Hyperbolic approximations in fluid mechanics: hypocoercivity and hybrid Besov spaces

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23th November 2023

Part 1: Hyperbolic Navier-Stokes equations

Paradox of heat conduction

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• A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

$\rightarrow$ Such behavior contradicts the principle of causality.

An alternative: Cattaneo's law

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- However, this leads to a non-Galilean invariant model. In '09, Christov formulated the following law

$$\varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q \right) + q = -\kappa \nabla T. \tag{2}$$

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- $\rightarrow$ Finite speed of propagation for the temperature.
- Question: How to justify rigorously the limit $\varepsilon \to 0$?
- Element of response to the *paradox of heat conduction*.
- Useful for numerics.

First-order partially dissipative coupling

Porous media approximation

- The compressible Euler equations with damping reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

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- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Fang-Xu '09, Kawashima-Xu '14
- Strong convergence in $\mathbb{R}^d$ with $d \geq 1$ for global-in-time strong solutions being small perturbations of $(\bar\rho, \bar u) = (\bar\rho, 0)$ with $\bar\rho > 0$: Danchin-CB '22.

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- **Tools**: Littlewood-Paley, Shizuta-Kawashima's theory and hypocoercivity theory.

Approximation of Cattaneo type

Cattaneo approximation:

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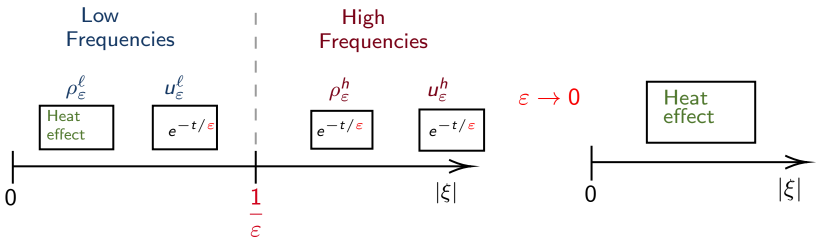

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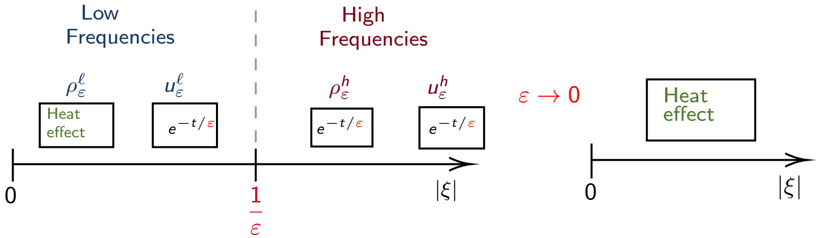

- We proved the strong relaxation limit in $\mathbb{R}^d$ in various contexts
  - Compressible Euler equations with damping (Danchin-CB, Math. Ann.).
  - Jin-Xin System (Shou-CB, JDE).
  - 2D-Boussinesq system (Bianchini-Paicu-CB, ARMA).
- How to show it for the Navier-Stokes-Cattaneo system?

A (partially) hyperbolic Navier-Stokes system

Hyperbolic Navier-Stokes equations

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε, by the following hyperbolic system

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Performing such approximation for the compressible Navier-Stokes system, one has

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Let us now see how to justify that the solution of this system converges to the solution of the classical Navier-Stokes equations.

Frequency splitting

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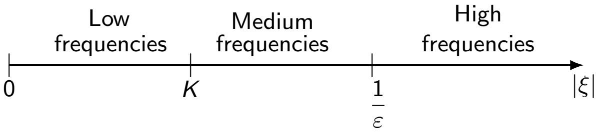

Formally, when $\varepsilon \to 0$, it means that:

- The low frequency regime is not modified.
- The mid-frequency regime becomes larger and larger and recovers the high-frequency regime.
- The high frequency regime disappears.

$\rightarrow$ We retrieve the behavior of the compressible Navier-Stokes-Fourier system in the limit.

Tools & Morale

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|^{\ell}_{\dot{B}^s_{2,1}} := \sum_{j\leq J_0} 2^{js}\|f_j\|_{L^2}, \qquad \|f\|^{m,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{J_0\leq j\leq J_\varepsilon} 2^{js}\|f_j\|_{L^p},$$

$$\|f\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} := \sum_{j\geq J_\varepsilon-1} 2^{js}\|f_j\|_{L^2}$$

  where $J_0=\log_2(K)$, for $K>0$ a constant, and $J_\varepsilon=-\kappa\log_2(\varepsilon)$.

