Hyperbolic approximation: Hypocoercivity and hybrid Besov spaces

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Paradox of heat conduction

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• A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

$\rightarrow$ Such behavior contradicts the principle of causality.

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- Justifying it would give an element of response to the *paradox of heat conduction* **and** allow to simplify the numerical analysis of the Navier-Stokes equations.

First-order partially dissipative coupling

Porous media approximation

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$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

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- Methods: Littlewood-Paley and hypocoercivity theory.

Toy-model analysis

Let us have a look at the one-dimensional damped p-system

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- **First difficulty:** how to handle the *partially dissipative* structure? Indeed, standard energy estimates leads to:

$$\frac{d}{dt}\|(\rho, u)\|_{L^2}^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2 \leq 0$$

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- Idea: Inspired by the hypocoercivity theory, consider the following perturbed functional

$$\mathcal{L}^2 = \|(\rho,u,\partial_x\rho,\partial_x u)\|_{L^2}^2 + \varepsilon\int_{\mathbb{R}} u\partial_x\rho.$$

Differentiating in time this functional, one obtains

$$\frac{d}{dt}\mathcal{L}^2 + \frac{1}{\varepsilon}\|(u,\partial_x u)\|_{L^2}^2 + \varepsilon\|\partial_x\rho\|_{L^2}^2 \leq 0.$$

Toy-model analysis (continued)

- **Second difficulty**: the decay rates depend on the frequencies and the relaxation parameter ε.

  From the previous estimate, one obtains formally

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- And, in high frequencies $|\xi| > \frac{1}{\varepsilon}$, the solution is exponentially damped.
- One has

$$\|(\rho, u)^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda\frac{t}{\varepsilon}}\|(\rho_0, u_0)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|(\rho, u)^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq C(\varepsilon t)^{-\frac{d}{2}}\|(\rho_0, u_0)\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)}$$

  where $(\rho, u)^h$ and $(\rho, u)^\ell$ correspond, respectively, to the high and low frequencies of the solution.

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For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

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$$\mathcal{L}^2 \triangleq \|U\|_{H^1}^2 + \int_{\mathbb{R}^d} \mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(BA^{k-1}\widehat{U} \cdot BA^k \widehat{U}\big).$$

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If the (SK) condition is satisfied, differentiating in time this functional leads to

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0 \quad \text{and } \mathcal{L} \sim \|U\|_{H^1}$$

- However, this *hyperbolic hypocoercivity* approach does not depict the full story for these systems.
- It is suitable to study the high frequencies of the solution but not the low frequencies.
- As we shall see, the distinction between these two regime is crucial: if you do not distinguish two frequency regimes then you can only recover the worst behavior of the two regimes for all the frequencies.

- Back to the damped *p*-system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \dfrac{v}{\varepsilon} = 0. \end{cases} \tag{3}$$

A spectral analysis of the matrix associated to the system:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \dfrac{1}{\varepsilon} \end{pmatrix}$$

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- The threshold between low and high frequencies is at $\frac{1}{\varepsilon}$
- $\rightarrow$ **The behavior of solution depend on the relation between ξ and ε and there is an extra property to use in low frequencies.**

Insights from the spectral analysis

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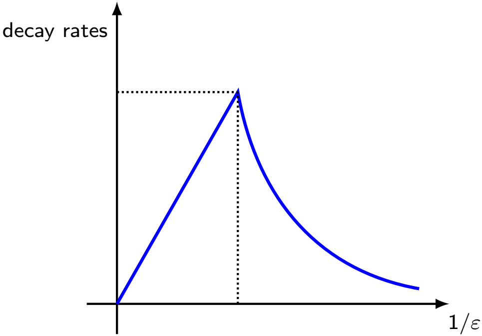
- The asymptotic behaviour of the solution when $\varepsilon \to 0$ is not so intuitive.
  - Naively, we expect that as the damping coefficient becomes larger the dissipation becomes more dominant.
  - However, the so-called *overdamping* effect occurs: **the decay rates are related to** $(\varepsilon, 1/\varepsilon)$**.**



- To handle this phenomenon, not mixing the frequencies and isolating the bad mode is crucial. (Not possible with hypocoercivity)

New goal: replicate what the spectral analysis tells us at the level of the a priori estimates.

