack of symmetry/symmetrizability
- The system (BN) is not a system of conservation laws.
- The entropy that is naturally associated with this system is only positive semi-definite. Thus, it is not clear if or how the entropy variables can be used to study the well-posedness of the system (BN).
- The associated quasilinear system does not satisfy the (SK) condition as it admits the eigenvalue 0.
- What are the dissipated components?

Strategy

- Lack of symmetry can be treated by constructing bi-linear energy functional, an idea used for instance in Bona-Collin-Guillopé 2013, Burtea 2015, 2016.
- The eigenspace associated to the eigenvalue 0 is of dimension 1 so it essentially means that only one mode.unknown is undamped.
- In our analysis this mode will correspond to the mass fraction $Y_+ = \dfrac{\alpha_+\rho_+}{\rho}$ verifying:
$$\partial_t Y_+ + u \cdot \nabla Y_+ = 0$$
- Separating this mode from the other, we get a partially dissipative system verifying the (SK) condition (ensuring the well-posedness of partially dissipative systems).

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- But what are the others unknowns one should consider?

Finding good unknowns by hand

Guiding principles that we followed:

- keep the velocity unchanged u;
- keep the mass fraction unchanged y;
- a third variable should be a positive multiple of the pressure difference $w = (?)\,(P_+ - P_-)$;
- "the pressure should not be written as a function of the mass fraction"; we are looking for a pressure that is linear w.r.t. w and a fourth variable.
- The contribution coming from the damping in the fourth variable, which we call R, should be at least quadratic w.r.t. w/ν

Reformulation of the system

We consider the following change of variables

$$\begin{cases} w = \Phi_1(\alpha_+, \rho_+, \rho_-) = \dfrac{P_+(\rho_+) - P_-(\rho_-)}{\frac{\gamma_+}{\alpha_+} + \frac{\gamma_-}{\alpha_-}}, \\ R = \Phi_2(\alpha_+, \rho_+, \rho_-) = \dfrac{\frac{\alpha_-}{\gamma_-}}{\frac{\alpha_+}{\gamma_+} + \frac{\alpha_-}{\gamma_-}} P_+ + \dfrac{\frac{\alpha_+}{\gamma_+}}{\frac{\alpha_+}{\gamma_+} + \frac{\alpha_-}{\gamma_-}} P_- \\ Y = \Phi_3(\alpha_+, \rho_+, \rho_-) = \dfrac{\alpha_+ \rho_+}{\alpha_+ \rho_+ + \alpha_- \rho_-}. \end{cases} \tag{6}$$

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The Jacobian of the transformation computed at $(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)$ is

$$J_{|(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)} = \frac{\bar{\rho}_+ \bar{\rho}_-}{\bar{\rho}^2} \cdot \frac{P'_+(\bar{\rho}_+) P'_-(\bar{\rho}_-)}{\frac{\gamma_+}{\bar{\alpha}_+} + \frac{\gamma_-}{\bar{\alpha}_-}} > 0.$$

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W.r.t. to these variables we obtain the system $(\bar{H}_1, \bar{H}_2, \bar{H}_3 > 0)$

$$\begin{cases} \partial_t y + u \cdot \nabla y = 0, \\ \partial_t w + u \cdot \nabla w + \left(\bar{H}_1 + H_1(w, r, y)\right) \operatorname{div} u + \left(\bar{H}_2 + H_2(w, r, y)\right) \dfrac{w}{\nu} = 0, \\ \partial_t r + u \cdot \nabla r + \left(\bar{H}_3 + H_3(w, r, y)\right) \operatorname{div} u = \left(\bar{H}_4 + H_4(w, r, y)\right) \dfrac{w^2}{\nu}, \\ \partial_t u + u \cdot \nabla u + \eta u + \dfrac{1}{\rho} \nabla r + (\gamma_+ - \gamma_-) \dfrac{1}{\rho} \nabla w = 0. \end{cases}$$

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Apply $\alpha \in \mathbb{N}^d$, $\alpha = |k|$ derivatives to the above equation

$$\begin{cases} \partial_t(D^\alpha\rho) + u\nabla(D^\alpha\rho) + \rho \operatorname{div} D^\alpha u = B_1, \\ \partial_t(D^\alpha u) + u\nabla(D^\alpha u) + \dfrac{p'(\rho)}{\rho}\nabla(D^\alpha\rho) = B_2, \end{cases}$$

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where B_1 and B_2 are commutator terms. Omitting these commutators one has

$$\frac{d}{dt} \left\{ \frac{1}{2} \int \frac{p'(\rho)}{\rho^2} |D^\alpha \rho|^2 + |D^\alpha u|^2 \right\} = \int \frac{\partial}{\partial t} \left\{ \frac{p'(\rho)}{\rho^2} \right\} |D^\alpha \rho|^2$$

$\rightarrow$ The terms that would cause a loss of derivative were cancelled thanks to the quasi-linear weight. The extra term appearing is absorbed with bounds on $(\partial_t u, \partial_t \rho)$.