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Morale

- The hyperbolic approximation *creates* a temporary high-frequency regime that disappears in the limit.
- The remaining frequency regimes correspond to the behaviour of the limit system.
- Difficulty: justify that the linear and nonlinear analysis can be done in the *new* high-frequency setting.

Linear a priori estimates scheme

Navier-Stokes-Cattaneo = Partially diffusive + Partially dissipative coupling.

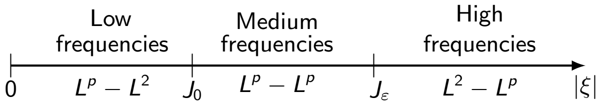

Figure: Frequency domain splitting for the hyperbolic approximation

Some linear analysis in high frequencies

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• Defining the effective velocity, as introduced by Hoff and Haspot, $w = u + (-\Delta)^{-1}\nabla\rho$, in high frequencies, the linear system we are interested in reads

$$\begin{cases} \partial_t\rho + \rho = \operatorname{div} w, \\ \partial_t w - \Delta w = w - (-\Delta)^{-1}\nabla\rho + \nabla\theta, \\ \partial_t\theta + \operatorname{div} q + \operatorname{div} w = 0, \\ \varepsilon^2\partial_t q + q + \nabla\theta = 0, \end{cases} \tag{5}$$

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• For the Cattaneo part, we introduce the Lyapunov (in the spirit of that of Beauchard and Zuazua and the hypocoercivity theory)

$$\mathcal{L}_j^h = \|(\theta_j, q_j)\|_{L^2}^2 + 2^{-2j}\int_{\mathbb{R}^d} q_j \cdot \nabla\theta_j \quad \text{for } j \geq J_\varepsilon. \tag{6}$$

$\to$ The blue term allows to recover dissipation for θ. Using that $\mathcal{L}_j^h \sim \|(\theta_j, q_j)\|_{L^2}^2$, direct computations gives

$$\frac{d}{dt}\mathcal{L}_j^h + \mathcal{L}_j^h \leq \|\operatorname{div} w_j\|_{L^2}\|\theta_j\|_{L^2}.$$

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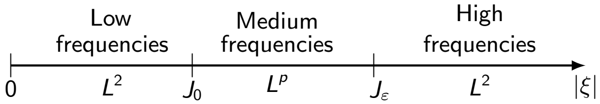

Figure: Frequency domain splitting for Navier-Stokes Cattaneo

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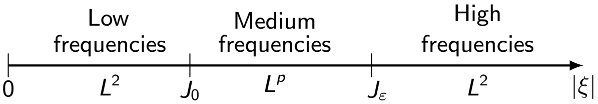

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- Due to the lack of embedding of the type $B^s_{p,1} \hookrightarrow B^s_{2,1}$ if $p > 2 \to$ it is difficult to absorb nonlinearities in the high and low-frequency regimes.
- Indeed, the medium frequencies are only bounded in L^p-based spaces.
- $\to$ Need to develop advanced product laws.

For instance: let $2 \leq p \leq 4$ and $p^* \triangleq 2p/(p-2)$. For all $s > 0$, we have

$$\|ab\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} + \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^{h,\varepsilon}_{\dot{B}^s_{2,1}}$$
$$+ \|a\|^{\ell,\varepsilon}_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^{\ell,\varepsilon}_{\dot{B}^{s+\frac{d}{p}-\frac{d}{2}}_{p,1}} + \|b\|^{\ell,\varepsilon}_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^{\ell,\varepsilon}_{\dot{B}^{s+\frac{d}{p}-\frac{d}{2}}_{p,1}}.$$

Tools: Bony paraproduct decomposition and precise frequency analysis.

Main results for the Navier-Stokes-Cattaneo system

We define $X_0^\varepsilon := X_0^\ell + X_0^{m,\varepsilon} + X_0^{h,\varepsilon}$, where

$$X_0^\ell = \|(\rho_0^\varepsilon - \bar\rho, v_0^\varepsilon, \theta_0^\varepsilon - \bar\theta, \varepsilon q_0^\varepsilon)\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}},$$

$$X_0^{m,\varepsilon} = \|v_0^\varepsilon\|^{m,\varepsilon}_{\dot B^{\frac{d}{p}-1}_{p,1}} + \|\rho_0^\varepsilon - \bar\rho\|^{m,\varepsilon}_{\dot B^{\frac{d}{p}}_{p,1}} + \|(\theta_0^\varepsilon - \bar\theta, \varepsilon q_0^\varepsilon)\|^{m,\varepsilon}_{\dot B^{\frac{d}{p}-2}_{p,1} \cap \dot B^{\frac{d}{p}-1}_{p,1}},$$