- We work with the following hybrid homogeneous Besov norms:

$$\|f\|^{h}_{\dot{\mathbb{B}}^{s}_{2,1}} \triangleq \sum_{j\geq \frac{1}{\varepsilon}} 2^{js}\|\dot{\Delta}_j f\|_{L^2} \quad \text{and} \quad \|f\|^{\ell}_{\dot{\mathbb{B}}^{s'}_{p,1}} \triangleq \sum_{j\leq \frac{1}{\varepsilon}} 2^{js'}\|\dot{\Delta}_j f\|_{L^p}$$

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- And to recover the "missing" property in low-frequency, let us look again at damped p-system:

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Defining the damped mode $w = v + \varepsilon\partial_x u$, the system can be rewritten

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$\rightarrow$ **Highlights the behavior observed in the spectral analysis, not just the heat effect as depicted in the previous references.**

Localizing in frequency the system, for $2^j \leq \dfrac{1}{\varepsilon}$, one has

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- And thus **in L^p spaces**!
- One easily obtains

$$\|(u,w)\|^{\ell}_{B^s_{p,1}} + \varepsilon \|u\|^{\ell}_{L^1_T(B^{s+2}_{p,1})} + \frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_T(B^s_{p,1})} \leq \|(u_0,w_0)\|^{\ell}_{B^s_{p,1}}$$

Localizing in frequency the system, for $2^j \leq \frac{1}{\varepsilon}$, one has

$$\begin{cases} \partial_t \dot{\Delta}_j u - \varepsilon \partial_{xx}^2 \dot{\Delta}_j u = -\partial_x \dot{\Delta}_j w, \\ \partial_t \dot{\Delta}_j w + \dfrac{\dot{\Delta}_j w}{\varepsilon} = -\varepsilon \partial_{xx}^2 \dot{\Delta}_j v. \end{cases}$$

- We can now study these two equations in a decoupled fashion thanks to the Bernstein inequality:

$$\|\partial_x f\|^{\ell}_{B^s_{p,1}} \lesssim \frac{1}{\varepsilon} \|f\|^{\ell}_{B^s_{p,1}}.$$

- And thus **in L^p spaces**!
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- Due to the non-zero imaginary part of the eigenvalues in high frequencies, such L^p procedure is not available in this region.
- Nevertheless, we can still perform our analysis in such hybrid $L^2 - L^p$ framework.

The system reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

Here the damped mode verifying better properties in low frequencies is $w = u + \dfrac{\nabla P(\rho)}{\rho}$ (note that $w = 0$ corresponds to the Darcy law for porous media).

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$$\partial_t \rho - \Delta P(\rho) = \operatorname{div} w.$$

Then, using that $\|w\|_{B^s_{p,1}} = \mathcal{O}(\varepsilon)$ (as it solves a purely damped equation), in the error estimates we can deduce that ρ converge strongly, at the rate ε, toward the solution of the porous media equation:

$$\partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0.$$

Theorem (Danchin, C-B, Math. Ann. 2022)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$.

- *Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution of the compressible Euler system with damping (constructed with the previous arguments)*
- *Let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:*

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \end{cases}$$

If we assume that

$$\|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Sobolev spaces

- Performing a similar analysis with Sobolev spaces does not allow (to the best of my knowledge) to exhibit an explicit convergence rate.
- It only leads to $\|w\|_{L^2_T(H^s)} = \mathcal{O}(1)$ (instead of $\|w\|_{L^1_T(B^s)} = \mathcal{O}(\varepsilon)$)

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Relaxation limit and threshold

- One has to be careful when justifying the limit $\varepsilon \to 0$. Indeed, recall that the threshold between low and high frequencies is situated at $\dfrac{1}{\varepsilon}$.
- Therefore, when $\varepsilon \to 0$, the low frequencies recovers the whole space of frequency and the high frequencies disappear.
- This is coherent as the behavior of the solution in low frequencies is similar to the one of the limit system.

Application to a (partially) hyperbolic Navier-Stokes system

Hyperbolic Navier-Stokes equations

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε, by the following hyperbolic system

$$\begin{cases} \partial_t u + \operatorname{div} v = 0 \\ \varepsilon \partial_t v + \nabla u + v = 0. \end{cases}$$

- Our aim is now to understand to what extent this approximation can be used to approximate systems modelling physical phenomena.