Recovering dissipation

The linear system we are interested in reads:

$$\begin{cases} \partial_t w + u \cdot \nabla w + \operatorname{div} u + \dfrac{w}{\nu} = 0, \\ \partial_t r + u \cdot \nabla r + \operatorname{div} u = \dfrac{w^2}{\nu}, \\ \partial_t u + u \cdot \nabla u + \eta u + \nabla r + \nabla w = 0. \end{cases}$$

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- Example in 1d and $w = 0$: consider the following perturbed functional

$$\mathcal{L}^2 = \|(r, u, \partial_x r, \partial_x u)\|_{L^2}^2 + \int_{\mathbb{R}} u\partial_x r,$$

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$$\mathcal{L}^2 = \|(r, u, \partial_x r, \partial_x u)\|_{L^2}^2 + \int_{\mathbb{R}} u \partial_x r,$$

which allows to recover dissipation properties on all the components. Indeed, one obtains

$$\frac{d}{dt}\mathcal{L}^2 + \|u\|_{L^2}^2 + \|(\partial_x r, \partial_x u)\|_{L^2}^2 \leq 0 \qquad \frac{d}{dt}\mathcal{L}^2 + \min(1, \xi^2)\mathcal{L}^2 \leq 0$$

And since $\mathcal{L}^2 \sim \|(r, u, \partial_x r, \partial_x u)\|_{L^2}^2$, one can derive time-decay estimates **depending on the frequencies!** $\rightarrow$ need to consider two different frequency-regime.

Statement of first main result

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Theorem

Let $d \geq 2$. There exists a constant $c_1 > 0$ independent of the parameters ν (and η) such that if the initial data verify:

$$\|(\alpha_{\pm 0} - \bar{\alpha}_{\pm}, \rho_{\pm 0} - \bar{\rho}_{\pm}, u_0)\|_{B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}} \leq c_1,$$

then System (BN) admits a unique global-in-time solution $(\alpha_{\pm}, \rho_{\pm}, u)$ such that

$$\begin{cases} (\alpha_{\pm} - \bar{\alpha}_{\pm}, \rho_{\pm} - \bar{\rho}_{\pm}, u) \in \mathcal{C}_b(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}), \\ \dfrac{P_+(\rho_+) - P_-(\rho_-)}{2\mu + \lambda} \in L^1(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}}) \quad \text{and} \quad u \in L^1(\mathbb{R}_+; B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}). \end{cases}$$

Moreover, the following estimate holds true uniformly with respect to the viscosity coefficient ν (and the friction parameter η)

$$\|(\alpha_{\pm} - \bar{\alpha}_{\pm}, \rho_{\pm} - \bar{\rho}_{\pm}, u)\|_{L^\infty(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} + \|u\|_{L^1(\mathbb{R}_+; B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1})}$$
$$\frac{1}{2\mu + \lambda} \|P_+(\rho_+) - P_-(\rho_-)\|_{L^1(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}})} \leq C c_1.$$

Second result : convergence rate for the vanishing viscosity and pressure-relaxation

Theorem (Burtea, C-B, Tan)

Let $d \geq 3$ and assume the same hypothesis on the parameters as in Theorem 1. Let $(\alpha_+^\nu, \alpha_-^\nu, \rho_+^\nu, \rho_+^\nu, u^\nu)$ and $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ be the solutions to the Cauchy problem (BN) and (K) associated with the initial data $(\alpha_{\pm+0}^\nu, \rho_{\pm 0}^\nu, u_0^\nu)$.

Then, there exists a constant $C > 0$ independent of ν and η such that

$$\|(\alpha_\pm^\nu - \alpha_\pm, \rho_\pm^\nu - \rho_\pm, \rho_-^\nu - \rho_-, u^\nu - u)\|_{L^\infty(B^{\frac{d}{2}-\frac{1}{2}})} + \|\rho_\pm^\nu - \rho_\pm\|_{L^2(B^{\frac{d}{2}-\frac{1}{2}})}$$
$$+ \|u^\nu - u\|_{L^1(B^{\frac{d}{2}-\frac{1}{2}})} \leq C \left(\|U_0\|_{B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}}} + \sqrt{\nu} \right).$$

And a similar result hold for $\eta \to 0$.

Thank you for your attention!