$$X_0^{h,\varepsilon} = \varepsilon\|w_0^\varepsilon\|^{h,\varepsilon}_{\dot B^{\frac{d}{2}}_{2,1}} + \varepsilon\|v_0^\varepsilon\|^{h,\varepsilon}_{\dot B^{\frac{d}{2}+1}_{2,1}} + \varepsilon\|\rho_0^\varepsilon - -\bar\rho\|^{h,\varepsilon}_{\dot B^{\frac{d}{2}+1}_{2,1}} + \varepsilon\|(\theta_0^\varepsilon - \bar\theta, \varepsilon q_0^\varepsilon)\|^{h,\varepsilon}_{\dot B^{\frac{d}{2}+1}_{2,1}}.$$

Theorem (Global well-posedness of System NSC)

Let $\varepsilon > 0$, $d \geq 3$, $p \in [2,4]$ and $P(\rho,\theta) = \pi(\rho)\theta$, $\bar\rho, \bar\theta > 0$. There exist $K, k \in \mathbb{Z}$ such that for all $\varepsilon > 0$ satisfying $J_0 \leq J_\varepsilon$, if we assume

$$X_0^\varepsilon \leq \eta_0,$$

then the Navier-Stokes-Cattaneo system admits a unique global-in-time solution $(\rho^\varepsilon - \bar\rho, u^\varepsilon, \theta^\varepsilon - \bar\theta, q^\varepsilon)$ in the space E_p^ε associated to the norm X^ε and the solution satisfies

$$X^\varepsilon(t) \leq X_0.$$

Ill-prepared relaxation result in a critical framework

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Theorem (Kawashima-Xu-Zuazua-CB '23)

Let $\varepsilon > 0$, $d \geq 3$, $p \in [2,4]$ and $P(\rho,\theta) = \pi(\rho)\theta$, $\bar\rho, \bar\theta > 0$.

- *Let $(\rho^\varepsilon - \bar\rho, v^\varepsilon, \theta^\varepsilon - \bar\theta, q^\varepsilon)$ be the global solution of Navier-Stokes-Cattaneo (constructed with the previous arguments) with initial data $(\rho_0^\varepsilon, v_0^\varepsilon, \theta_0^\varepsilon, q_0^\varepsilon)$.*
- *Let $(\rho - \bar\rho, v, \theta - \bar\theta)$ be the global solution of Navier-Stokes-Fourier with initial data (ρ_0, v_0, θ_0).*

We define the error unknowns $(\widetilde\rho, \widetilde v, \widetilde\theta) := (\rho^\varepsilon - \rho, v^\varepsilon - v, \theta^\varepsilon - \theta)$. If we assume that

$$\|(\widetilde\rho_0, \widetilde v_0, \widetilde\theta_0)\|^\ell_{B^{\frac{d}{2}-1}_{2,1}} + \|\widetilde\rho_0\|^h_{B^{\frac{d}{p}-1}_{p,1}} + \|(\widetilde v_0, \widetilde\theta_0)\|^h_{B^{\frac{d}{p}-1}_{p,1}} \lesssim \varepsilon. \tag{7}$$

Then, we have the strong convergence result:

$$\|(\widetilde\rho, \widetilde v, \widetilde\theta)\|^\ell_{L^\infty_T(B^{\frac{d}{2}-2}_{2,1})} + \|(\widetilde\rho, \widetilde v, \widetilde\theta)\|^\ell_{L^1_T(B^{\frac{d}{2}}_{2,1})} + \|q^\varepsilon + \kappa\nabla\theta^\varepsilon\|_{L^1_T(B^{\frac{d}{p}-1}_{p,1})}$$
$$+ \|\widetilde\rho\|^h_{L^\infty_T\cap L^1_T(B^{\frac{d}{p}-1}_{p,1})} + \|(\widetilde v, \widetilde\theta)\|^h_{L^\infty_T(B^{\frac{d}{p}-2}_{p,1})} + \|(\widetilde v, \widetilde\theta)\|^h_{L^1_T(B^{\frac{d}{p}}_{p,1})} \lesssim \varepsilon$$

Extensions

- A fully hyperbolic Navier-Stokes system.
- Hyperbolic approximation in 2D.