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Performing such approximation for the compressible Navier-Stokes system, one has

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + up) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \partial_t q + q + \kappa \nabla T = 0, \end{cases} \tag{NSCC}$$

Let us now see how to justify that the solution of this system converge to the solution of the classical Navier-Stokes equations.

Frequency splitting

- First of all, knowledge about the limit system is necessary. Danchin showed the existence of solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.

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Therefore, in order to obtain the complete picture, it appears natural to divide the frequency space as follows

Formally, when $\varepsilon \to 0$, it means that:

- The low frequency regime is not modified.
- The mid-frequency regime becomes larger and larger and recovers the high-frequency regime.
- The high frequency regime disappears.

And, in the limit, we retrieve the behavior of the compressible Navier-Stokes system.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|^{\ell}_{\dot{B}^s_{p,1}} := \sum_{j\leq J_0} 2^{js}\|f_j\|_{L^2}, \qquad \|f\|^{m,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{J_0\leq j\leq J_\varepsilon} 2^{js}\|f_j\|_{L^2}$$

$$\text{and} \quad \|f\|^{h,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{j\geq J_\varepsilon -1} 2^{js}\|f_j\|_{L^2}$$

  where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\log_2(\varepsilon)$.

- Then, in each regime, different methods have to be developed to derive a priori estimates. Hypocoercivity + efficient unknowns + tools similar to the one that were used to deal with the underlying limit system.

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Morale

- The hyperbolic approximation *creates* a temporary high frequencies regimes that disappears in the limit.
- The remaining low frequencies corresponds to the behavior of the limit system.
- Due to the partially dissipative nature of this approximation, hypocoercive tools have to be employed. Difficulty: justify that the nonlinear terms can be handled for the *medium* frequencies.

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$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

  can be approximated by the 0-th order stratified Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \tag{2DB}$$

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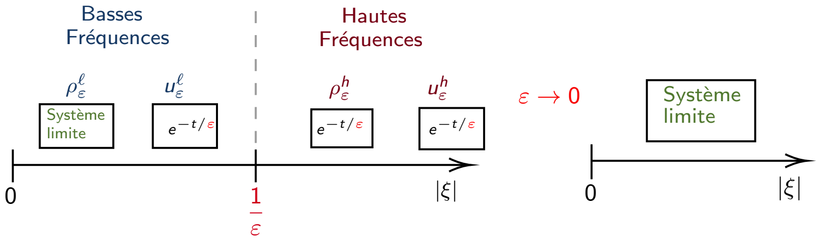
- Question: under what conditions can an operator be approximated in this fashion?
- Interplay of partial dissipation, anisotropy and special nonlinear structure on the time evolution of solutions to hyperbolic and dispersive systems.

Thank you for your attention!

- T. Crin-Barat, R. Danchin, Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit, Mathematatische Annalen, 2022.
- T. Crin-Barat, R. Danchin, Partially dissipative hyperbolic systems in the critical regularity setting: The multi-dimensional case. Journal de Mathématiques Pures et Appliquées, 2022.
- R. Bianchini, T. Crin-Barat, M. Paicu, Relaxation approximation and asymptotic stability of stratified solutions to the IPM equation. arXiv:2210.02118

Approximation de type Cattaneo :

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x u_\varepsilon = 0 \\ \varepsilon \partial_t u_\varepsilon + \partial_x \rho_\varepsilon + u_\varepsilon = 0 \end{cases} \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \partial_t \rho - \Delta \rho = 0$$



Projet 1a : Quid du passage à la limite de relaxation

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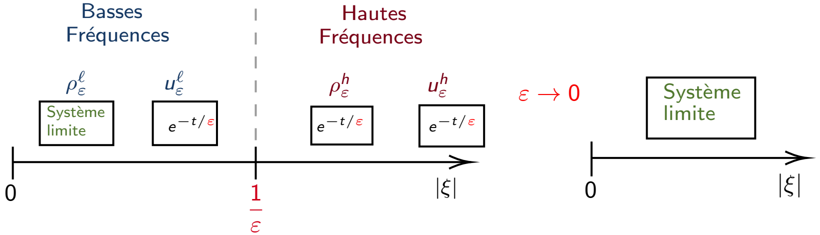

- Nos précédents travaux : justification de **la relaxation forte** dans $\mathbb{R}^d$ de
  - Système d'Euler compressible amorti → Équation des milieux poreux
  - Système hyperbolique-parabolique de chemotaxis → Eqs. de Keller-Segel.