Part 2: 2d-Boussinesq equations and incompressible porous media equation

Presentation of the model

The two-dimensional Incompressible Porous Media (IPM) system is the active scalar equation:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \mathbf{u} = -\kappa \nabla P + \mathbf{g}\rho, \qquad \mathbf{g} = (0, -g)^T, \qquad \text{(Darcy law)} \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{IPM}$$

It models the dynamics of a fluid of density $\rho = \rho(t, x, y) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ through a porous medium according to the Darcy law.
The constants $\kappa > 0$ and $g > 0$ are the permeability coefficient and the gravity acceleration respectively, which hereafter are assumed to be $\kappa = g = 1$.

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- Application: transport of a dissolved contaminant in porous media where the contaminant is convected with the subsurface water. For instance, one could be interested in the time taken by the pollutant to reach the water table below.

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The two-dimensional Incompressible Porous Media (IPM) system is the active scalar equation:

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It models the dynamics of a fluid of density $\rho = \rho(t, x, y) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ through a porous medium according to the Darcy law.
The constants $\kappa > 0$ and $g > 0$ are the permeability coefficient and the gravity acceleration respectively, which hereafter are assumed to be $\kappa = g = 1$.

- Application: transport of a dissolved contaminant in porous media where the contaminant is convected with the subsurface water. For instance, one could be interested in the time taken by the pollutant to reach the water table below.
- Mathematical motivation: Less regular than 2D Euler.

Litterature and stratification

The incompressibility condition together with Darcy's law implies that

$$\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\partial_x \rho = (\mathcal{R}_1\mathcal{R}_2\rho, -\mathcal{R}_1^2\rho)$$

where $(\mathcal{R}_1, \mathcal{R}_2)$ is the two-dimensional homogeneous Riesz transform of order 0:

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- For this system, Córdoba, Gancedo and Orive (07') proved the local well-posedness in Hölder space C^{δ} with $0 < \delta < 1$ by the particle-trajectory method.
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- What about global-in-time solutions close to equilibrium?
- Due to the form of the velocity $\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\partial_x \rho$, all the steady states of (IPM) are stratified: constant in x.
- Among these steady states $\bar{\rho}_{eq} = g(y)$, there are only some for which one can hope to stabilise the system around. Here we focus on the linear and stable ones:

$$\bar{\rho}_{eq}(y) = \rho_0 - y$$

where $\rho_0 > 0$ is a constant averaged density.

Stability by stratification

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- Linearizing (IPM) around $\bar{\rho}_{eq}(y) = \rho_0 - y$, one obtains

$$\partial_t \widetilde{\rho} - \mathcal{R}_1^2 \widetilde{\rho} = (\mathcal{R}_2 \mathcal{R}_1 \widetilde{\rho}, -\mathcal{R}_1^2 \widetilde{\rho}) \cdot \nabla \widetilde{\rho}. \qquad \text{(IPM-diss)}$$

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  And, as we shall see, the linear operator $-\mathcal{R}_1^2$ will provide crucial information to justify global well-posedness results.
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- This should lead to instability as it can be related to the Rayleigh–Bénard convection instability occurring even in the presence of diffusion.
- To sum-up: here, in a sense, we will rely on the fact that the stratification inherent in the model serves as a stabilising mechanism to derive global-in-time results.

Literature

- We refer to the work of Elgindi (17') about the justification of asymptotic stability of (IPM) in the whole space $\mathbb{R}^2$ for initial data in $H^{20}(\mathbb{R}^2)$.
- The analogous result in the periodic finite channel in $H^{10}(\mathbb{T} \times [-\pi, \pi])$ is due to Castro, Córdoba and Lear (19').
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Our contributions are:

- The asymptotic stability of (IPM) in $\dot{H}^{1^-}(\mathbb{R}^2)\cap\dot{H}^s(\mathbb{R}^2)$ with $s>3$.
- A new relaxation approximation of (IPM) by the two-dimensional Boussinesq system with damped velocity.
- And, as a byproduct of the above two results, an existence result for the two-dimensional Boussinesq system with damped velocity.

Existence result for (IPM)

Theorem (Bianchini-CB-Paicu (ARMA '24))

Let $0 < \tau < 1$ *and* $s \geq 3 + \tau$. *For any initial datum* $\rho_{\mathrm{in}} \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$, *there exists a constant value* $0 < \delta_0 \ll 1$ *such that, under the assumption*

$$\|\rho_{\mathrm{in}} - \bar{\rho}_{eq}\|_{\dot{H}^{1-\tau} \cap \dot{H}^s} \leq \delta_0,$$

there exists a unique global-in-time smooth solution $\widetilde{\rho}$ *to system* (IPM-diss) *satisfying the following inequality for all times* $t > 0$

$$\|\widetilde{\rho}\|_{L^\infty_T(\dot{H}^{1-\tau} \cap \dot{H}^s)} + \|\mathcal{R}_1 \widetilde{\rho}\|_{L^2_T(\dot{H}^{1-\tau} \cap \dot{H}^s)} + \|\nabla \mathcal{R}_1^2 \widetilde{\rho}\|_{L^1_T(L^\infty)} \lesssim \|\widetilde{\rho}_{\mathrm{in}}\|_{\dot{H}^{1-\tau} \cap \dot{H}^s},$$

where $\widetilde{\rho} = \rho - \bar{\rho}_{eq}$.

Idea of proof

Recall that the equation we are interested in reads:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = (\mathcal{R}_2 \mathcal{R}_1 \rho, -\mathcal{R}_1^2 \rho) \cdot \nabla \rho.$$

To justify the global-in-time existence of this equation, one way is to recover the following bound

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Let us investigate the toy-model:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0. \tag{8}$$

In a Sobolev framework, performing standard energy estimates leads to, for any $s \in \mathbb{R}$,

$$\|\rho\|_{L^\infty_T(H^s)} + \|\mathcal{R}_1 \rho\|_{L^2_T(H^s)} \leq \|\rho_{in}\|_{H^s} \tag{9}$$

Issue: this only gives a L^2-in-time bound that is not enough to control the advection term (except if one assumes $s \geq 20$).

Anisotropic Besov spaces

Anisotropic Besov approach

To derive additional properties from $\partial_t \rho - \mathcal{R}_1^2 \rho = 0$, we will use Littlewood-Paley decompositions adapted to $\mathcal{R}_1$ whose symbol is $\dfrac{\xi_1}{|\xi|}$.

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We introduce the following anisotropic Littlewood-Paley decompositions: for $j, q \in \mathbb{Z}$, we denote

- $\dot{\Delta}_j$ the blocks associated to the direction $|\xi|$;
- $\dot{\Delta}_q^h$ the blocks associated to the direction ξ_1,

and we define the following *homogeneous anisotropic* Besov semi-norms:

$$\|f\|_{\dot{B}^{s_1,s_2}} \triangleq \left\|2^{js_1} 2^{qs_2} \|\dot{\Delta}_j \dot{\Delta}_q^h f\|_{L^2(\mathbb{R}^d)}\right\|_{\ell^1(j\in\mathbb{Z}, k\in\mathbb{Z})}.$$

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• Recall that $\dot{\Delta}_j$ localises the support of the Fourier transform of a distribution in an annulus and $\dot{\Delta}_q^h$ localises it in a stripe. Therefore $\dot{\Delta}_j \dot{\Delta}_q^h$ localises in the intersection of an annulus and a stripe.

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• Such spaces have been used in the past by Chemin, Paicu, Zhang, Xin et al., for instance in the context of the anisotropic Navier-Stokes system and the MHD system.

A new bound

- Applying both localisations, we get

$$\partial_t \dot{\Delta}_j \dot{\Delta}_q^h \rho - \mathcal{R}_1^2 \dot{\Delta}_j \dot{\Delta}_q^h \rho = 0.$$

Standard energy estimates yield

$$\frac{d}{dt} \|\dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2}^2 + \|\mathcal{R}_1 \dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2}^2 = 0$$

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Now, one can apply Gronwall-like inequality to "simplify the squares":

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Then, for any $s_1, s_2 \in \mathbb{R}$, multiplying by $2^{js_1} 2^{qs_2}$ and summing on $j, q \in \mathbb{Z}$:

$$\|\rho\|_{L_T^\infty(\dot{B}^{s_1,s_2})} + \|\rho\|_{L_T^1(\dot{B}^{s_1-2,s_2+2})} \lesssim \|\rho_{in}\|_{\dot{B}^{s_1,s_2}} \qquad (10)$$

Using the embedding $\dot{B}^{\frac{3}{2},\frac{1}{2}} \hookrightarrow \dot{W}^{1,\infty}$, one has:

$$\|\nabla \mathcal{R}_1^2 \rho\|_{L^\infty} \lesssim \|\mathcal{R}_1^2 \rho\|_{\dot{B}_{2,1}^{\frac{3}{2},\frac{1}{2}}} \lesssim \|\rho\|_{\dot{B}_{2,1}^{-\frac{1}{2},\frac{5}{2}}}.$$

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• And to deal with the nonlinearities, we develop new product laws, adapted to this anisotropic framework, and absorb them by the linear left-hand side side.

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- Nevertheless, we are not able to close the estimates in this anisotropic setting. This is due to a lack of commutator estimates adapted to this double localisation, which engenders a loss of one derivative.

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- Therefore, additionally to our anisotropic analysis, we need to perform Sobolev estimates at one regularity higher than the Besov indexes so as to absorb high-order nonlinearities in the Besov analysis.

Lemma (Embedding in Sobolev space)

Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$ such that $\tau_1 < s_1 + s_2 < \tau_2$ and $s_2 > 0$. If $a \in \dot{H}^{\tau_1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2}(\mathbb{R}^2)$ and $a \in B^{s_1, s_2}$, then

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$\longrightarrow$ Reason why $s > 3$ in our work, even though we only involve "2 derivatives" in our Besov framework.

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• For initial data **below** H^2, Kiselev and Yao ('22) proved the time-growth of the norm $\|\rho - \bar{\rho}\|_{H^{s'}}$ for any $s' \geq 1$ and any stratified smooth steady state $\bar{\rho}$ arbitrarily close to the solution, in the bounded strip $\mathbb{T} \times [-\pi, \pi]$.

Relaxation approximation of (IPM)

2D Boussinesq system

The two-dimensional Boussinesq system reads

$$\begin{cases} \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \eta \mathbf{g}, \qquad \mathbf{g} = (0, -g), \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{E}$$

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Considering a damping in the equation of the vorticity and linearizing around the same linear steady states as before, it is shown by Bianchini and Natalini that (E) can be recast into

$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = (\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla b), \\ \partial_t \Omega - \mathcal{R}_1 b + \frac{\Omega}{\varepsilon} - = \Lambda^{-1}[(\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla \Lambda \Omega)], \end{cases} \tag{2D-B}$$

with $\Omega = \Lambda^{-1}\omega$ where ω is the vorticity.

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For this system:

- Wan (19') proved the global well-posedness in H^s with $s \geq 5$.
- Bianchini and Natalini (21') derived time-decay estimates in the same setting.
- Quid of $\varepsilon \to 0$?

Formal link between (IPM) and (2D-B)

Let us have a closer look at the linear structure:

$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = 0, \\ \partial_t \Omega - \mathcal{R}_1 b + \dfrac{\Omega}{\varepsilon} = 0. \end{cases} \tag{11}$$

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Taking inspiration from the theory of partially dissipative systems, in the following "diffusive" scaling:

$$(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)(\tau, x) \triangleq (b, \frac{\Omega}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t, \tag{12}$$

the system (2D-B), in the scaled unknowns $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$, reads:

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Formally, as $\varepsilon \to 0$, the second equation gives the Darcy's law $\widetilde{\Omega}^\varepsilon = \mathcal{R}_1 \widetilde{b}^\varepsilon$ and inserting it in the first one gives the linear part of the incompressible porous media equation:

$$\partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1^2 \widetilde{b}^\varepsilon = 0.$$

Relaxation Theorem

Theorem (Bianchini-CB-Paicu ('22))

Let $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ *be the unique solution of (2D-B) associated to* $(b_{\rm in}, \Omega_{in}) \in H^{3^+}$. *Then, for any* $0 < s' < s$ *and* $0 < \tau < \tau' < 1$, *as* $\varepsilon \to 0$,

$$\widetilde{b}^\varepsilon \to \rho \text{ strongly in } C([0,T], \dot{H}^{1-\tau'}_{loc} \cap \dot{H}^{s-s'}_{loc}),$$

where ρ *is the unique solution of (IPM) associated to the initial data* b_{in}.

Moreover, we recover the Darcy law in the following sense:

$$\|\widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}} \cap B^{\frac{1}{2},\frac{1}{2}})} \leq \varepsilon \mathcal{M}(0).$$

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Proof: follows from uniform estimates established for the system (2D-B). Again, we extract crucial a priori bounds for the solution thanks to the use of anisotropic Besov spaces.

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Future researches:

- Explicit convergence rate?
- What about more general operators?
- Which nonlinearities can we handle for general operators?

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- Hyperbolic relaxation on bounded domains.
- Numerical schemes preserving the asymptotics and the relaxation.

Thank you!